Math 478: Midterm — Tuesday, March 20, 2007

SHOW ALL WORK! USE THESE SHEETS ONLY.

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1. Find the Lagrange interpolation polynomial for the data given in the following table

x	7	1	2
y	146	2	1

Use $n=2$ since 3 data points given.

$$P_2(x) = \sum_{j=0}^{2} l_j(x) f(x_j) = 146\, l_0(x) + 2\, l_1(x) + l_2(x)$$

where

$$l_j(x) = \prod_{\substack{k=0 \\ k\neq j}}^{2} \frac{x-x_k}{x_j-x_k}, \quad j=0,1,2$$

So

$$l_0(x) = \frac{x-x_1}{x_0-x_1}\,\frac{x-x_2}{x_0-x_2} = \frac{x-1}{6}\,\frac{x-2}{5} = \frac{1}{30}(x-1)(x-2)$$

$$l_1(x) = \frac{x-x_0}{x_1-x_0}\,\frac{x-x_2}{x_1-x_2} = \frac{x-7}{-6}\,\frac{x-2}{-1} = \frac{1}{6}(x-2)(x-7)$$

$$l_2(x) = \frac{x-x_0}{x_2-x_0}\,\frac{x-x_1}{x_2-x_1} = \frac{x-7}{-5}\,\frac{x-1}{1} = -\frac{1}{5}(x-7)(x-1)$$

Together

$$\underline{\underline{P_2(x)}} = \frac{146}{30}(x-1)(x-2) + \frac{2}{6}(x-2)(x-7) - \frac{1}{5}(x-1)(x-7)$$

$$= \underline{\underline{5x^2 - 16x + 13}}$$

2. In a *Chebyshev* quadrature rule all of the weights are fixed (as opposed to Newton-Cotes formulas where the nodes are fixed), and then the nodes are chosen to make the rule exact for polynomials of maximal degree. Moreover, in order to reduce the number of multiplications all weights are chosen equal, i.e., $A_i = A$. Thus $\int_a^b f(x)dx \approx A \sum_{i=0}^{n} f(x_i)$.

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(a) Determine the nodes and the value of A for a three-point Chebyshev quadrature rule on $[-1, 1]$. You may assume that the nodes are symmetric.

(b) What is the degree of accuracy of this method?

(a) Exact for constants:

want $\int_{-1}^{1} dx = A + A + A \implies \underline{A = \frac{2}{3}}$

Exact for linear polynomials:

want $\int_{-1}^{1} x\,dx = A(x_0 + x_1 + x_2) \overset{\text{symmetry of nodes, i.e. } x_0 = -x_2}{=} A x_1$

$\implies 0 = \frac{2}{3} x_1 \implies \underline{x_1 = 0}$

Exact for quadratic polynomials:

want $\int_{-1}^{1} x^2 dx = A(x_0^2 + x_1^2 + x_2^2) \overset{\text{symmetry and the known values}}{=} \frac{2}{3}\, 2 x_0^2$

$\implies \frac{2}{3} = \frac{2}{3} 2x_0^2 \implies x_0^2 = \frac{1}{2}$ or $\underline{x_0 = \frac{\sqrt{2}}{2}}$

Then $\underline{x_2 = -\frac{\sqrt{2}}{2}}$

$$\implies \int_{-1}^{1} f(x)\,dx \approx \frac{2}{3}\left[f\left(-\frac{\sqrt{2}}{2}\right) + f(0) + f\left(\frac{\sqrt{2}}{2}\right)\right]$$

(b) Check accuracy for higher-order polynomials:

$\int_{-1}^{1} x^3 dx = 0$ and $\frac{2}{3}\left[\left(-\frac{\sqrt{2}}{2}\right)^3 + 0^3 + \left(\frac{\sqrt{2}}{3}\right)^3\right] = 0$ ✓ exact for cubics

$\int_{-1}^{1} x^4 dx = \frac{2}{5}$ and $\frac{2}{3}\left[\left(-\frac{\sqrt{2}}{2}\right)^4 + 0^4 + \left(\frac{\sqrt{2}}{2}\right)^4\right] = \frac{1}{3}$ not exact for quartics

Via polynomial interpolation: $\int_{-1}^{1} f(x)\,dx \approx \sum_{i=0}^{n} f(x_i) \underbrace{\int_{-1}^{1} l_i(x)\,dx}_{=:A_i}$

Want $\int_{-1}^{1} l_0(x)\,dx = \int_{-1}^{1} l_1(x)\,dx = \int_{-1}^{1} l_2(x)\,dx = A$

where $l_j(x) = \prod_{\substack{k=0 \\ k\neq j}}^{2} \frac{x - x_k}{x_j - x_k}$ $\quad j = 0, 1, 2$

By symmetry $x_0 = x_2 = z$, $x_1 = 0$

So, solve $\int_{-1}^{1} l_0(x)\,dx = \int_{-1}^{1} l_1(x)\,dx$ for z

and then solve $\int_{-1}^{1} l_2(x)\,dx = A$ for A

$$\int_{-1}^{1} l_0(x)\,dx = \frac{(x-z)x}{2z^2} \overset{!}{=} \int_{-1}^{1} l_1(x)\,dx = -\frac{(x-z)(x+z)}{z^2}$$

$$\Rightarrow \quad \underline{\underline{z = \pm \frac{\sqrt{2}}{2}}}$$

Then

$$\int_{-1}^{1} l_2(x)\,dx = \int_{-1}^{1} \frac{\left(x + \frac{\sqrt{2}}{2}\right)x}{2\,\frac{1}{2}}\,dx = \underline{\underline{\frac{2}{3} = A}}$$

