Chapter 4

Compactly Supported Radial Basis Functions

As we saw earlier, compactly supported functions Φ that are truly strictly conditionally positive definite of order $m > 0$ do not exist. The compact support automatically ensures that Φ is strictly positive definite. Another observation was that compactly supported radial functions can be strictly positive definite on $\mathbb{R}^s$ only for a fixed maximal s-value. It is not possible for a function to be strictly positive definite and radial on $\mathbb{R}^s$ for all s and also have a compact support. Therefore we focus our attention on the characterization and construction of functions that are compactly supported, strictly positive definite and radial on $\mathbb{R}^s$ for some fixed s.

According to our earlier work (Bochner's Theorem and generalizations thereof), a function is strictly positive definite and radial on $\mathbb{R}^s$ if its s-variate Fourier transform is non-negative. Theorem 2.1.2 gives the Fourier transform of $\Phi = \varphi(\|\cdot\|)$ as

$$\hat{\Phi}(\boldsymbol{x}) = \mathcal{F}_s\varphi(r) = r^{-(s-2)/2}\int_0^\infty \varphi(t)t^{s/2}J_{(s-2)/2}(rt)dt.$$

4.1 Operators for Radial Functions and Dimension Walks

Schaback and Wu [564] defined an integral operator and its inverse differential operator, and discussed an entire calculus for how these operators act on radial functions. These operators will facilitate the construction of compactly supported radial functions.

Definition 4.1.1 *1. Let φ be such that $t \mapsto t\varphi(t) \in L_1[0,\infty)$, then we define*

$$(\mathcal{I}\varphi)(r) = \int_r^\infty t\varphi(t)dt, \qquad r \geq 0.$$

2. For even $\varphi \in C^2(\mathbb{R})$ we define

$$(\mathcal{D}\varphi)(r) = -\frac{1}{r}\varphi'(r), \qquad r \geq 0.$$

In both cases the resulting functions are to be interpreted as even functions using even extension.

Remark: Note that the operator $\mathcal{I}$ differs from the operator I introduced earlier by a factor t in the integrand. However, the two operators are related. In fact, we have $\mathcal{I}\varphi(\cdot^2/2) = I\varphi(\cdot)$, i.e.,

$$\int_r^\infty t\varphi(t^2/2)dt = \int_{r^2/2}^\infty \varphi(t)dt.$$

The most important properties of these operators are (see, e.g., [564] or [627]):

Theorem 4.1.2 *1. Both $\mathcal{D}$ and $\mathcal{I}$ preserve compact support, i.e., if φ has compact support, then so do $\mathcal{D}\varphi$ and $\mathcal{I}\varphi$.*

2. If $\varphi \in C^(\mathbb{R})$ and $t \mapsto t\phi(t) \in L_1[0,\infty)$, then $\mathcal{D}\mathcal{I}\varphi = \varphi$.

3. If $\varphi \in C^2(\mathbb{R})$ is even and $\varphi' \in L_1[0,\infty)$, then $\mathcal{I}\mathcal{D}\varphi = \varphi$.

4. If $t \mapsto t^{s-1}\varphi(t) \in L_1[0,\infty)$ and $s \geq 3$, then $\mathcal{F}_s(\varphi) = \mathcal{F}_{s-2}(\mathcal{I}\varphi)$.

5. If $\varphi \in C^2(\mathbb{R})$ is even and $t \mapsto t^s\varphi'(t) \in L_1[0,\infty)$, then $\mathcal{F}_s(\varphi) = \mathcal{F}_{s+2}(\mathcal{D}\varphi)$.

