MATH 100 – Introduction to the Profession

Logic

Greg Fasshauer

Department of Applied Mathematics
Illinois Institute of Technology

Fall 2012

Outline[1]

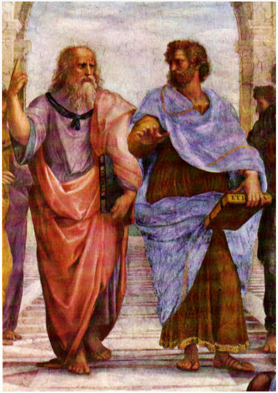
1. "That's Logic"
2. Logical Connectives
3. Conditionals and Biconditionals
4. Truth Tables
5. Looking Ahead Toward Proofs
6. Quantifiers

[1] Most of this discussion is closely linked to [Devlin, Chapter 2], but we also discuss connections to MATLAB and Mathematica where appropriate.

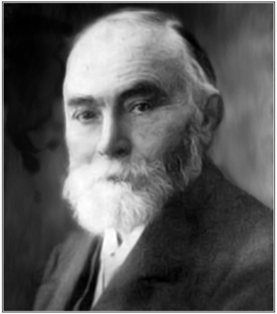
“I know what you’re thinking about,” said Tweedledum: “but it isn’t so, nohow.”
“Contrariwise,” continued Tweedledee, “if it was so, it might be; and if it were so, it would be; but as it isn’t, it ain’t. That’s logic.”

From “Through the Looking Glass” by Lewis Carroll (see `http://www.online-literature.com/carroll/lookingglass/4/` for the complete Chapter 4: Tweedledee and Tweedledum)

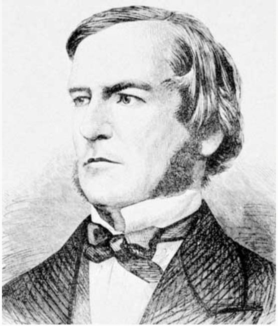
Famous (western) Logicians



Artistole (right, with Plato)



Gottlob Frege



George Boole

"And", "Or" and "Not"

And: used when two "sentences" hold simultaneously

- Mathematical notation: $\wedge$, &
- In words: $\phi \wedge \psi$ is T only if both ϕ and ψ are T

[2] It is recommended that you run all the examples using either MATLAB or Mathematica (or possibly Wolfram Alpha).

"And", "Or" and "Not"

And: used when two "sentences" hold simultaneously

- Mathematical notation: $\wedge$, &
- In words: $\phi \wedge \psi$ is T only if both ϕ and ψ are T
- Example[2](in MATLAB):

```
is_true = (3<pi) && (4>pi)
is_true = (3<pi) & (4>pi)
is_true = and(3<pi, 4>pi)
```

[2] It is recommended that you run all the examples using either MATLAB or Mathematica (or possibly Wolfram Alpha).

"And", "Or" and "Not"

And: used when two "sentences" hold simultaneously

- Mathematical notation: $\wedge$, &
- In words: $\phi \wedge \psi$ is T only if both ϕ and ψ are T
- Example[2](in MATLAB):

```
is_true = (3<pi) && (4>pi)
is_true = (3<pi) & (4>pi)
is_true = and(3<pi, 4>pi)
```

- Example (in Mathematica):

```
(3<Pi) && (4>Pi)
And[3<Pi, 4>Pi]
```

[2] It is recommended that you run all the examples using either MATLAB or Mathematica (or possibly Wolfram Alpha).

“And”, “Or” and “Not”

And: used when two “sentences” hold simultaneously

- Mathematical notation: $\wedge$, &
- In words: $\phi \wedge \psi$ is T only if both ϕ and ψ are T
- Example[2] (in MATLAB):

```
is_true = (3<pi) && (4>pi)
is_true = (3<pi) & (4>pi)
is_true = and(3<pi, 4>pi)
```

- Example (in Mathematica):

```
(3<Pi) && (4>Pi)
And[3<Pi, 4>Pi]
```

- Truth table: `http://www.wolframalpha.com/input/?i=P+%26%26+Q`

[2] It is recommended that you run all the examples using either MATLAB or Mathematica (or possibly Wolfram Alpha).

Or: used when either one of two "sentences" holds (*inclusive "or"*)

- Mathematical notation: $\vee$, $|$
- In words: $\phi \vee \psi$ is T if at least one of ϕ and ψ is T

Or: used when either one of two "sentences" holds (*inclusive "or"*)

- Mathematical notation: $\vee$, $|$
- In words: $\phi \vee \psi$ is T if at least one of ϕ and ψ is T
- Example (in MATLAB):

```
is_true = (pi<3) || (4<pi)
is_true = (pi<3) | (4<pi)
is_true = or(pi<3, 4<pi)
```

Or: used when either one of two “sentences” holds (*inclusive “or”*)

- Mathematical notation: $\vee$, |
- In words: $\phi \vee \psi$ is T if at least one of ϕ and ψ is T
- Example (in MATLAB):

```
is_true = (pi<3) || (4<pi)
is_true = (pi<3) | (4<pi)
is_true = or(pi<3, 4<pi)
```

- Example (in Mathematica):

```
(Pi<3) || (4<Pi)
Or[Pi<3, 4<Pi]
```

Or: used when either one of two “sentences” holds (*inclusive “or”*)

- Mathematical notation: $\vee$, |
- In words: $\phi \vee \psi$ is T if at least one of ϕ and ψ is T
- Example (in MATLAB):

```
is_true = (pi<3) || (4<pi)
is_true = (pi<3) | (4<pi)
is_true = or(pi<3, 4<pi)
```

- Example (in Mathematica):

```
(Pi<3) || (4<Pi)
Or[Pi<3, 4<Pi]
```

- Truth table: `http://www.wolframalpha.com/input/?i=P+||+Q`

Not: negates the "sentence"

- Mathematical notation: $\neg$, $\sim$
- In words: $\neg\phi$ is T if ϕ is F

Not: negates the “sentence”

- Mathematical notation: $\neg$, $\sim$
- In words: $\neg\phi$ is T if ϕ is F
- Example (in MATLAB):

```
is_true = ~(3<pi)
is_true = not(3<pi)
```

Not: negates the "sentence"

- Mathematical notation: $\neg$, $\sim$
- In words: $\neg\phi$ is T if ϕ is F
- Example (in MATLAB):

```
is_true = ~(3<pi)
is_true = not(3<pi)
```

- Example (in Mathematica):

```
!(3<Pi)
Not[3<Pi]
```

Not: negates the “sentence”

- Mathematical notation: $\neg$, $\sim$
- In words: $\neg\phi$ is T if ϕ is F
- Example (in MATLAB):

```
is_true = ~(3<pi)
is_true = not(3<pi)
```

- Example (in Mathematica):

```
!(3<Pi)
Not[3<Pi]
```

- Truth table:

ϕ	$\neg\phi$
T	F
F	T

Example (Negating a statement)

Being a German who loves a good beer, I've been asked by my German friends how I can live in the U.S., claiming

> *All American beer tastes dreadful.*

Example (Negating a statement)

Being a German who loves a good beer, I've been asked by my German friends how I can live in the U.S., claiming

All American beer tastes dreadful.

Obviously, this is not true. So how do I reply if I want to negate this statement?

Example (Negating a statement)

Being a German who loves a good beer, I've been asked by my German friends how I can live in the U.S., claiming

> *All American beer tastes dreadful.*

Obviously, this is not true. So how do I reply if I want to negate this statement?

1. Not all American beer tastes dreadful.
2. All non-American (i.e., German!) beer tastes great.
3. All American beer tastes great.
4. All American beer does not taste dreadful.
5. At least one American beer tastes great.
6. At least one American beer does not taste dreadful.

We will see below which of these statements is *logically most appropriate*.

