

MATH 100 – Introduction to the Profession

Modeling and the Exponential Function in MATLAB

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Outline¹

- 1 Mathematical Models
- 2 Modeling Growth
- 3 The Modeling Process
- 4 Other Types of Problems to Model
- 5 Modeling and Simulation

¹Most of this discussion is linked to [T. Gowers: Mathematics: A Very Short Introduction, Chapter 1] and [ExM, Chapter 8].



What is a mathematical model?

A mathematical model is an abstract, simplified, mathematical construct related to a part of reality and created for a particular purpose.

from [E. A. Bender: An Introduction to Mathematical Modeling]

When devising a model, one tries to ignore as much as possible about the phenomenon under consideration, abstracting from it only those features that are essential to understanding its behaviour.

from [T. Gowers: Mathematics: A Very Short Introduction]



Earlier Example: Rabbits

Processes that **change** over time may be one of the most common types of modeling problems we encounter in practice.

Such change can happen

- at **discrete** instances, and thus lead to **difference equations** such as the recursion

$$f_n = f_{n-1} + f_{n-2}, \quad \text{for } n \geq 3 \text{ and } f_1 = f_2 = 1,$$

which describes the Fibonacci sequence (growth of a rabbit population), and can also be written as

$$\Delta f_{n-1} = f_{n-2}, \quad \text{for } n \geq 3 \text{ and } f_1 = f_2 = 1,$$

using **forward differences** $\Delta f_{n-1} = f_n - f_{n-1}$;

- continuously**, so that we obtain a **differential equation** such as

$$P'(t) = rP(t), \quad P(0) = P_0,$$

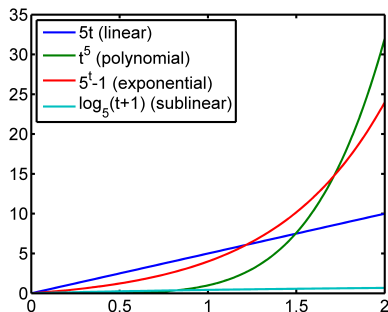
see below.



Types of Growth

We often differentiate between **linear** and **nonlinear** models. In particular, when discussing the growth of a quantity/function/sequence we may encounter

- **linear growth** of the type
 $P(t) = at$
- **polynomial growth** of the type
 $P(t) = t^a$
- **exponential growth** of the type
 $P(t) = a^t$
- **sublinear growth** of the type
 $P(t) = \log_a(t)$



The Modeling Process

- 1 **Formulate the problem.** What do we want to achieve with our model? The answer to this question may suggest a specific mathematical technique to be used.
- 2 Identify which quantities are known/available, which you want to compute/predict, i.e., **introduce variables and consider how they are interrelated.**
- 3 Assess the complexity of your model and consider making simplifying assumptions to **ensure that you have a manageable approach** to solving the problem. Possibly iterate steps 1-3.
- 4 **Validate the model.** Calibrate parameters if needed by comparing with available data. Make sure the model works for simple/standard situations before applying to something more challenging. Possibly iterate steps 1-4.



An Example: Population Growth

1 Let's assume we want to model **long-term** population growth.

2 The **quantities of interest** are:

(a) **Given**

r : the net population change per individual (i.e., reproduction rate = birth rate – death rate), can be expressed as

$$r = \frac{P'(t)}{P(t)}$$

t : time

P_0 : initial population, i.e., $P(0) = P_0$.

(b) **Desired**

$P(t)$: population at any given time t , with a **special interest** in large values of t



3 Let's **make some assumptions** that allow us to solve the problem.
Let's say that

- (a) the **growth rate is constant**, and
- (b) the **past has no effect on the future**, i.e., the growth process depends only on the current population $P(t)$ and its rate of growth $P'(t)$.

The resulting ▶ differential equation

$$P'(t) = rP(t)$$

has general solution

$$P(t) = ce^{rt},$$

where the constant of integration c is obtained using the initial condition, i.e.,

$$P(0) = ce^{r \cdot 0} \stackrel{!}{=} P_0 \implies c = P_0,$$

so that we get the specific solution

$$P(t) = P_0 e^{rt}, \quad \text{for all } t \geq 0.$$



4 In order to **validate our model**, let's consider what happens for large values of t :

(a) For $r > 0$ we have

$$\lim_{t \rightarrow \infty} P(t) = \lim_{t \rightarrow \infty} P_0 e^{rt} = \infty,$$

so the population grows without bounds for any (positive) initial population P_0 and any positive growth rate r . This is **unrealistic**.