The operators $\mathcal{I}$ and $\mathcal{D}$ allow us to express s-variate Fourier transforms as $(s-2)$- or $(s+2)$-variate Fourier transforms, respectively. In particular, a direct consequence of the above properties and the characterization of strictly positive definite radial functions (Theorem 2.4.1) is

Theorem 4.1.3 *1. Suppose $\varphi \in C(\mathbb{R})$. If $t \mapsto t^{s-1}\varphi(t) \in L_1[0,\infty)$ and $s \geq 3$, then φ is strictly positive definite and radial on $\mathbb{R}^s$ if and only if $\mathcal{I}\varphi$ is strictly positive definite and radial on $\mathbb{R}^{s-2}$.*

2. If $\varphi \in C^2(\mathbb{R})$ is even and $t \mapsto t^s\varphi'(t) \in L_1[0,\infty)$, then φ is strictly positive definite and radial on $\mathbb{R}^s$ if and only if $\mathcal{D}\varphi$ is strictly positive definite and radial on $\mathbb{R}^{s+2}$.

This allows us to construct new strictly positive definite radial functions from given ones by a "dimension-walk" technique that steps through multivariate Euclidean space in even increments.

4.2 Wendland's Compactly Supported Functions

In [627] Wendland constructed a popular family of compactly supported radial functions by starting with the truncated power function (which we know to be strictly positive definite and radial on $\mathbb{R}^s$ for $s \leq 2\ell - 1$), and then walking through dimensions by repeatedly applying the operator I.

Definition 4.2.1 *With $\varphi_\ell(r) = (1-r)_+^\ell$ we define*

$$\varphi_{s,k} = \mathcal{I}^k \varphi_{\lfloor s/2 \rfloor + k + 1}.$$

It turns out that the functions $\varphi_{s,k}$ are all supported on $[0,1]$ and have a polynomial representation there. More precisely,

Theorem 4.2.2 *The functions $\varphi_{s,k}$ are strictly positive definite and radial on $\mathbb{R}^s$ and are of the form*

$$\varphi_{s,k}(r) = \begin{cases} p_{s,k}(r), & r \in [0,1], \\ 0, & r > 1, \end{cases}$$

with a univariate polynomial $p_{s,k}$ of degree $\lfloor s/2 \rfloor + 3k + 1$. Moreover, $\varphi_{s,k} \in C^{2k}(\mathbb{R})$ are unique up to a constant factor, and the polynomial degree is minimal for given space dimension s and smoothness $2k$.

Wendland gave recursive formulas for the functions $\varphi_{s,k}$ for all s, k. We instead list the explicit formulas of [195]

Theorem 4.2.3 *The functions $\varphi_{s,k}$, $k = 0, 1, 2, 3$, have the form*

$$\begin{aligned}
\varphi_{s,0}(r) &= (1-r)_+^{\ell}, \\
\varphi_{s,1}(r) &\doteq (1-r)_+^{\ell+1}\left[(\ell+1)r + 1\right], \\
\varphi_{s,2}(r) &\doteq (1-r)_+^{\ell+2}\left[(\ell^2+4\ell+3)r^2 + (3\ell+6)r + 3\right], \\
\varphi_{s,3}(r) &\doteq (1-r)_+^{\ell+3}\left[(\ell^3+9\ell^2+23\ell+15)r^3 + (6\ell^2+36\ell+45)r^2 + (15\ell+45)r + 15\right],
\end{aligned}$$

where $\ell = \lfloor s/2 \rfloor + k + 1$, and the symbol $\doteq$ denotes equality up to a multiplicative positive constant.

Proof: The case $k = 0$ follows directly from the definition. Application of the definition for the case $k = 1$ yields

$$\begin{aligned}
\varphi_{s,1}(r) &= (\mathcal{I}\varphi_\ell)(r) = \int_r^\infty t\varphi_\ell(t)dt \\
&= \int_r^\infty t(1-t)_+^{\ell} dt \\
&= \int_r^1 t(1-t)^{\ell} dt \\
&= \frac{1}{(\ell+1)(\ell+2)}(1-r)^{\ell+1}\left[(\ell+1)r + 1\right],
\end{aligned}$$

where the compact support of φ_ℓ reduces the improper integral to a definite integral which can be evaluated using integration by parts. The other two cases are obtained similarly by repeated application of I. □