Example (Leap year calculation, from Ch. 3 [ExM])

Using *this* year (1st element of vector returned by `clock` function)

```
c = clock, y = c(1)
mod(y,4) == 0 && mod(y,100) ~= 0 || mod(y,400) == 0
```

Example (Leap year calculation, from Ch. 3 [ExM])

Using *this* year (1st element of vector returned by `clock` function)

```
c = clock, y = c(1)
mod(y,4) == 0 && mod(y,100) ~= 0 || mod(y,400) == 0
```

Using two years at once

```
y = [2000 2100]
mod(y,4) == 0 && mod(y,100) ~= 0 || mod(y,400) == 0
```

Example (Leap year calculation, from Ch. 3 [ExM])

Using *this* year (1st element of vector returned by `clock` function)

```
c = clock, y = c(1)
mod(y,4) == 0 && mod(y,100) ~= 0 || mod(y,400) == 0
```

Using two years at once

```
y = [2000 2100]
mod(y,4) == 0 && mod(y,100) ~= 0 || mod(y,400) == 0
```

leads to an error.

Example (Leap year calculation, from Ch. 3 [ExM])

Using *this* year (1st element of vector returned by `clock` function)

```
c = clock, y = c(1)
mod(y,4) == 0 && mod(y,100) ~= 0 || mod(y,400) == 0
```

Using two years at once

```
y = [2000 2100]
mod(y,4) == 0 && mod(y,100) ~= 0 || mod(y,400) == 0
```

leads to an error. Replace the "short-circuit"[a] operators `&&` and `||` by the "elementwise" operators `&` and `|`:

```
y = [2000 2100]
mod(y,4) == 0 & mod(y,100) ~= 0 | mod(y,400) == 0
```

Example (Leap year calculation, from Ch. 3 [ExM])

Using *this* year (1st element of vector returned by `clock` function)

```
c = clock, y = c(1)
mod(y,4) == 0 && mod(y,100) ~= 0 || mod(y,400) == 0
```

Using two years at once

```
y = [2000 2100]
mod(y,4) == 0 && mod(y,100) ~= 0 || mod(y,400) == 0
```

leads to an error. Replace the "short-circuit"[a] operators `&&` and `||` by the "elementwise" operators `&` and `|`:

```
y = [2000 2100]
mod(y,4) == 0 & mod(y,100) ~= 0 | mod(y,400) == 0
```

Note: `and` and `or` have "elementwise" interpretations, not "short-circuit". `not` doesn't care. Try it!

[a] The 2nd operand, e.g., `mod(y,100)`, is evaluated only when the result is not fully determined by the 1st operand, e.g., `mod(y,4)`.

Example (Logical operations and arrays in MATLAB)

```
R=rand(4,3)
(R > 0.3) & (R < 0.7)
```

Note: here the "elementwise" operator is required.

Example (Logical operations and arrays in MATLAB)

```
R=rand(4,3)
(R > 0.3) & (R < 0.7)
```

Note: here the “elementwise” operator is required.

Example (Logical operations and `find` in MATLAB)

```
find(R > 0.3 & R < 0.7)
```

Note: `find` goes through matrix in column-major order and returns vector of indices.
More detailed:

Example (Logical operations and arrays in MATLAB)

```
R=rand(4,3)
(R > 0.3) & (R < 0.7)
```

Note: here the “elementwise” operator is required.

Example (Logical operations and `find` in MATLAB)

```
find(R > 0.3 & R < 0.7)
```

Note: `find` goes through matrix in column-major order and returns vector of indices.
More detailed:

```
[row,col,val] = find(R > 0.3 & R < 0.7)
```

Conditional: for logical implications; *if* . . . *then*

- Mathematical notation: $\Rightarrow$, $\rightarrow$

Conditional: for logical implications; *if . . . then*

- Mathematical notation: $\Rightarrow$, $\to$
- Example[3] (in MATLAB):

```
x = -3
if x < 0
    abs_x = -x
else
    abs_x = x
end
```

[3] This computer code use is different than the mathematical use since we can't assign a truth value to the "if . . . then" code fragment.

Conditional: for logical implications; *if* ... *then*

- Mathematical notation: $\Rightarrow$, $\rightarrow$
- Example[3] (in MATLAB):

```
x = -3
if x < 0
    abs_x = -x
else
    abs_x = x
end
```

or – if you want a function – using logical multipliers

```
abs = @(x) (x<0)*(-x) + (x>=0)*x
abs(-3), abs(4)
```

[3] This computer code use is different than the mathematical use since we can't assign a truth value to the "if ... then" code fragment.

Conditional: for logical implications; *if ... then*

- Mathematical notation: $\Rightarrow$, $\rightarrow$
- Example[3] (in MATLAB):

```
x = -3
if x < 0
    abs_x = -x
else
    abs_x = x
end
```

or – if you want a function – using logical multipliers

```
abs = @(x) (x<0)*(-x) + (x>=0)*x
abs(-3), abs(4)
```

- Example (in Mathematica):

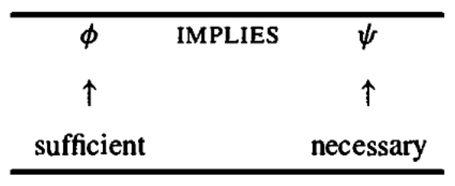
```
abs[x_] := If[x < 0, -x, x]
abs[-3]
abs[-4]
```

[3] This computer code use is different than the mathematical use since we can't assign a truth value to the "if ... then" code fragment.

Truth table:

`http://www.wolframalpha.com/input/?i=P+Implies+Q`

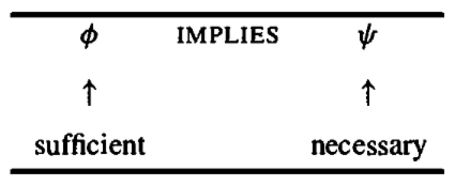
Important concepts for all of mathematics:



Truth table:

`http://www.wolframalpha.com/input/?i=P+Implies+Q`

Important concepts for all of mathematics:



- ϕ implies ψ
- if ϕ then ψ
- ϕ is sufficient for ψ
- ϕ only if ψ
- ψ if ϕ
- ψ whenever ϕ
- ψ is necessary for ϕ

Are conditionals confusing?

Probably so. Consider the following interpretation of the truth table[4]:

- If Springfield is the capital of Illinois, then Springfield is the capital of Illinois

[4] The state capital of Illinois is Springfield.

Are conditionals confusing?

Probably so. Consider the following interpretation of the truth table[4]:

- If Springfield is the capital of Illinois, then Springfield is the capital of Illinois (T $\Rightarrow$ T is T).

[4] The state capital of Illinois is Springfield.

Are conditionals confusing?

Probably so. Consider the following interpretation of the truth table[4]:

- If Springfield is the capital of Illinois, then Springfield is the capital of Illinois (T $\Rightarrow$ T is T). . . . OK

[4]The state capital of Illinois is Springfield.

Are conditionals confusing?

Probably so. Consider the following interpretation of the truth table[4]:

- If Springfield is the capital of Illinois, then Springfield is the capital of Illinois (T $\Rightarrow$ T is T). . . . OK
- If Springfield is the capital of Illinois, then Chicago is the capital of Illinois

[4] The state capital of Illinois is Springfield.

Are conditionals confusing?

Probably so. Consider the following interpretation of the truth table[4]:

- If Springfield is the capital of Illinois, then Springfield is the capital of Illinois (T $\Rightarrow$ T is T). . . . OK
- If Springfield is the capital of Illinois, then Chicago is the capital of Illinois (T $\Rightarrow$ F is F).

[4] The state capital of Illinois is Springfield.

Are conditionals confusing?

Probably so. Consider the following interpretation of the truth table[4]:

- If Springfield is the capital of Illinois, then Springfield is the capital of Illinois (T $\Rightarrow$ T is T). . . . OK
- If Springfield is the capital of Illinois, then Chicago is the capital of Illinois (T $\Rightarrow$ F is F). . . . This better be false!

[4] The state capital of Illinois is Springfield.