(b) For $r < 0$ we have

$$\lim_{t \rightarrow \infty} P(t) = \lim_{t \rightarrow \infty} P_0 e^{rt} = 0,$$

so the species becomes extinct for any (positive) initial population P_0 and any negative growth rate r . This is also **unrealistic**.

It seems **virtually impossible to make a prediction of a reasonable population size**. Moreover, the growth behavior depends dramatically on (the sign of) the growth rate r .



- 5 This model may still be (and in fact is) useful for relatively short-term growth predictions.

For example, we can apply it to interest calculations in finance:

- see [▶ compound interest](#)
- see [▶ student loan](#)

However, for biological populations and long-term predictions we need to **rethink/refine our model!**



Solving an ODE in MATLAB

We can solve many kinds of ODE initial value problems using, e.g., `ode23()` in MATLAB.

Here's the solution of

$$P'(t) = rP(t), \quad \text{with } r = 0.3 \text{ and } P_0 = 1000$$

```

r = .3;      % growth rate
P0 = 1000;   % initial population
tend = 5;    % final time for simulation
timespan = [0 tend]; % time interval to simulate
tt = linspace(0,tend,100); % for plotting
ode23(@(t,P) r*P, timespan, P0) % MATLAB ODE solver
Pexact = @(t) P0*exp(r*t) % analytical solution
hold on
plot(tt,Pexact(tt),'r.')
hold off

```



Refined Population Growth Models

For example, we might consider a model in which:

- The **growth rate depends on the size of the population**, i.e., $r = r(P)$, so that **growth slows down when things get crowded and resources become sparse**.
- A popular such model is the **logistic differential equation**

$$P'(t) = \underbrace{\left(r - r \frac{P(t)}{C} \right)}_{r=r(P)} P(t).$$

Here C denotes the **carrying capacity** of the environment.

- This equation can also be solved analytically (a little bit harder, but still basic Calculus), so that we get the solution

$$P(t) = \frac{CP_0 e^{rt}}{C + P_0(e^{rt} - 1)}, \quad \text{for all } t \geq 0,$$

where P_0 is again the initial population.



MATLAB Solution of Logistic Equation

We solve

$$P'(t) = \left(r - r \frac{P(t)}{C} \right) P(t), \quad \text{with } r = 1, C = 1500 \text{ and } P_0 = 1000$$

again using `ode23()`.

```
r = 1;      % growth rate
P0 = 1000; % initial population
tend = 5;  % final time for simulation
timespan = [0 tend]; % time interval to simulate
tt = linspace(0,tend,100); % for plotting
C = 1500;  % capacity
ode23(@(t,P) r*P*(1-P/C), timespan, P0) % MATLAB soln
Pexact = @(t) C*P0*exp(r*t)./(C+P0*(exp(r*t)-1))
hold on
plot(tt,Pexact(tt),'r.')
hold off
```



Alternatively, we might consider a model in which:

- The mortality rate depends on the current population, but the birth rate depends on the population that was mature some time earlier.
- This leads to a delay differential equation of the type

$$P'(t) = -mP(t) + bP(t - \ell),$$

where the mortality rate m and birth rate b are both positive quantities, and time lag ℓ tells us how far to go back to account for maturity².

- This problem is much harder to solve. For $m = 0$, $b = 1$ and $\ell = 1$, and initial history $P(t) = 1$ for $t \leq 0$ one gets for $t \in [0, 5]$

$$P(t) = \begin{cases} t + 1 & 0 \leq t \leq 1 \\ \frac{t^2 + 3}{2} & 1 \leq t \leq 2 \\ \frac{t^3 - 3t^2 + 12t + 1}{6} & 2 \leq t \leq 3 \\ \frac{t^4 - 8t^3 + 42t^2 - 60t + 85}{24} & 3 \leq t \leq 4 \\ \frac{t^5 - 15t^4 + 120t^3 - 430t^2 + 980t - 599}{120} & 4 \leq t \leq 5 \end{cases}$$

²Recall the Fibonacci recursion.