Examples: For $s = 3$ we get some of the most commonly used functions as

$$\begin{aligned}
\varphi_{3,0}(r) &= (1-r)_+^2, & &\in C^0 \cap SPD(\mathbb{R}^3) \\
\varphi_{3,1}(r) &\doteq (1-r)_+^4\left(4r+1\right), & &\in C^2 \cap SPD(\mathbb{R}^3) \\
\varphi_{3,2}(r) &\doteq (1-r)_+^6\left(35r^2+18r+3\right), & &\in C^4 \cap SPD(\mathbb{R}^3) \\
\varphi_{3,3}(r) &\doteq (1-r)_+^8\left(32r^3+25r^2+8r+1\right), & &\in C^6 \cap SPD(\mathbb{R}^3).
\end{aligned}$$

4.3 Wu's Compactly Supported Functions

In [656] Wu presents another way to construct strictly positive definite radial functions with compact support. He starts with the function

$$\psi(r) = (1-r^2)_+^\ell, \qquad \ell \in \mathbb{N},$$

which is strictly positive definite and radial since we know that the truncated power function $\psi(\sqrt{\cdot})$ is multiply monotone. Wu then constructs another function that is strictly positive definite and radial on $\mathbb{R}$ by convolution, i.e.,

$$\begin{aligned}
\psi_\ell(r) &= (\psi * \psi)(2r) \\
&= \int_{-\infty}^{\infty} (1-t^2)_+^\ell (1-(2r-t)^2)_+^\ell dt \\
&= \int_{-1}^{1} (1-t^2)^\ell (1-(2r-t)^2)^\ell dt.
\end{aligned}$$

This function is strictly positive definite since its Fourier transform is essentially the square of the Fourier transform of ψ. Just like the Wendland functions, this function is a polynomial on its support. In fact, the degree of the polynomial is $4\ell+1$, and $\psi_\ell \in C^{2\ell}(\mathbb{R})$.

Now, a family of strictly positive definite radial functions is constructed by a dimension walk using the $\mathcal{D}$ operator, i.e.,

$$\psi_{k,\ell} = \mathcal{D}^k \psi_\ell.$$

The functions $\psi_{k,\ell}$ are strictly positive definite and radial in $\mathbb{R}^s$ for $s \le 2k+1$, are polynomials of degree $4\ell - 2k + 1$ on their support and in $C^{2(\ell-k)}$ in the interior of the support. On the boundary the smoothness increases to $C^{2\ell-k}$.

Example: For $\ell = 3$ we can compute the three functions

$$\psi_{k,3}(r) = \mathcal{D}^k \psi_3(r) = \mathcal{D}^k((1-\cdot^2)_+^3 * (1-\cdot^2)_+^3)(2r), \qquad k = 0,1,2,3.$$

This results in

$$\begin{aligned}
\psi_{0,3}(r) &\doteq \left(5 - 39r^2 + 143r^4 - 429r^6 + 429r^7 - 143r^9 + 39r^{11} - 5r^{13}\right)_+ \\
&\doteq (1-r)_+^7(5 + 35r + 101r^2 + 147r^3 + 101r^4 + 35r^5 + 5r^6) \qquad \in C^6 \cap SPD(\mathbb{R}) \\
\psi_{1,3}(r) &\doteq \left(6 - 44r^2 + 198r^4 - 231r^5 + 99r^7 - 33r^9 + 5r^{11}\right)_+ \\
&\doteq (1-r)_+^6(6 + 36r + 82r^2 + 72r^3 + 30r^4 + 5r^5) \qquad \in C^4 \cap SPD(\mathbb{R}^3) \\
\psi_{2,3}(r) &\doteq \left(8 - 72r^2 + 105r^3 - 63r^5 + 27r^7 - 5r^9\right)_+ \\
&\doteq (1-r)_+^5(8 + 40r + 48r^2 + 25r^3 + 5r^4) \qquad \in C^2 \cap SPD(\mathbb{R}^5) \\
\psi_{3,3}(r) &\doteq \left(16 - 35r + 35r^3 - 21r^5 + 5r^7\right)_+ \\
&\doteq (1-r)_+^4(16 + 29r + 20r^2 + 5r^3) \qquad \in C^0 \cap SPD(\mathbb{R}^7).
\end{aligned}$$

Remarks:

1. For a prescribed smoothness the polynomial degree of Wendland's functions is lower than that of Wu's functions. For example, both Wendland's function $\varphi_{3,2}$ and Wu's function $\psi_{1,3}$ are C^4 smooth and strictly positive definite and radial in $\mathbb{R}^3$. However, the polynomial degree of Wendland's function is 8, whereas that of Wu's function is 11.