Are conditionals confusing?

Probably so. Consider the following interpretation of the truth table[4]:

- If Springfield is the capital of Illinois, then Springfield is the capital of Illinois (T $\Rightarrow$ T is T). . . . OK
- If Springfield is the capital of Illinois, then Chicago is the capital of Illinois (T $\Rightarrow$ F is F). . . . This better be false!
- If Chicago is the capital of Illinois, then Springfield is the capital of Illinois

[4]The state capital of Illinois is Springfield.

Are conditionals confusing?

Probably so. Consider the following interpretation of the truth table[4]:

- If Springfield is the capital of Illinois, then Springfield is the capital of Illinois (T $\Rightarrow$ T is T). . . . OK
- If Springfield is the capital of Illinois, then Chicago is the capital of Illinois (T $\Rightarrow$ F is F). . . . This better be false!
- If Chicago is the capital of Illinois, then Springfield is the capital of Illinois (F $\Rightarrow$ T is T).

[4] The state capital of Illinois is Springfield.

Are conditionals confusing?

Probably so. Consider the following interpretation of the truth table[4]:

- If Springfield is the capital of Illinois, then Springfield is the capital of Illinois (T $\Rightarrow$ T is T). . . . OK
- If Springfield is the capital of Illinois, then Chicago is the capital of Illinois (T $\Rightarrow$ F is F). . . . This better be false!
- If Chicago is the capital of Illinois, then Springfield is the capital of Illinois (F $\Rightarrow$ T is T). . . . Weird!

[4] The state capital of Illinois is Springfield.

Are conditionals confusing?

Probably so. Consider the following interpretation of the truth table[4]:

- If Springfield is the capital of Illinois, then Springfield is the capital of Illinois (T $\Rightarrow$ T is T). . . . OK
- If Springfield is the capital of Illinois, then Chicago is the capital of Illinois (T $\Rightarrow$ F is F). . . . This better be false!
- If Chicago is the capital of Illinois, then Springfield is the capital of Illinois (F $\Rightarrow$ T is T). . . . Weird!
- If Chicago is the capital of Illinois, then Rockford is the capital of Illinois

[4] The state capital of Illinois is Springfield.

Are conditionals confusing?

Probably so. Consider the following interpretation of the truth table[4]:

- If Springfield is the capital of Illinois, then Springfield is the capital of Illinois (T $\Rightarrow$ T is T). . . . OK
- If Springfield is the capital of Illinois, then Chicago is the capital of Illinois (T $\Rightarrow$ F is F). . . . This better be false!
- If Chicago is the capital of Illinois, then Springfield is the capital of Illinois (F $\Rightarrow$ T is T). . . . Weird!
- If Chicago is the capital of Illinois, then Rockford is the capital of Illinois (F $\Rightarrow$ F is T).

[4] The state capital of Illinois is Springfield.

Are conditionals confusing?

Probably so. Consider the following interpretation of the truth table[4]:

- If Springfield is the capital of Illinois, then Springfield is the capital of Illinois (T $\Rightarrow$ T is T). . . . OK
- If Springfield is the capital of Illinois, then Chicago is the capital of Illinois (T $\Rightarrow$ F is F). . . . This better be false!
- If Chicago is the capital of Illinois, then Springfield is the capital of Illinois (F $\Rightarrow$ T is T). . . . Weird!
- If Chicago is the capital of Illinois, then Rockford is the capital of Illinois (F $\Rightarrow$ F is T). . . . Very weird!

[4] The state capital of Illinois is Springfield.

Are conditionals confusing?

Probably so. Consider the following interpretation of the truth table[4]:

- If Springfield is the capital of Illinois, then Springfield is the capital of Illinois (T $\Rightarrow$ T is T). . . . OK
- If Springfield is the capital of Illinois, then Chicago is the capital of Illinois (T $\Rightarrow$ F is F). . . . This better be false!
- If Chicago is the capital of Illinois, then Springfield is the capital of Illinois (F $\Rightarrow$ T is T). . . . Weird!
- If Chicago is the capital of Illinois, then Rockford is the capital of Illinois (F $\Rightarrow$ F is T). . . . Very weird!

Read (carefully!) the discussion in [Devlin, pp. 18-19].

Conditional in words:

$\phi \Rightarrow \psi$ is only then not T if ψ is F in spite of ϕ being T

More colloquially: $\phi \Rightarrow \psi$ is considered true until proven false ("innocent until proven guilty").

[4] The state capital of Illinois is Springfield.

Another example for the conditional

Let's consider the following promise I make to you:

> *If you average 85% or above in this class, then you will get an A.*

Another example for the conditional

Let's consider the following promise I make to you:

> *If you average 85% or above in this class, then you will get an A.*

At the end of the semester we could have various outcomes:

- You average 90% and get an A (T $\rightarrow$ T is T, i.e., I kept my promise)

Another example for the conditional

Let's consider the following promise I make to you:

> *If you average 85% or above in this class, then you will get an A.*

At the end of the semester we could have various outcomes:

- You average 90% and get an A (T $\rightarrow$ T is T, i.e., I kept my promise)
- You average 90% and get a B (T $\rightarrow$ F is F, i.e., I broke my promise)

Another example for the conditional

Let's consider the following promise I make to you:

> *If you average 85% or above in this class, then you will get an A.*

At the end of the semester we could have various outcomes:

- You average 90% and get an A (T $\to$ T is T, i.e., I kept my promise)
- You average 90% and get a B (T $\to$ F is F, i.e., I broke my promise)
- You average 80% and get an A (F $\to$ T is T, i.e., I didn't break my promise $\ddot\smile$)

Another example for the conditional

Let's consider the following promise I make to you:

If you average 85% or above in this class, then you will get an A.

At the end of the semester we could have various outcomes:

- You average 90% and get an A ($T \to T$ is T, i.e., I kept my promise)
- You average 90% and get a B ($T \to F$ is F, i.e., I broke my promise)
- You average 80% and get an A ($F \to T$ is T, i.e., I didn't break my promise ⌣̈)
- You average 80% and get a B ($F \to F$ is T, i.e., I didn't break my promise – and you received what you expected)

Another example for the conditional

Let's consider the following promise I make to you:

> *If you average 85% or above in this class, then you will get an A.*

At the end of the semester we could have various outcomes:

- You average 90% and get an A (T $\rightarrow$ T is T, i.e., I kept my promise)
- You average 90% and get a B (T $\rightarrow$ F is F, i.e., I broke my promise)
- You average 80% and get an A (F $\rightarrow$ T is T, i.e., I didn't break my promise ☺)
- You average 80% and get a B (F $\rightarrow$ F is T, i.e., I didn't break my promise – and you received what you expected)

This illustrates perfectly that scoring 85% or higher was sufficient for an A, but not necessary.

And a mathematical example

Let's assume that n is a positive integer. Then

$$(n \text{ is a perfect square with last digit 7}) \Rightarrow (n \text{ is a prime number})$$

is a true statement.

And a mathematical example

Let's assume that n is a positive integer. Then

$$(n \text{ is a perfect square with last digit 7}) \Rightarrow (n \text{ is a prime number})$$

is a true statement.
This is so, because we have both (by **definition**)

- F $\Rightarrow$ T is T
- F $\Rightarrow$ F is T

and we know that no perfect square ends in 7 (so it is irrelevant that all we know about n is that it is a positive integer).

And a mathematical example

Let's assume that n is a positive integer. Then

$$(n \text{ is a perfect square with last digit 7}) \Rightarrow (n \text{ is a prime number})$$

is a true statement.
This is so, because we have both (by **definition**)

- F $\Rightarrow$ T is T
- F $\Rightarrow$ F is T

and we know that no perfect square ends in 7 (so it is irrelevant that all we know about n is that it is a positive integer).

Remark

Statements like these (and the earlier ones about the IL state capital) do not agree with common sense. Usually we work with statements that are in some logical context (see the discussion of causation in [Devlin, p. 17]).