MATLAB Solution of DDE

We solve the delay differential equation

$$P'(t) = P(t - 1), \quad \text{with } P(t) = 1 \text{ for } t \leq 0$$

using `dde23()`.

```
tend = 5; % final time for simulation
timespan = [0 tend]; % time interval to simulate
tt = linspace(0,tend,100); % for plotting
lag = 1; % time delay
hist = 1; % initial history
sol = dde23(@(t,P,d) d, lag, hist, timespan)
plot(sol.x,sol.y)
% analytical solution
Pexact = @(t) (tt>=0 & tt<=1).* (tt+1) + (tt>=1 & tt<=2).* (tt.^2+3)/2 + ...
    (tt>=2 & tt<=3).* (tt.^3-3*tt.^2+12*tt+1)/6 + ...
    (tt>=3 & tt<=4).* (tt.^4-8*tt.^3+42*tt.^2-60*tt+85)/24 + ...
    (tt>=4 & tt<=5).* (tt.^5-15*tt.^4+120*tt.^3-430*tt.^2+980*tt-599)/120
hold on
plot(tt,Pexact(tt),'r.')
hold off
```

Note the piecewise defined solution using logical indexing.



Other refinements of our population growth model might consider

- seasonable variations
- random fluctuations
- partially discrete models, e.g., depending on age groups or gender



Other Types of Problems to Model

- Many engineering problems, such as in mechanics, electronics, or in materials science are described by mathematical models – often involving systems of differential equations describing change.
- Probabilistic/stochastic models are used, e.g., in gambling/games, or for complicated natural or social phenomena such as weather prediction, or financial forecasting.
- Modeling the behavior of gases uses systems of differential equations (Boltzmann equations) to describe the kinetics of the molecules of the gas, but also uses stochastic models since some quantities can only be described in the average sense (due to the Heisenberg uncertainty principle) (read [T. Gowers: Mathematics: A Very Short Introduction, Chapter 1]).
- Modeling of scheduling problems uses techniques of discrete mathematics (such as graphs, and graph coloring), but also optimization algorithms.



Other Types of Problems to Model (cont.)

- Modeling of complex networks in, e.g., biology or neuroscience often requires combinations of many different techniques (discrete, differential equations, probabilistic).
- Logic serves as a tool to model many formal systems, such as in artificial intelligence or formal languages.
- Many other situation in everyday life can be subjected to a mathematical model, e.g., in economics, sociology, politics (voting), etc.

Just about all complex models are simulated using computational techniques. The use of (sophisticated) analytical techniques may greatly improve the efficiency of computational models.

Frequently, we can apply an “abstract” mathematical model to many different practical applications.



The “Third Pillar of Science”

*“Together with theory and experimentation, **computational science now constitutes the “third pillar” of scientific inquiry**, enabling researchers to build and test models of complex phenomena – such as multi-century climate shifts, multidimensional flight stresses on aircraft, and stellar explosions – that cannot be replicated in the laboratory, and to manage huge volumes of data rapidly and economically.”*

President's Information Technology Advisory Committee [PITAC (2005)]



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The simplest differential equation

Investigate the exponential growth model for $P(t) = a^t$ by playing with `expgui.m` from [ExM].

We see that $P(t) = P'(t)$ when

$$P(t) = e^t, \quad \text{where } e = 2.71828182845904 \dots \text{ is Euler's number.}$$

This means we have found a solution of the differential equation

$$P'(t) = P(t).$$

This can also be seen from

$$\frac{d}{dt}e^t = e^t.$$

The most basic differentiation rule tells us that any constant multiple $P(t) = ce^t$ works as well.



Using the chain rule we see that

$$\frac{d}{dt}e^{rt} = re^{rt},$$

so that $P(t) = ce^{rt}$ is the general solution of

$$P'(t) = rP(t).$$

◀ Return



Compound interest

- 1 If your parents had invested \$10,000 for your college education at an average annual interest rate of 5% 20 years ago, **how much would be in the account now?**
- 2 We use t to denote **time** (measured in years), $r = 0.05$ as the **annual interest rate** and $A_0 = A(0) = 10000$ as the **initial amount**. **What is $A(20)$, and more generally $A(t)$?**
- 3 If **interest is compounded annually**, then

$$A(1) = A(0) + rA(0) = (1 + r)A(0)$$

$$A(2) = A(1) + rA(1) = (1 + r)A(1) = (1 + r)^2 A(0)$$

$$\vdots$$

$$A(n) = (1 + r)^n A(0)$$

Using $h = 1$, this can also be viewed as³

$$A(t + h) = A(t) + rhA(t).$$

³Note polynomial growth as for Fibonacci.



