MATH 100 – Introduction to the Profession

Random Affine Transformations and Fractals in MATLAB

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Fall 2012

Linear Transformations

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- shear maps (distorted version of our house)

In particular, the origin $\boldsymbol{x} = [0\ 0]^T$ is mapped to $\mathsf{A}\boldsymbol{x} = [0\ 0]^T$, so is kept fixed.

Affine Transformations

Now we also allow a possible translation by a vector $\boldsymbol{b}$ in addition to the matrix multiplication, i.e.,

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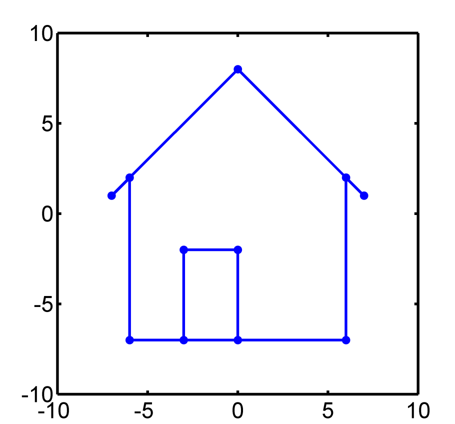
2D affine transformations

- include all linear transformations (when $\boldsymbol{b} = [0\ 0]^T$)
- allow the origin to be moved (translated)

We begin with the house as before:

```
X = house
dot2dot(X)
```



The matrix G $\left(\frac{\pi}{4}\right)$

$$G\left(\frac{\pi}{4}\right) = \begin{bmatrix} \cos\frac{\pi}{4} & -\sin\frac{\pi}{4} \\ \sin\frac{\pi}{4} & \cos\frac{\pi}{4} \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$$

rotates the house counterclockwise by a 45° angle:

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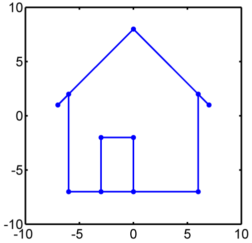
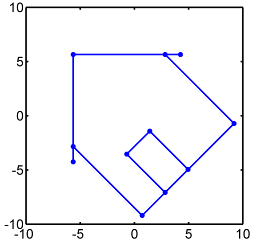
This is a linear transformation (as in `wiggle.m`).





To obtain an affine transformation we add a nonzero vector $\boldsymbol{b}$.

However, since in our house example the “vector” $\boldsymbol{x}$ is actually a matrix X (a collection of many points listed in the columns of X), we need to use a translation matrix B consisting of many copies of the (same) translation vector $\boldsymbol{b}$.

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This can be done by using MATLAB’s `repmat()` command:

```
G = [sqrt(2)/2 -sqrt(2)/2; sqrt(2)/2 sqrt(2)/2]
b = [1; 2]     % 1 to the right, 2 up
n = size(X,2)  % number of columns/points in X
% make as many copies of b as are needed to match X
B = repmat(b,1,n)
dot2dot(G*X + B)
```

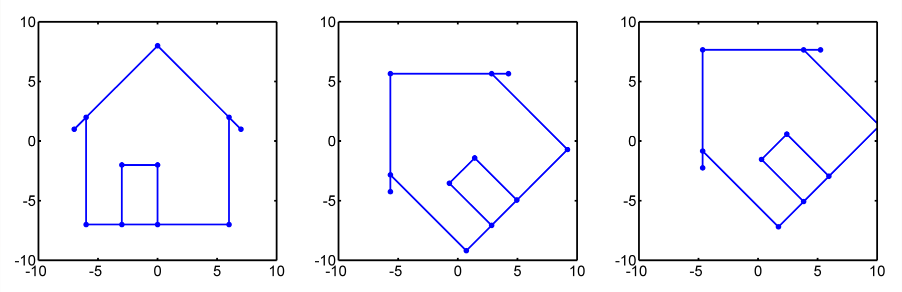

Figure : The original house (left), rotated by $G(\frac{\pi}{4})$ about the origin (middle), and rotated by $G(\frac{\pi}{4})$ about the origin and then translated by $\boldsymbol{b} = [1\ 2]^T$ (right).

Fractal fern

The MATLAB script `fern.m` from [ExM] uses

- three different affine transformations
- and one linear transformation

that are performed randomly with different probabilities to generate a fractal shape that looks like a real-life fern.

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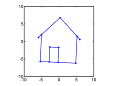
In particular, `fern.m` uses (plots show effects of A only)

- 85% of the time: a small clockwise rotation and small rescaling with an upward shift:

$$\boldsymbol{x} \mapsto A_1\boldsymbol{x} + \boldsymbol{b}_1 = \begin{bmatrix} 0.85 & 0.04 \\ -0.04 & 0.85 \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} 0 \\ 1.6 \end{bmatrix}$$



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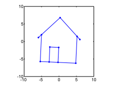
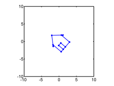


- 7% of the time: a larger counter-clockwise rotation and larger rescaling with the same upward shift:

$$\boldsymbol{x} \mapsto A_2\boldsymbol{x} + \boldsymbol{b}_2 = \begin{bmatrix} 0.20 & -0.26 \\ 0.23 & 0.22 \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} 0 \\ 1.6 \end{bmatrix}$$



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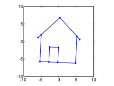

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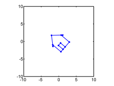
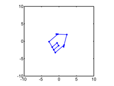


- 7% of the time: a larger clockwise rotation, rescaling and shear with a smaller upward shift:

$$\boldsymbol{x} \mapsto A_3\boldsymbol{x} + \boldsymbol{b}_3 = \begin{bmatrix} -0.15 & 0.28 \\ 0.26 & 0.24 \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} 0 \\ 0.44 \end{bmatrix}$$



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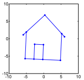

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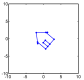

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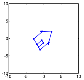
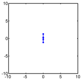


- 1% of the time: a projection and rescaling onto the stem:

$$\boldsymbol{x} \mapsto A_4\boldsymbol{x} = \begin{bmatrix} 0 & 0 \\ 0 & 0.16 \end{bmatrix} \boldsymbol{x}$$



Other (mathematically) interesting parts of the MATLAB script `fern.m` are:

- Use of negation to control the loop that keeps adding points (it runs until the “stop” button is pressed, i.e., its value is 1):

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while ~get(stop,'value')
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- Use of a random number generator to generate a random number (the probability of switching between transformations) uniformly distributed in $(0, 1)$:

```
r = rand;
```

Summary scripts

Look at `fern_recap.m` (on the ExM website).

In particular, `finitefern(n,'s')` produces a fern picture in which *n* points are highlighted and added one at a time.

A group at the University of Calgary [Algorithmic Botany] around Przemyslaw Prusinkiewicz has been using so-called L-systems (similar to the system of transformations that generated the fractal fern) to create entire synthetic landscapes:

They have many publications, such as [PalubickiEtAl], from which the above image is taken.

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