Biconditional: for logical equivalence; *if and only if*

- Mathematical notation: $\Leftrightarrow$, $\leftrightarrow$
- Does not make sense in MATLAB (only in Symbolic Toolbox)

Biconditional: for logical equivalence; *if and only if*

- Mathematical notation: $\Leftrightarrow$, $\leftrightarrow$
- Does not make sense in MATLAB (only in Symbolic Toolbox)
- Example (in Mathematica):

```
Equivalent[3<Pi, 4>Pi]
```

Biconditional: for logical equivalence; *if and only if*

- Mathematical notation: $\Leftrightarrow$, $\leftrightarrow$
- Does not make sense in MATLAB (only in Symbolic Toolbox)
- Example (in Mathematica):

  `Equivalent[3<Pi, 4>Pi]`

- Truth table: `http://www.wolframalpha.com/input/?i=P+Equivalent+Q`

Further insights via truth tables

Example

Prove $\neg(\phi \wedge \psi)$ is equivalent to $(\neg\phi) \vee (\neg\psi)$.

Further insights via truth tables

Example

Prove $\neg(\phi \wedge \psi)$ is equivalent to $(\neg\phi) \vee (\neg\psi)$.

ϕ	ψ	$\phi \wedge \psi$	$\neg(\phi \wedge \psi)$
T	T		
T	F		
F	T		
F	F		

ϕ	ψ	$\neg\phi$	$\neg\psi$	$(\neg\phi) \vee (\neg\psi)$
T	T			
T	F			
F	T			
F	F			

Further insights via truth tables

Example

Prove $\neg(\phi \wedge \psi)$ is equivalent to $(\neg\phi) \vee (\neg\psi)$.

ϕ	ψ	$\phi \wedge \psi$	$\neg(\phi \wedge \psi)$
T	T	T	
T	F	F	
F	T	F	
F	F	F	

ϕ	ψ	$\neg\phi$	$\neg\psi$	$(\neg\phi) \vee (\neg\psi)$
T	T			
T	F			
F	T			
F	F			

Further insights via truth tables

Example

Prove $\neg(\phi \wedge \psi)$ is equivalent to $(\neg\phi) \vee (\neg\psi)$.

ϕ	ψ	$\phi \wedge \psi$	$\neg(\phi \wedge \psi)$
T	T	T	F
T	F	F	T
F	T	F	T
F	F	F	T

ϕ	ψ	$\neg\phi$	$\neg\psi$	$(\neg\phi) \vee (\neg\psi)$
T	T			
T	F			
F	T			
F	F			

Further insights via truth tables

Example

Prove $\neg(\phi \wedge \psi)$ is equivalent to $(\neg\phi) \vee (\neg\psi)$.

ϕ	ψ	$\phi \wedge \psi$	$\neg(\phi \wedge \psi)$
T	T	T	F
T	F	F	T
F	T	F	T
F	F	F	T

ϕ	ψ	$\neg\phi$	$\neg\psi$	$(\neg\phi) \vee (\neg\psi)$
T	T	F	F	
T	F	F	T	
F	T	T	F	
F	F	T	T	

Further insights via truth tables

Example

Prove $\neg(\phi \wedge \psi)$ is equivalent to $(\neg\phi) \vee (\neg\psi)$.

ϕ	ψ	$\phi \wedge \psi$	$\neg(\phi \wedge \psi)$
T	T	T	F
T	F	F	T
F	T	F	T
F	F	F	T

ϕ	ψ	$\neg\phi$	$\neg\psi$	$(\neg\phi) \vee (\neg\psi)$
T	T	F	F	F
T	F	F	T	T
F	T	T	F	T
F	F	T	T	T

Example (Exercise 2.2.9(a) in [Devlin])

Prove $\neg(\phi \Rightarrow \psi)$ is equivalent to $\phi \wedge (\neg\psi)$.

ϕ	ψ	$\phi \Rightarrow \psi$	$\neg(\phi \Rightarrow \psi)$
T	T		
T	F		
F	T		
F	F		

ϕ	ψ	$\phi \wedge (\neg\psi)$
T	T	
T	F	
F	T	
F	F	

Example (Exercise 2.2.9(a) in [Devlin])

Prove $\neg(\phi \Rightarrow \psi)$ is equivalent to $\phi \wedge (\neg\psi)$.

ϕ	ψ	$\phi \Rightarrow \psi$	$\neg(\phi \Rightarrow \psi)$
T	T	T	
T	F	F	
F	T	T	
F	F	T	

ϕ	ψ	$\phi \wedge (\neg\psi)$
T	T	
T	F	
F	T	
F	F	

Example (Exercise 2.2.9(a) in [Devlin])

Prove $\neg(\phi \Rightarrow \psi)$ is equivalent to $\phi \wedge (\neg\psi)$.

ϕ	ψ	$\phi \Rightarrow \psi$	$\neg(\phi \Rightarrow \psi)$
T	T	T	F
T	F	F	T
F	T	T	F
F	F	T	F

ϕ	ψ	$\phi \wedge (\neg\psi)$
T	T	
T	F	
F	T	
F	F	

Example (Exercise 2.2.9(a) in [Devlin])

Prove $\neg(\phi \Rightarrow \psi)$ is equivalent to $\phi \wedge (\neg\psi)$.

ϕ	ψ	$\phi \Rightarrow \psi$	$\neg(\phi \Rightarrow \psi)$
T	T	T	F
T	F	F	T
F	T	T	F
F	F	T	F

ϕ	ψ	$\phi \wedge (\neg\psi)$
T	T	F
T	F	T
F	T	F
F	F	F

Example (Exercise 2.2.9(a) in [Devlin])

Prove $\neg(\phi \Rightarrow \psi)$ is equivalent to $\phi \wedge (\neg\psi)$.

ϕ	ψ	$\phi \Rightarrow \psi$	$\neg(\phi \Rightarrow \psi)$
T	T	T	F
T	F	F	T
F	T	T	F
F	F	T	F

ϕ	ψ	$\phi \wedge (\neg\psi)$
T	T	F
T	F	T
F	T	F
F	F	F

How can we then interpret $\phi \Rightarrow \psi$ in terms of $\wedge$, $\vee$ and $\neg$?

Example (Exercise 2.2.9(a) in [Devlin])

Prove $\neg(\phi \Rightarrow \psi)$ is equivalent to $\phi \wedge (\neg\psi)$.

ϕ	ψ	$\phi \Rightarrow \psi$	$\neg(\phi \Rightarrow \psi)$
T	T	T	F
T	F	F	T
F	T	T	F
F	F	T	F

ϕ	ψ	$\phi \wedge (\neg\psi)$
T	T	F
T	F	T
F	T	F
F	F	F

How can we then interpret $\phi \Rightarrow \psi$ in terms of $\wedge$, $\vee$ and $\neg$? $(\neg\phi) \vee \psi$

Example (Wason Selection Task, Exercise 2.2.20 in [Devlin])

E K 4 7

Assuming each card has a letter on one face and a number on the other, which card(s) do you have to turn over in order to test the truth of the proposition that if a card has a vowel on one face, then its opposite shows an even number?[a]

Example (Wason Selection Task, Exercise 2.2.20 in [Devlin])

E	K	4	7

Assuming each card has a letter on one face and a number on the other, which card(s) do you have to turn over in order to test the truth of the proposition that if a card has a vowel on one face, then its opposite shows an even number?[a]

Answer: Cards “E” and “7”.

Example (Wason Selection Task, Exercise 2.2.20 in [Devlin])

E	K	4	7

Assuming each card has a letter on one face and a number on the other, which card(s) do you have to turn over in order to test the truth of the proposition that if a card has a vowel on one face, then its opposite shows an even number?[a]

Answer: Cards "E" and "7".

- Checking the other side of "E" is obvious (we need an even number to show for the proposition to hold).

