Math 472: Midterm — Wednesday, Oct. 12, 2005

1. (a) Consider the differential equation $y(t) + ty'(t) + yy'(t) = 1$. Find the function f needed by any numerical IVP solver.
(b) Write the *Van der Pol* equation $y''(t) = y'(t)\,(1 - y(t)^2) - y(t)$ as a system of first-order equations.

a) $y + ty' + yy' = 1 \iff y' = \dfrac{1-y}{t+y}$

6 Therefore, equivalent to

$$\underline{y'(t) = f(t,y) = \frac{1-y}{t+y}}$$

6 b) $y'' = y'(1-y^2) - y$

Let $\left.\begin{aligned} y_1 &= y \\ y_2 &= y' \end{aligned}\right\} \Rightarrow \begin{aligned} &\underline{y_1' = y_2} \\ &\underline{y_2' = y_2(1-y_1^2) - y_1} \end{aligned}$

2. (a) What is the basic difference between an *explicit* and an *implicit* method for solving IVPs numerically? List at least one advantage for each type. Give an example of a method for each type.
(b) What is the basic difference between a *single-step* and a *multistep* method for solving IVPs numerically? List at least one advantage for each type. Give an example of a method for each type.
(c) What is the main advantage of Runge-Kutta methods compared with Taylor series methods?

a) • implicit has y_{n+1} on rhs, explicit does not

6 • implicit is more stable, explicit easier to implement

• implicit: trapezoid (AM2), backward Euler,
explicit: AB2, Euler

b) • multistep requires additional startup values

6 • multistep usually higher order, single step easier to implement (does not need additional starting values)

• multistep: AM, AB
single step: Euler

2 c) RK does not require higher derivatives

3. Consider the linear functional

$$Lf = f(x+h) - f(x) - \frac{h}{2}\left[3f'(x) - f'(x-h)\right].$$

(a) Show that L annihilates polynomials in $\mathbb{P}_2$.

(b) Use the Peano kernel theorem to show that $|Lf| \le \frac{5}{12}h^3\|f'''\|_\infty$ on the interval $(x-h, x+h)$.

8 a) Show $Lf = 0$ for $f \in \{1, x, x^2\}$ (a basis for $\mathbb{P}_2$)

$f(x)=1$: $Lf = 1 - 1 - \frac{h}{2}[3(0) - 0] = \underline{\underline{0}}$

$f(x)=x$: $Lf = (x+h) - x - \frac{h}{2}[3-1] = \underline{\underline{0}}$

$f(x)=x^2$: $Lf = (x+h)^2 - x^2 - \frac{h}{2}[6x - 2(x-h)]$

$$= 2xh + h^2 - 3xh + xh - h^2 = \underline{\underline{0}}$$

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b) $$K_2(\xi) = L\left[(x-\xi)_+^2\right]$$

$$= (x-\xi+h)_+^2 - (x-\xi)_+^2 - \frac{h}{2}\left[3\cdot 2(x-\xi)_+ - 2(x-\xi-h)_+\right]$$

$$= (x-\xi+h)_+^2 - (x-\xi)_+^2 - 3h(x-\xi)_+ + h(x-\xi-h)_+$$

Then

$$Lf = \frac{1}{2}\int_{x-h}^{x+h} K_2(\xi) f'''(\xi)\,d\xi$$

so

$$|Lf| \le \frac{1}{2}\|f'''\|_\infty \left|\int_{x-h}^{x+h}\left[(x-\xi+h)_+^2 - (x-\xi)_+^2 - 3h(x-\xi)_+ + h(x-\xi-h)_+\right]d\xi\right|$$

$$= \frac{1}{2}\|f'''\|_\infty \left|\int_{x-h}^{x+h}(x-\xi+h)^2\,d\xi - \int_{x-h}^{x}\left((x-\xi)^2 + 3h(x-\xi)\right)d\xi\right|$$

$$= \frac{1}{2}\|f'''\|_\infty \left|\left[-\frac{(x-\xi+h)^3}{3}\right]_{x-h}^{x+h} + \left[\frac{(x-\xi)^3}{3} - \frac{3h(x-\xi)^2}{2}\right]_{x-h}^{x}\right| = \frac{1}{2}\|f'''\|_\infty \underbrace{\left|\frac{8h^3}{3} - \frac{h^3}{3} + \frac{3h}{2}\right.}_{=\frac{5}{6}h^3}$$

4. Consider the IVP

$$y'(t) = -y(t), \qquad y(0) = 1.$$

(a) What is the value of the approximate solution at $t_1 = 1$ (i.e., $h = 1$) for Euler's method?
(b) What is the value of the approximate solution at $t_1 = 1$ (i.e., $h = 1$) for the backward Euler method?