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3. Determine whether the natural cubic spline that interpolates the table

x	0	1	2	3
y	1	1	0	10

is or is not the function

$$S(x) = \begin{cases} 1 + x - x^3, \quad =: S_0(x) & 0 \le x \le 1, \\ 1 - 2(x-1) - 3(x-1)^2 + 4(x-1)^3, \quad =: S_1(x) & 1 \le x \le 2, \\ 4(x-2) + 9(x-2)^2 - 3(x-2)^3, \quad =: S_2(x) & 2 \le x \le 3. \end{cases}$$

Need to check

$$\left.\begin{aligned} S_0(1) &= S_1(1) = 1 \ ✓ \\ S_1(2) &= S_2(2) = 0 \ ✓ \end{aligned}\right\} C^0 \text{ continuity } ✓$$

$$\left.\begin{aligned} S_0'(1) &= 1 - 3x^2\Big|_{x=1} = -2 = S_1'(1) \ ✓ \\ S_1'(2) &= -2 - 6(x-1) + 12(x-1)^2\Big|_{x=2} = 4 = S_2'(2) \ ✓ \end{aligned}\right\} C^1 \text{ continuity } ✓$$

$$\left.\begin{aligned} S_0''(1) &= -6x\Big|_{x=1} = -6 = S_1''(1) \ ✓ \\ S_1''(2) &= -6 + 24(x-1)\Big|_{x=2} = 18 = S_2''(2) \ ✓ \end{aligned}\right\} C^2 \text{ continuity } ✓$$

Interpolation:

$$\begin{aligned} S(0) &= 1 = y_0 \\ S(1) &= 1 = y_1 \\ S(2) &= 0 = y_2 \\ S(3) &= 10 = y_3 \end{aligned} \quad ✓$$

Natural:

$$S_0''(0) = -6x\Big|_{x=0} = 0 \quad ✓$$

$$S_2''(3) = 18 - 18(x-2)\Big|_{x=3} = 0 \quad ✓$$

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4. (a) Is the following implicit two-step method convergent

$$y_{n+2} - y_n = \frac{1}{3}h\left[f_{n+2} + 4f_{n+1} + f_n\right]?$$

(b) What is the order of the method?

(a) Check consistency:

$$\sum_{m=0}^{s} a_m = a_2 + a_1 + a_0 = 1-1 = 0$$

and $$\sum_{m=0}^{s} m\, a_m = a_1 + 2a_2 = 2 = \sum_{m=0}^{s} b_m = \frac{1}{3}[1+4+1] \quad \checkmark$$

consistent

or $\rho(1) = 0$ and $\rho'(1) = \sigma(1)$

where $\rho(w) = -1 + w^2$ and $\sigma(w) = \frac{1}{3}(1 + 4w + w^2)$

Check root condition:

Roots of ρ are $w = \pm 1$, so simple on unit circle

ok

Together, convergent

(b) Order, continue with $$\sum_{m=0}^{s} \frac{m^k}{k!} a_k \overset{?}{=} \sum_{m=0}^{s} \frac{m^{k-1}}{(k-1)!} b_m$$

$k=2$: $\frac{2^2}{2!} a_2 = 2$, $b_1 + 2b_2 = \frac{4}{3} + 2\frac{1}{3} = \frac{6}{3} = 2$ ✓

$k=3$: $\frac{2^3}{3!} a_2 = \frac{4}{3}$, $\frac{1}{2} b_1 + \frac{2^2}{2!} b_2 = \frac{1}{2}\frac{4}{3} + 2\frac{1}{3} = \frac{4}{3}$ ✓

$k=4$: $\frac{2^4}{4!} a_2 = \frac{2}{3}$, $\frac{1}{6} b_1 + \frac{2^3}{3!} b_2 = \frac{1}{6}\frac{4}{3} + \frac{4}{3}\frac{1}{3} = \frac{2}{3}$ ✓

order 4

$k=5$: $\frac{2^5}{5!} a_2 = \frac{4}{15}$, $\frac{1}{24} b_1 + \frac{2^4}{4!} b_2 = \frac{1}{24}\frac{4}{3} + \frac{2}{3}\frac{1}{3} = \frac{5}{18}$ fails

5. Is there any reason to distrust the following numerical scheme for solving the IVP $y' = f(t, y)$

$$y_{n+3} + y_{n+2} - y_{n+1} - y_n = 2h\left[f_{n+2} + f_{n+1}\right]?$$

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Explain.

<u>Check consistency:</u>

$\sum_{m=0}^{s} a_m = a_3 + a_2 + a_1 + a_0 = 1 + 1 - 1 - 1 = 0$ ✓

and $\sum_{m=0}^{s} m a_m = a_1 + 2a_2 + 3a_3 = -1 + 2 + 3 = 4$

$\sum_{m=0}^{s} b_m = 2 + 2 = 4$

> same, so consistent

<u>Root condition:</u>

$\rho(w) = -1 - w + w^2 + w^3$ has roots $w_0 = 1$

$= (w-1)(w+1)^2$

$w_1 = w_2 = -1$

<u>double</u> root on unit circle

$\Rightarrow$ BUT <u>not</u> stable