Example (Wason Selection Task, Exercise 2.2.20 in [Devlin])

E	K	4	7

Assuming each card has a letter on one face and a number on the other, which card(s) do you have to turn over in order to test the truth of the proposition that if a card has a vowel on one face, then its opposite shows an even number?[a]

Answer: Cards “E” and “7”.

- Checking the other side of “E” is obvious (we need an even number to show for the proposition to hold).
- It doesn’t matter what’s on the other side of “K” (the proposition makes no claim about cards with a consonant).

Example (Wason Selection Task, Exercise 2.2.20 in [Devlin])

E	K	4	7

Assuming each card has a letter on one face and a number on the other, which card(s) do you have to turn over in order to test the truth of the proposition that if a card has a vowel on one face, then its opposite shows an even number?[a]

Answer: Cards “E” and “7”.

- Checking the other side of “E” is obvious (we need an even number to show for the proposition to hold).
- It doesn’t matter what’s on the other side of “K” (the proposition makes no claim about cards with a consonant).
- Nor does it matter what’s on the other side of “4” (if it’s a vowel, great; if it’s a consonant, no rule was violated either).

Example (Wason Selection Task, Exercise 2.2.20 in [Devlin])

E	K	4	7

Assuming each card has a letter on one face and a number on the other, which card(s) do you have to turn over in order to test the truth of the proposition that if a card has a vowel on one face, then its opposite shows an even number?[a]

Answer: Cards “E” and “7”.

- Checking the other side of “E” is obvious (we need an even number to show for the proposition to hold).
- It doesn’t matter what’s on the other side of “K” (the proposition makes no claim about cards with a consonant).
- Nor does it matter what’s on the other side of “4” (if it’s a vowel, great; if it’s a consonant, no rule was violated either).
- We need to check “7” (if a vowel shows up, the proposition is false).

[a] Only about 10% of the population get this right.

When we do a mathematical proof, then we need to decide whether a certain mathematical statement is logically true or not.

When we do a mathematical proof, then we need to decide whether a certain mathematical statement is logically true or not.

The direct approach (known as *modus ponens*, i.e., method by affirmation) is to conclude

If ϕ and $(\phi \Rightarrow \psi)$, then ψ.

When we do a mathematical proof, then we need to decide whether a certain mathematical statement is logically true or not.

The direct approach (known as *modus ponens*, i.e., method by affirmation) is to conclude

$$\text{If } \phi \text{ and } (\phi \Rightarrow \psi), \text{ then } \psi.$$

Example

Show that if n is an even integer, then $(-1)^n = 1$.

When we do a mathematical proof, then we need to decide whether a certain mathematical statement is logically true or not.

The direct approach (known as *modus ponens*, i.e., method by affirmation) is to conclude

$$\text{If } \phi \text{ and } (\phi \Rightarrow \psi), \text{ then } \psi.$$

Example

Show that if n is an even integer, then $(-1)^n = 1$.

Proof.

n is an arbitrary even integer, so $n = 2k$ for some integer k.

□

When we do a mathematical proof, then we need to decide whether a certain mathematical statement is logically true or not.

The direct approach (known as *modus ponens*, i.e., method by affirmation) is to conclude

$$\text{If } \phi \text{ and } (\phi \Rightarrow \psi), \text{ then } \psi.$$

Example

Show that if n is an even integer, then $(-1)^n = 1$.

Proof.

n is an arbitrary even integer, so $n = 2k$ for some integer k. But then

$$(-1)^n = (-1)^{2k}$$

□

When we do a mathematical proof, then we need to decide whether a certain mathematical statement is logically true or not.

The direct approach (known as *modus ponens*, i.e., method by affirmation) is to conclude

If ϕ and $(\phi \Rightarrow \psi)$, then ψ.

Example

Show that if n is an even integer, then $(-1)^n = 1$.

Proof.

n is an arbitrary even integer, so $n = 2k$ for some integer k. But then

$$(-1)^n = (-1)^{2k} = \Big(\underbrace{(-1)^2}\Big)^k$$

□

When we do a mathematical proof, then we need to decide whether a certain mathematical statement is logically true or not.

The direct approach (known as *modus ponens*, i.e., method by affirmation) is to conclude

$$\text{If } \phi \text{ and } (\phi \Rightarrow \psi), \text{ then } \psi.$$

Example

Show that if n is an even integer, then $(-1)^n = 1$.

Proof.

n is an arbitrary even integer, so $n = 2k$ for some integer k. But then

$$(-1)^n = (-1)^{2k} = \Big(\underbrace{(-1)^2}_{=1}\Big)^k$$

□

When we do a mathematical proof, then we need to decide whether a certain mathematical statement is logically true or not.

The direct approach (known as *modus ponens*, i.e., method by affirmation) is to conclude

If ϕ and $(\phi \Rightarrow \psi)$, then ψ.

Example

Show that if n is an even integer, then $(-1)^n = 1$.

Proof.

n is an arbitrary even integer, so $n = 2k$ for some integer k. But then

$$(-1)^n = (-1)^{2k} = \Big(\underbrace{(-1)^2}_{=1}\Big)^k = 1^k = 1.$$

□

Contrapositive

Sometimes it is difficult to prove an implication directly. In this case we can try to show that the contrapositive of the conditional holds, i.e., we use (see HW 2.2.11)

$$\phi \Rightarrow \psi \qquad \text{is equivalent to} \qquad (\neg\psi) \Rightarrow (\neg\phi)$$

and conclude (*modus tollens*, method by denial)

$$\text{If } \phi \text{ and } ((\neg\psi) \Rightarrow (\neg\phi)), \text{ then } \psi.$$

Contrapositive

Sometimes it is difficult to prove an implication directly. In this case we can try to show that the contrapositive of the conditional holds, i.e., we use (see HW 2.2.11)

$$\phi \Rightarrow \psi \qquad \text{is equivalent to} \qquad (\neg\psi) \Rightarrow (\neg\phi)$$

and conclude (*modus tollens*, method by denial)

$$\text{If } \phi \text{ and } ((\neg\psi) \Rightarrow (\neg\phi)), \text{ then } \psi.$$

Note that this has nothing to do with proof by contradiction (more on that later).

Contrapositive

Sometimes it is difficult to prove an implication directly. In this case we can try to show that the contrapositive of the conditional holds, i.e., we use (see HW 2.2.11)

$$\phi \Rightarrow \psi \qquad \text{is equivalent to} \qquad (\neg\psi) \Rightarrow (\neg\phi)$$

and conclude (*modus tollens*, method by denial)

$$\text{If } \phi \text{ and } ((\neg\psi) \Rightarrow (\neg\phi)), \text{ then } \psi.$$

Note that this has nothing to do with proof by contradiction (more on that later).

Example

- Conditional: If you average 85% or above, then you will get an A.
- Contrapositive: If you don't/didn't get an A, then you don't/didn't average 85% or above.

Converse

Do not confuse the contrapositive of a conditional with its converse:

If $\phi \Rightarrow \psi$ then its converse is $\psi \Rightarrow \phi$

There is no logical connection between a conditional and its converse.

Converse

Do not confuse the contrapositive of a conditional with its converse:

If $\phi \Rightarrow \psi$ then its converse is $\psi \Rightarrow \phi$

There is no logical connection between a conditional and its converse.

The only link is that if both are true, then the biconditional holds, i.e.,

$$(\phi \Rightarrow \psi) \wedge (\psi \Rightarrow \phi) \quad \text{is equivalent to} \quad (\phi \Leftrightarrow \psi)$$

Converse

Do not confuse the contrapositive of a conditional with its converse:

$$\text{If } \phi \Rightarrow \psi \quad \text{then its converse is} \quad \psi \Rightarrow \phi$$

There is no logical connection between a conditional and its converse.

The only link is that if both are true, then the biconditional holds, i.e.,

$$(\phi \Rightarrow \psi) \wedge (\psi \Rightarrow \phi) \quad \text{is equivalent to} \quad (\phi \Leftrightarrow \psi)$$

Example

- Conditional: If you average 85% or above, then you will get an A.
- Contrapositive: If you don't/didn't get an A, then you don't/didn't average 85% or above.
- Converse: If you get/got an A, then you average/averaged 85% or above.