6 a) Euler, $h=1$

$$\underline{y_1} = y_0 + h\, f(t_0, y_0)$$
$$= 1 + \underbrace{h}_{=1}(\underbrace{-y_0}_{=1}) = \underline{\underline{0}}$$

6 b) backward Euler, $h=1$

$$y_1 = y_0 + h\, f(t_1, y_1)$$
$$= 1 + \underbrace{h}_{=1}(-y_1)$$

$$\Leftrightarrow \quad 2y_1 = 1 \quad \Rightarrow \quad \underline{\underline{y_1 = \tfrac{1}{2}}}$$

5. (a) What does the Butcher tableaux look like for the explicit second-order Runge-Kutta method with $c_2 = \frac{2}{3}$?
(b) What is(are) the formula(s) to compute the "new" value $\boldsymbol{y}_{n+1}$ for this method?

a) Have $b_1 + b_2 = 1$ for general second-order explicit RK.

$c_2 b_2 = \frac{1}{2}$

8 $a_{21} b_2 = \frac{1}{2}$ $\qquad c_2 = \frac{2}{3} \Rightarrow b_2 = \frac{3}{4},\ b_1 = \frac{1}{4},\ a_{21} = \frac{2}{3}$

$$\begin{array}{c|cc} 0 & 0 & 0 \\ \frac{2}{3} & \frac{2}{3} & 0 \\ \hline & \frac{1}{4} & \frac{3}{4} \end{array}$$

6 b) $\bar{y}_{n+1} = \bar{y}_n + \frac{h}{4}\left[\bar{k}_1 + 3\bar{k}_2\right]$

with $\bar{k}_1 = \bar{f}(t_n, \bar{y}_n)$

$\bar{k}_2 = \bar{f}\left(t_n + \frac{2}{3}h, \bar{y}_n + \frac{2}{3}h\bar{k}_1\right)$

6. The two-step *leapfrog method* is defined as

$$y_{n+2} = y_n + 2hf(t_{n+1}, y_{n+1}).$$

(a) Is this a Taylor, Theta, Adams, BDF, Runge-Kutta method? List all that are appropriate.
(b) What is the order of the leapfrog method?
(c) What is its local truncation error?
(d) Use the Dahlquist Equivalence Theorem to show that it is convergent.

2 a) None apply.

b) Have $a_2 = 1,\ a_1 = 0,\ a_0 = -1,\ b_2 = 0,\ b_1 = 2,\ b_0 = 0$

So $\sum_{m=0}^{2} a_m = -1 + 0 + 1 = 0$

8 $$\left.\begin{aligned} &\sum_{m=0}^{2} m a_m - \sum_{m=0}^{2} b_m = 2 - 2 = 0 \\ &\sum_{m=0}^{2} \frac{m^2}{2} a_m - \sum_{m=0}^{2} m b_m = 2 - 2 = 0 \\ &\sum_{m=0}^{2} \frac{m^3}{6} a_m - \sum_{m=0}^{2} \frac{m^2}{2} b_m = \frac{8}{6} - \frac{1}{2}(2) = \frac{1}{3} \neq 0 \end{aligned}\right\} \Rightarrow \underline{\text{order } 2}$$

$\underline{\text{or}}$ $\rho(w) = w^2 - 1 \overset{w = \xi + 1}{\Longrightarrow} \rho(\xi) = (\xi+1)^2 - 1 = \xi^2 + 2\xi$

$\sigma(w) = 2w \Rightarrow \sigma(\xi) = 2(\xi + 1)$

Then $\rho(\xi) - \sigma(\xi)\ln(\xi+1) = \xi^2 + 2\xi - 2(\xi+1)\left[\xi - \frac{\xi^2}{2} + \frac{\xi^3}{3} - \dots\right]$

$= \xi^2 - 2\xi^2 + \xi^2 + 2\xi - 2\xi + \xi^3 - \frac{2}{3}\xi^3 + O(\xi^4)$

$= \frac{1}{3}\xi^3 + O(\xi^4) \Rightarrow \underline{\text{order } 2}$

c) From either method

4 $\underline{\text{LTE} = \frac{1}{3} h^3 f'''(\eta)}$

d) Already know consistent. Check root condition.

4 $\rho(w) = w^2 - 1 = (w+1)(w-1)$ has simple roots on $|w| = 1$

$\Rightarrow$ convergent

7. Show that the BDF

$$y_{n+2} - \frac{4}{3}y_{n+1} + \frac{1}{3}y_n = \frac{2}{3}hf(t_{n+2}, y_{n+2})$$

is convergent.

$$\rho(w) = w^2 - \frac{4}{3}w + \frac{1}{3} = \frac{1}{3}(3w-1)(w-1)$$

6

has simple zero on $|w|=1$

other zero, $w = \frac{1}{3}$, has $|w|<1$

$$\sigma(w) = \frac{2}{3}w^2$$

Now

$$\rho(1) = 1 - \frac{4}{3} + \frac{1}{3} = 0$$

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$$\left.\begin{array}{l} \\ \rho'(w) = 2w - \frac{4}{3} \Rightarrow \rho'(1) = \frac{2}{3} = \sigma(1) \end{array}\right\} \Rightarrow \text{consistent}$$

Together, we get convergence by Dahlquist Equivalence Thm.