③ If interest is compounded monthly, then

$$A\left(\frac{1}{12}\right) = A(0) + \frac{r}{12}A(0) = \left(1 + \frac{r}{12}\right)A(0)$$

$$A\left(\frac{2}{12}\right) = A\left(\frac{1}{12}\right) + \frac{r}{12}A\left(\frac{1}{12}\right) = \left(1 + \frac{r}{12}\right)A\left(\frac{1}{12}\right) = \left(1 + \frac{r}{12}\right)^2A(0)$$

$$\vdots$$

$$A(t) = \left(1 + \frac{r}{12}\right)^{12t}A(0)$$

Using $h = \frac{1}{12}$, this can again be viewed as⁴

$$A(t + h) = A(t) + rhA(t).$$

⁴Also polynomial growth.



- ③ If interest is compounded continuously, then we consider

$$A(t+h) = A(t) + rhA(t) \iff \frac{A(t+h) - A(t)}{h} = rA(t)$$

and letting $h \rightarrow 0$ and using the **definition of the derivative**,

$$A'(t) = \lim_{h \rightarrow 0} \frac{A(t+h) - A(t)}{h},$$

we get⁵

$$A'(t) = rA(t) \implies A(t) = A(0)e^{rt}$$

⁵Now we have **exponential** growth.



- 4 For the **validation** we look at

```

fprintf('          t          annually')
fprintf('          monthly      continuously\n')
format bank
format compact
r = 0.05;
A0 = 10000;
for t = 0:20
    A_annual = (1+r)^t*A0;
    A_month = (1+r/12)^(12*t)*A0;
    A_cont = exp(r*t)*A0;
    disp([t A_annual A_month A_cont])
end

```

and see that the **models are reasonable**.



Student Loan

- Let's assume you have a \$20,000 student loan at 10% annual interest, you plan to make monthly payments, and want to pay off the loan in 3 years. **What should your monthly payments be?**
- We use n to denote **number of months** or **number of payments**, $r = 0.1$ as the **annual interest rate** and $A_0 = A(0) = 20000$ as the **initial amount**. We also use p to denote the **monthly payment**.
- Each month your **payment reduces the current balance**, but **interest is still added until the loan is paid off**. Therefore, following the same line of thought as earlier, after one time period (think $h = \frac{1}{12}$, i.e., one month) the loan amount has been reduced to

$$A(h) = A(0) + rhA(0) - p = (1 + rh)A(0) - p.$$



Continuing month for month we get

$$A(h) = (1 + rh)A(0) - p$$

$$\begin{aligned} A(2h) &= (1 + rh)A(h) - p = (1 + rh)((1 + rh)A(0) - p) - p \\ &= (1 + rh)^2 A(0) - p((1 + rh) + 1) \end{aligned}$$

$$\begin{aligned} A(3h) &= (1 + rh)A(2h) - p = (1 + rh)((1 + rh)^2 A(0) - p((1 + rh) + 1)) - p \\ &= (1 + rh)^3 A(0) - p((1 + rh)^2 + (1 + rh) + 1) \end{aligned}$$

⋮

$$A(nh) = (1 + rh)^n A(0) - p((1 + rh)^{n-1} + \dots + (1 + rh) + 1)$$



Since

$$(1 + rh)^{n-1} + \dots + (1 + rh) + 1$$

is a **geometric sum** we have⁶

$$(1 + rh)^{n-1} + \dots + (1 + rh) + 1 = \frac{(1 + rh)^n - 1}{(1 + rh) - 1} = \frac{(1 + rh)^n - 1}{rh}$$

and therefore

$$\begin{aligned} A(nh) &= (1 + rh)^n A(0) - p \left((1 + rh)^{n-1} + \dots + (1 + rh) + 1 \right) \\ &= (1 + rh)^n A(0) - p \frac{(1 + rh)^n - 1}{rh}. \end{aligned}$$

⁶ $\sum_{k=1}^n q^{k-1} = \frac{q^n - 1}{q - 1}$



3 Solving

$$A(nh) = (1 + rh)^n A(0) - p \frac{(1 + rh)^n - 1}{rh} = 0$$

for p yields

$$p = \frac{(1 + rh)^n}{(1 + rh)^n - 1} rh A(0).$$

4 Without validating the model, we evaluate this using MATLAB:

```
A0 = 20000;
```

```
r = .10;
```

```
h = 1/12;
```

```
n = 36;
```

```
p = (1+r*h)^n / ((1+r*h)^n - 1) * r * h * A0
```

and see that you should make monthly payments of \$645.34.