Note: The converse assures us that scoring 85% or above is necessary for an A (not so good for you ☹).

Also, do not confuse a statement with its converse!

More Lewis Carroll:

> *'Not the same thing a bit!' said the Hatter. 'You might just as well say that "I see what I eat" is the same thing as "I eat what I see"!'*

From "Alice in Wonderland" by Lewis Carroll (see http://www.online-literature.com/carroll/alice/7/ for the complete Chapter 7: A Mad Tea Party)

Inverse

One could also consider the contrapositive of the converse[5] of a conditional, also known as the inverse of the conditional:

[5] Or, equivalently, the converse of the contrapositive.

Inverse

One could also consider the contrapositive of the converse[5] of a conditional, also known as the inverse of the conditional:

Example

- Conditional: If you average 85% or above, then you will get an A.
- Contrapositive: If you don't/didn't get an A, then you don't/didn't average 85% or above.
- Converse: If you get/got an A, then you average/averaged 85% or above.
- Inverse: If you don't average 85% or above, then you won't get an A.

Note: The inverse (which is equivalent to the converse) perhaps shows even better that scoring 85% or above is necessary for an A.

[5] Or, equivalently, the converse of the contrapositive.

Negated Conditional

You also might be tempted to confuse the contrapositive of a conditional with its negation. As we showed earlier, the negation of a conditional

$$\neg(\phi \Rightarrow \psi) \quad \text{is equivalent to} \quad \phi \wedge (\neg\psi)$$

which is different from the contrapositive of $\phi \Rightarrow \psi$:

$$(\neg\psi) \Rightarrow (\neg\phi)$$

Negated Conditional

You also might be tempted to confuse the contrapositive of a conditional with its negation. As we showed earlier, the negation of a conditional

$$\neg(\phi \Rightarrow \psi) \quad \text{is equivalent to} \quad \phi \wedge (\neg\psi)$$

which is different from the contrapositive of $\phi \Rightarrow \psi$:

$$(\neg\psi) \Rightarrow (\neg\phi)$$

The latter would be equivalent to (verify this!)

$$(\neg\phi) \vee \psi$$

We just said that the negated conditional is

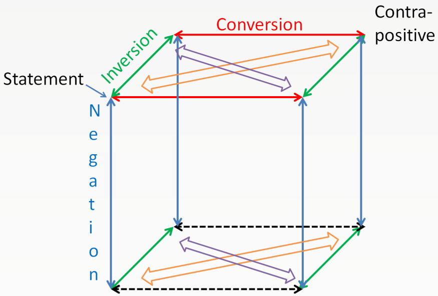
$$\phi \wedge (\neg\psi)$$

Example

- Conditional: If you average 85% or above, then you will get an A.
- Contrapositive: If you don't/didn't get an A, then you don't/didn't average 85% or above.
- Converse: If you get/got an A, then you average/averaged 85% or above.
- Inverse: If you don't average 85% or above, then you won't get an A.
- Negated conditional: You average 85% or above and you won't get an A.

Note: The negated conditional is only true if the conditional is false, i.e., in the case when I break my promise.

The perverse cube of reversed implications



$$\neg\phi \Rightarrow \neg\psi \quad \longleftrightarrow \quad \neg\psi \Rightarrow \neg\phi$$
$$\phi \vee \neg\psi \qquad\qquad \psi \vee \neg\phi$$
$$\updownarrow \qquad\qquad \updownarrow$$
$$\phi \Rightarrow \psi \quad \longleftrightarrow \quad \psi \Rightarrow \phi$$
$$\neg\phi \vee \psi \qquad\qquad \neg\psi \vee \phi$$

$\updownarrow$ Negation $\updownarrow$

$$\neg(\neg\phi \Rightarrow \neg\psi) \quad \cdots \quad \neg(\neg\psi \Rightarrow \neg\phi)$$
$$\neg\phi \wedge \psi \qquad\qquad \neg\psi \wedge \phi$$
$$\updownarrow \qquad\qquad \updownarrow$$
$$\neg(\phi \Rightarrow \psi) \quad \cdots \quad \neg(\psi \Rightarrow \phi)$$
$$\phi \wedge \neg\psi \qquad\qquad \psi \wedge \neg\phi$$

Create the truth tables for all these statements to verify their relations.

"For all" and "there exists"

Universal quantifier: $\forall x$, "for all x it is the case that ..."
Existential quantifier: $\exists x$, "there exists an x such that ..."

“For all” and “there exists”

Universal quantifier: $\forall x$, “for all x it is the case that ...”
Existential quantifier: $\exists x$, “there exists an x such that ...”

Example (in Mathematica – quantifiers don’t exist in MATLAB)

```
ForAll[x, Element[x,Reals], x^2 + 2 x + c > 0]
Reduce[%, c]     (* find c s.t. the statement is T *)
```

In words:

“For all” and “there exists”

Universal quantifier: $\forall x$, “for all x it is the case that ...”
Existential quantifier: $\exists x$, “there exists an x such that ...”

Example (in Mathematica – quantifiers don’t exist in MATLAB)

```
ForAll[x, Element[x,Reals], x^2 + 2 x + c > 0]
Reduce[%, c]     (* find c s.t. the statement is T *)
```

In words: For what values of c is the inequality $x^2 + 2x + c > 0$ true for all values of x?

"For all" and "there exists"

Universal quantifier: $\forall x$, "for all x it is the case that ..."
Existential quantifier: $\exists x$, "there exists an x such that ..."

Example (in Mathematica – quantifiers don't exist in MATLAB)

```
ForAll[x, Element[x,Reals], x^2 + 2 x + c > 0]
Reduce[%, c]     (* find c s.t. the statement is T *)
```

In words: For what values of c is the inequality $x^2 + 2x + c > 0$ true for all values of x?

Example (in Mathematica)

```
Exists[x, x^2 + 2 x + c == 0 && x > 0]
Reduce[%, c, Reals]  (*consider only real numbers*)
```

In words:

"For all" and "there exists"

Universal quantifier: $\forall x$, "for all x it is the case that ..."
Existential quantifier: $\exists x$, "there exists an x such that ..."

Example (in Mathematica – quantifiers don't exist in MATLAB)

```
ForAll[x, Element[x,Reals], x^2 + 2 x + c > 0]
Reduce[%, c]     (* find c s.t. the statement is T *)
```

In words: For what values of c is the inequality $x^2 + 2x + c > 0$ true for all values of x?

Example (in Mathematica)

```
Exists[x, x^2 + 2 x + c == 0 && x > 0]
Reduce[%, c, Reals]  (*consider only real numbers*)
```

In words: For what values of c does the equation $x^2 + 2x + c = 0$ have a positive solution x? When does a positive solution exist[a]?

[a] Existence (and uniqueness) of a solution are fundamental issues in math.

We can also nicely visualize what's going on in Mathematica

```
Manipulate[
   Plot[x^2 + 2 x + c, {x, -5, 5},
   PlotRange -> {-6, 10}], {c, -5, 5}]
```

This shows that

- for $c \leq 0$ there is an intersection with the positive x-axis, so a positive solution to the equation $x^2 + 2x + c = 0$ exists.
- for $c > 0$ the parabola does not intersect the positive x-axis, so the equation $x^2 + 2x + c = 0$ is false for positive values of x.
- for $c > 1$ the parabola does not intersect the x-axis at all, so the inequality $x^2 + 2x + c > 0$ is true for all values of x (but the equation $x^2 + 2x + c = 0$ is false).