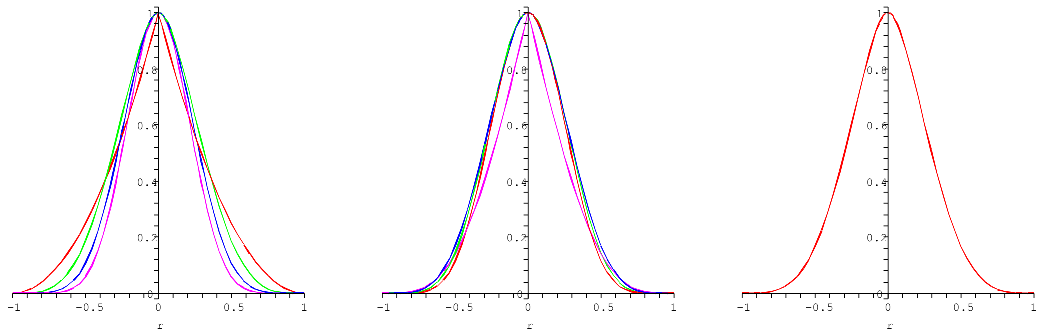

Figure 4.1: Plot of Wendland's functions (left), Wu's functions (center), and Buhmann's function (right) listed as examples.

2. While both families of strictly positive definite compactly supported functions are constructed via dimension walk, Wendland uses integration (and thus obtains a family of increasingly smoother functions), whereas Wu needs to start with a function of sufficient smoothness, and then obtains successively less smooth functions (via differentiation).

4.4 Buhmann's Compactly Supported Functions

A third family of compactly supported strictly positive definite radial functions that has appeared in the literature is due to Buhmann (see [84]). Buhmann's functions contain a logarithmic term in addition to a polynomial. His functions have the general form

$$\phi(r) = \int_0^\infty (1 - r^2/t)_+^\lambda t^\alpha (1 - t^\delta)_+^\rho dt.$$

Here $0 < \delta \leq \frac{1}{2}$, $\rho \geq 1$, and in order to obtain functions that are strictly positive definite and radial on $\mathbb{R}^s$ for $s \leq 3$ the constraints for the remaining parameters are $\lambda \geq 0$, and $-1 < \alpha \leq \frac{\lambda - 1}{2}$.

Example: An example with $\alpha = \delta = \frac{1}{2}$, $\rho = 1$ and $\lambda = 2$ is listed in [85]:

$$\phi(r) \doteq 12r^4 \log r - 21r^4 + 32r^3 - 12r^2 + 1, \qquad 0 \leq r \leq 1, \ \in C^2 \cap SPD(\mathbb{R}^3).$$

Remarks:

1. While Buhmann [85] claims that his construction encompasses both Wendland's and Wu's functions, Wendland [634] gives an even more general theorem that shows that integration of a positive function $f \in L_1[0, \infty)$ against a strictly positive definite (compactly supported) kernel K results in a (compactly supported) strictly positive definite function, i.e.,

$$\varphi(r) = \int_0^\infty K(t, r) f(t) dt$$

is strictly positive definite. Buhmann's construction then corresponds to choosing $f(t) = t^{\alpha}(1 - t^{\delta})_{+}^{\rho}$ and $K(t, r) = (1 - r^2/t)_{+}^{\lambda}$.

2. Multiply monotone functions are covered by this general theorem by taking $K(t, r) = (1 - rt)_{+}^{k-1}$ and f an arbitrary positive function in L_1 so that $d\mu(t) = f(t)dt$ in Williamson's characterization Theorem 2.6.2. Also, functions that are strictly positive definite and radial in $\mathbb{R}^s$ for all s (or equivalently completely monotone functions) are covered by choosing $K(t, r) = e^{-rt}$.