We can also nicely visualize what's going on in Mathematica

```
Manipulate[
   Plot[x^2 + 2 x + c, {x, -5, 5},
   PlotRange -> {-6, 10}], {c, -5, 5}]
```

This shows that

- for $c \leq 0$ there is an intersection with the positive x-axis, so a positive solution to the equation $x^2 + 2x + c = 0$ exists.
- for $c > 0$ the parabola does not intersect the positive x-axis, so the equation $x^2 + 2x + c = 0$ is false for positive values of x.
- for $c > 1$ the parabola does not intersect the x-axis at all, so the inequality $x^2 + 2x + c > 0$ is true for all values of x (but the equation $x^2 + 2x + c = 0$ is false).

Remark

We could also use a restricted domain for the existence example, i.e.,

```
Exists[x, x>0, x^2 + 2 x + c == 0]
Reduce[%, c, Reals]  (*consider only real numbers*)
```

Example (Comparing the use of $\exists$ and $\forall$ in common language)

Consider the two statements:

- Everybody likes at least one drink, namely water.
- Everybody likes at least one drink; I myself go for beer.

Example (Comparing the use of $\exists$ and $\forall$ in common language)

Consider the two statements:

- Everybody likes at least one drink, namely water.
  (one drink liked by everybody)
- Everybody likes at least one drink; I myself go for beer.
  (everybody likes some drink)

How can we formalize these two statements using quantifiers?

Example (Comparing the use of $\exists$ and $\forall$ in common language)

Consider the two statements:

- Everybody likes at least one drink, namely water.
  (one drink liked by everybody)
- Everybody likes at least one drink; I myself go for beer.
  (everybody likes some drink)

How can we formalize these two statements using quantifiers?

- There exists a drink D such that, for every person P, P likes D.

$$(\exists D)\,(\forall P)\,[P \text{ likes } D]$$

Example (Comparing the use of $\exists$ and $\forall$ in common language)

Consider the two statements:

- Everybody likes at least one drink, namely water.
  (one drink liked by everybody)
- Everybody likes at least one drink; I myself go for beer.
  (everybody likes some drink)

How can we formalize these two statements using quantifiers?

- There exists a drink D such that, for every person P, P likes D.

$$(\exists D)\,(\forall P)\,[P \text{ likes } D]$$

- For every person P there exists a drink D such that P likes D.

$$(\forall P)\,(\exists D)\,[P \text{ likes } D]$$

Example (Comparing the use of $\exists$ and $\forall$ in common language)

Consider the two statements:

- Everybody likes at least one drink, namely water. (one drink liked by everybody)
- Everybody likes at least one drink; I myself go for beer. (everybody likes some drink)

How can we formalize these two statements using quantifiers?

- There exists a drink D such that, for every person P, P likes D.

$$(\exists D)\,(\forall P)\,[P \text{ likes } D]$$

- For every person P there exists a drink D such that P likes D.

$$(\forall P)\,(\exists D)\,[P \text{ likes } D]$$

The order of quantifiers matters!

Example (Comparing the use of $\exists$ and $\forall$ in common language)

Consider the two statements:

- Everybody likes at least one drink, namely water.
  (one drink liked by everybody)
- Everybody likes at least one drink; I myself go for beer.
  (everybody likes some drink)

How can we formalize these two statements using quantifiers?

- There exists a drink D such that, for every person P, P likes D.

$$(\exists D)\,(\forall P)\,[P \text{ likes } D]$$

- For every person P there exists a drink D such that P likes D.

$$(\forall P)\,(\exists D)\,[P \text{ likes } D]$$

The order of quantifiers matters! What about the other two cases?

A mathematical example

Let P be the set of prime numbers and $\mathbb{N}$ the set of natural numbers. Then

$$(\forall n \in \mathbb{N})\,(\exists m \in \mathbb{N})\,[(m > n) \wedge (m \in P)]$$

says that

A mathematical example

Let P be the set of prime numbers and $\mathbb{N}$ the set of natural numbers. Then

$$(\forall n \in \mathbb{N})\,(\exists m \in \mathbb{N})\,[(m > n) \wedge (m \in P)]$$

says that there are infinitely many primes.

A mathematical example

Let P be the set of prime numbers and $\mathbb{N}$ the set of natural numbers. Then

$$(\forall n \in \mathbb{N})\,(\exists m \in \mathbb{N})\,[(m > n) \wedge (m \in P)]$$

says that there are infinitely many primes.

Actually, it says that

> *For whatever natural number, n, we pick*
> *we can find another natural number, m,*
> *that is both larger than n and a prime number.*

A mathematical example

Let P be the set of prime numbers and $\mathbb{N}$ the set of natural numbers. Then

$$(\forall n \in \mathbb{N})\,(\exists m \in \mathbb{N})\,[(m > n) \wedge (m \in P)]$$

says that there are infinitely many primes.

Actually, it says that

> *For whatever natural number, n, we pick*
> *we can find another natural number, m,*
> *that is both larger than n and a prime number.*

The statement $m \in P$ (or "m is a prime number") can be formalized as

A mathematical example

Let P be the set of prime numbers and $\mathbb{N}$ the set of natural numbers. Then

$$(\forall n \in \mathbb{N})\,(\exists m \in \mathbb{N})\,[(m > n) \wedge (m \in P)]$$

says that there are infinitely many primes.

Actually, it says that

For whatever natural number, n, we pick
we can find another natural number, m,
that is both larger than n and a prime number.

The statement $m \in P$ (or "m is a prime number") can be formalized as

$$(\forall a, b \in \mathbb{N})\,[(ab = m) \Rightarrow ((a = 1) \vee (b = 1))]$$

Negation of Universal Statements

Earlier we used the example (in Mathematica)

$$(\forall x \in \mathbb{R}) \left[x^2 + 2x + c > 0 \right]$$

Let's assume it is true, i.e.,

For all x, the inequality $x^2 + 2x + c > 0$ is true.

Negation of Universal Statements

Earlier we used the example (in Mathematica)

$$(\forall x \in \mathbb{R})\left[x^2 + 2x + c > 0\right]$$

Let's assume it is true, i.e.,

For all x, the inequality $x^2 + 2x + c > 0$ is true.

What if we want to negate this statement? We get

$$\neg\left[(\forall x \in \mathbb{R})\left[x^2 + 2x + c > 0\right]\right]$$

Negation of Universal Statements

Earlier we used the example (in Mathematica)

$$(\forall x \in \mathbb{R}) \left[x^2 + 2x + c > 0 \right]$$

Let's assume it is true, i.e.,

For all x, the inequality $x^2 + 2x + c > 0$ is true.

What if we want to negate this statement? We get

$$\neg \left[(\forall x \in \mathbb{R}) \left[x^2 + 2x + c > 0 \right] \right]$$

```
Not[ForAll[x, Element[x,Reals], x^2 + 2 x + c > 0]]
Reduce[%, c]     (* find c s.t. the statement is T *)
```

Negation of Universal Statements

Earlier we used the example (in Mathematica)

$$(\forall x \in \mathbb{R}) \left[x^2 + 2x + c > 0\right]$$

Let's assume it is true, i.e.,

For all x, the inequality $x^2 + 2x + c > 0$ is true.

What if we want to negate this statement? We get

$$\neg \left[(\forall x \in \mathbb{R}) \left[x^2 + 2x + c > 0\right]\right]$$

```
Not[ForAll[x, Element[x,Reals], x^2 + 2 x + c > 0]]
Reduce[%, c]     (* find c s.t. the statement is T *)
```

What does the statement mean? (Mathematica resolved it!)

Negation of Universal Statements

Earlier we used the example (in Mathematica)

$$(\forall x \in \mathbb{R}) \left[x^2 + 2x + c > 0\right]$$

Let's assume it is true, i.e.,

For all x, the inequality $x^2 + 2x + c > 0$ is true.

What if we want to negate this statement? We get

$$\neg \left[(\forall x \in \mathbb{R}) \left[x^2 + 2x + c > 0\right]\right]$$

```
Not[ForAll[x, Element[x,Reals], x^2 + 2 x + c > 0]]
Reduce[%, c]     (* find c s.t. the statement is T *)
```

What does the statement mean? (Mathematica resolved it!)

There exists some x for which the inequality is not true.

Note that the values of c found by Mathematica are complementary to those for which the original statement was true.

Negation of Existential Statements

Let's use our second earlier example

$$(\exists x \in \mathbb{R}) \left[(x^2 + 2x + c = 0) \wedge (x > 0) \right]$$

Let's assume it is true, i.e.,

> *There exists an x, such that the equation $x^2 + 2x + c = 0$ is true and $x > 0$.*

Negation of Existential Statements

Let's use our second earlier example

$$(\exists x \in \mathbb{R}) \left[(x^2 + 2x + c = 0) \wedge (x > 0) \right]$$

Let's assume it is true, i.e.,

> *There exists an x, such that the equation $x^2 + 2x + c = 0$ is true and $x > 0$.*

What if we want to negate this statement? We get

$$\neg \left[(\exists x \in \mathbb{R}) \left[(x^2 + 2x + c = 0) \wedge (x > 0) \right] \right]$$

Negation of Existential Statements

Let's use our second earlier example

$$(\exists x \in \mathbb{R}) \left[(x^2 + 2x + c = 0) \wedge (x > 0) \right]$$

Let's assume it is true, i.e.,

> *There exists an x, such that the equation $x^2 + 2x + c = 0$ is true and $x > 0$.*

What if we want to negate this statement? We get

$$\neg \left[(\exists x \in \mathbb{R}) \left[(x^2 + 2x + c = 0) \wedge (x > 0) \right] \right]$$

```
Not[Exists[x, x^2 + 2 x + c == 0 && x > 0]]
Reduce[%, c, Reals]  (*consider only real numbers*)
```

Negation of Existential Statements

Let's use our second earlier example

$$(\exists x \in \mathbb{R}) \left[(x^2 + 2x + c = 0) \wedge (x > 0) \right]$$

Let's assume it is true, i.e.,

There exists an x, such that the equation $x^2 + 2x + c = 0$ is true and $x > 0$.

What if we want to negate this statement? We get

$$\neg \left[(\exists x \in \mathbb{R}) \left[(x^2 + 2x + c = 0) \wedge (x > 0) \right] \right]$$

```
Not[Exists[x, x^2 + 2 x + c == 0 && x > 0]]
Reduce[%, c, Reals]  (*consider only real numbers*)
```

What does the statement mean?

Negation of Existential Statements

Let's use our second earlier example

$$(\exists x \in \mathbb{R}) \left[(x^2 + 2x + c = 0) \wedge (x > 0) \right]$$

Let's assume it is true, i.e.,

> *There exists an x, such that the equation $x^2 + 2x + c = 0$ is true and $x > 0$.*

What if we want to negate this statement? We get

$$\neg \left[(\exists x \in \mathbb{R}) \left[(x^2 + 2x + c = 0) \wedge (x > 0) \right] \right]$$

```
Not[Exists[x, x^2 + 2 x + c == 0 && x > 0]]
Reduce[%, c, Reals]  (*consider only real numbers*)
```

What does the statement mean?

> *For all x, the equation is not true or $x \leq 0$.*

Again, the values of c found by Mathematica are complementary to those for which the original statement was true.

Using a restricted domain, things are a little simpler:

$$(\exists x \in \mathbb{R}^+)\left[x^2 + 2x + c = 0\right]$$

Let's assume it is true, i.e.,

There exists a positive x, the equation $x^2 + 2x + c = 0$ is true.

Using a restricted domain, things are a little simpler:

$$(\exists x \in \mathbb{R}^+) \left[x^2 + 2x + c = 0\right]$$

Let's assume it is true, i.e.,

There exists a positive x, the equation $x^2 + 2x + c = 0$ is true.

What if we want to negate this statement? We get

$$\neg \left[(\exists x \in \mathbb{R}^+) \left[x^2 + 2x + c = 0\right]\right]$$

Using a restricted domain, things are a little simpler:

$$(\exists x \in \mathbb{R}^+) \left[x^2 + 2x + c = 0 \right]$$

Let's assume it is true, i.e.,

There exists a positive x, the equation $x^2 + 2x + c = 0$ is true.

What if we want to negate this statement? We get

$$\neg \left[(\exists x \in \mathbb{R}^+) \left[x^2 + 2x + c = 0 \right] \right]$$

```
Not[Exists[x, x > 0, x^2 + 2 x + c == 0]]
Reduce[%, c, Reals]  (*consider only real numbers*)
```

Using a restricted domain, things are a little simpler:

$$(\exists x \in \mathbb{R}^+) \left[x^2 + 2x + c = 0\right]$$

Let's assume it is true, i.e.,

There exists a positive x, the equation $x^2 + 2x + c = 0$ is true.

What if we want to negate this statement? We get

$$\neg \left[(\exists x \in \mathbb{R}^+) \left[x^2 + 2x + c = 0\right]\right]$$

```
Not[Exists[x, x > 0, x^2 + 2 x + c == 0]]
Reduce[%, c, Reals]  (*consider only real numbers*)
```

What does the statement mean?

Using a restricted domain, things are a little simpler:

$$(\exists x \in \mathbb{R}^+) \left[x^2 + 2x + c = 0\right]$$

Let's assume it is true, i.e.,

There exists a positive x, the equation $x^2 + 2x + c = 0$ is true.

What if we want to negate this statement? We get

$$\neg \left[(\exists x \in \mathbb{R}^+) \left[x^2 + 2x + c = 0\right]\right]$$

```
Not[Exists[x, x > 0, x^2 + 2 x + c == 0]]
Reduce[%, c, Reals]  (*consider only real numbers*)
```

What does the statement mean?

For all positive x, the equation is not true.

Again, the values of c found by Mathematica are complementary to those for which the original statement was true.

Back to Beer

We end by determining the "correct" formulation for the beer example. Let's formalize the statement

All American beer tastes dreadful.

We introduce the following notation:

$\mathcal{B}$: the set of all beers

$A(x)$: the statement "x is American"

$D(x)$: the statement "x tastes dreadful"

Then we get

$$(\forall x \in \mathcal{B})\,[A(x) \Rightarrow D(x)]$$

i.e.,

For all beers, if the beer is American then it tastes dreadful.

Now we need to negate

$$(\forall x \in \mathcal{B})\,[A(x) \Rightarrow D(x)]$$

Now we need to negate

$$(\forall x \in \mathcal{B})\,[A(x) \Rightarrow D(x)]$$

According to our earlier discussion we have

$$(\exists x \in \mathcal{B})\,\neg\,[A(x) \Rightarrow D(x)]$$

Now we need to negate

$$(\forall x \in \mathcal{B})\,[A(x) \Rightarrow D(x)]$$

According to our earlier discussion we have

$$(\exists x \in \mathcal{B})\,\neg\,[A(x) \Rightarrow D(x)]$$

which, using the fact that $\neg\,[\phi \Rightarrow \psi]$ is equivalent to $\phi \wedge (\neg\psi)$, yields

$$(\exists x \in \mathcal{B})\,[A(x) \wedge (\neg D(x))]$$

Now we need to negate

$$(\forall x \in \mathcal{B})\, [A(x) \Rightarrow D(x)]$$

According to our earlier discussion we have

$$(\exists x \in \mathcal{B})\, \neg\, [A(x) \Rightarrow D(x)]$$

which, using the fact that $\neg\, [\phi \Rightarrow \psi]$ is equivalent to $\phi \wedge (\neg\psi)$, yields

$$(\exists x \in \mathcal{B})\, [A(x) \wedge (\neg D(x))]$$

In words:

> *There exists a beer which is American and does not taste dreadful.*

References I

Devlin, Keith J.
Set, Functions and Logic (3rd Ed.).
Chapman & Hall/CRC, 2004.

Moler, Cleve.
Experiments with MATLAB.
http://www.mathworks.com/moler/exm/chapters.html