Buletinul A.Ş. a R.M. 1998 Известия АН РМ
Matematica №3(28) Математика

FINITENESS OF THE POINT SPECTRUM OF SOME NONSELFADJOINT OPERATORS CLOSE TO THE OPERATORS GENERATED BY JACOBI MATRICES

IG. CIALENCO, P. COJUHARI

ABSTRACT. The main purpose of the present paper is to give sufficient conditions for the finiteness of the point spectrum of some nonselfadjoint operators.

In the present paper the results on the finiteness of the point spectrum of perturbed operator of the form $H = V + V^* + B$, where V is an isometric operator (nonunitary) on the Hilbert space $\mathfrak{H}$, and $B \in \mathbb{B}(\mathfrak{H})$ ($\mathbb{B}(\mathfrak{H})$ denotes the class of all linear and bounded operators on $\mathfrak{H}$) are obtained. In particular, we study the point spectrum of the operator generated by Jacobi matrix. The main results are obtained using direct method of the perturbation theory of spectrum of linear operators.

The obtained results are in concordance with those established for the Friedrichs model [1], differential operators of the second and forth order [2, 3], Wiener–Hopf integral operator [4] etc. As a rule, in the case of nonselfadjoint perturbations, the problem of finiteness of the point spectrum is reduced to the theorem of uniqueness of analytical function. Thus, the Weyl's function that corresponds to differential operators of the second order [2], the resolvent function in the case of Wiener–Hopf integral operators [4] etc. are studied. In this note, we suggest a suitable method, and more simple, generalize substantially the results from [5], where the author investigated the operator L generated by the difference expression

$$(Ly)_j = \frac{1}{2}(y_{j-1} + y_{j+1}) + b_j y_j \quad (j = 1, 2, \dots)$$

where $(y_j) \in l_2$; $y_0 = \theta y_1$; $b_j \in \mathbb{C}$ $(j = 1, 2, \dots)$; $\theta \in \mathbb{C}$.

In the first section, it is proved an abstract theorem about finiteness of the point spectrum of some perturbed operators (generally speaking nonselfadjoint). In Sections 2–4, we show some applications of the abstract theorem for the operator generated by Jacobi matrix, and respectively in the case when the unperturbed operator is with finite differences.

It should be mentioned that the abstract scheme can be used for more general cases. For example, in the case of operators generated by Jacobi matrices of arbitrary order, for operators with finite differences of any order, Wiener–Hopf integral operators etc. The results concerning these operators will be published in author papers afterwards.

The results on finiteness of the set of eigenvalues of the selfadjoint operators similar to those examined in this note, have been obtained in [6–8].

1. Throughout the paper, $\mathfrak{H}$ will denote a Hilbert space, $\mathbb{B}(\mathfrak{H})$ the class of all linear and bounded operators on $\mathfrak{H}$, $\mathbb{B}_\infty(\mathfrak{H})$ the class of all compact operators on $\mathfrak{H}$.

Let V be an isometric operator defined on $\mathfrak{H}$, such that the operator V^* has no eigenvalues on the unit circle $\mathbb{T} = \{z \in \mathbb{C} \mid \|z\| = 1\}$. Consider the operator H of the form

$$H = H_0 + B, \tag{1}$$

where $H_0 = V + V^*$, $B \in \mathbb{B}(\mathfrak{H})$.

The operator H will be considered as a perturbation of the operator H_0 by the operator B. It is known that the spectrum of the operator H_0 is the set of values of the function $a(z) = z + z^{-1}$, called the symbol of the operator H_0 (see [9]), on the unit circle $\mathbb{T}$. Since $\min\limits_{z\in\mathbb{T}} a(z) = -2$ and $\max\limits_{z\in\mathbb{T}} a(z) = 2$, the spectrum of operator H_0 is the interval $[-2, 2]$ and since $\sigma_p(V^*) \cap \mathbb{T} = \emptyset$, it follows that $\sigma_p(H_0) = \emptyset$.

After perturbation the spectrum of operator H_0 may change substantially. Due to Weyl-type theorem, in the case when the perturbation is compact, the essential spectrum remains invariable, but eigenvalues may appear on essential spectrum as well as outside of it. Note that in this situation the discrete spectrum may have accumulation points only on the essential spectrum.

Further, the conditions on the operator B for finiteness of the point spectrum $\sigma_p(H)$ of the operator H will be indicated.

Theorem 1. *Let B be an operator that satisfies the following conditions:*

(i) The operator B can be represented in the form $B = RTS$, where $R, S \in \mathbb{B}(\mathfrak{H})$, $T \in \mathbb{B}_\infty(\mathfrak{H})$;

(ii) $\overline{\lim\limits_{n\to\infty}} \sqrt[n]{\|SV^n\|} < 1$, $\overline{\lim\limits_{n\to\infty}} \sqrt[n]{\|RV^n\|} < 1$.

Then the set of all eigenvalues of the operator H is finite. Moreover, the possible eigenvalues have finite multiplicity.

The proof of this theorem is based both on the straightforward studying of the holomorphic extension of the resolvent of the unperturbed operator H_0 and on the theorem about the operator valued analytic function (see [10, Theorem XIII.13] or [11, Theorem 5.1]).

The properties of the function $a(z) = z + z^{-1}$ $(z \in \mathbb{C})$ are well known (see for instance [12]). This function is a one-to-one transformation of $\mathbb{T}_- = \{z \in \mathbb{C} \mid |z| > 1\}$ onto $\mathbb{C} \setminus [-2, 2]$. Let us denote by $\mu(\lambda)$ the zero of the function $a(z) - \lambda$, that transforms $\mathbb{C} \setminus [-2, 2]$ on the set $\mathbb{T}_-$. Since

$$a(z) - \lambda = -\frac{1}{\mu(\lambda)}(z^{-1} - \mu(\lambda))(z - \mu(\lambda)) \quad (\lambda \in \mathbb{C} \setminus [-2, 2])$$

one gets

$$R_0(\lambda) = -\mu(\lambda)(V - \mu(\lambda))^{-1}(V^* - \mu(\lambda))^{-1} \quad (\lambda \in \mathbb{C}\backslash[-2,2]), \tag{2}$$

where $R_0(\lambda)$ denotes the resolvent function of the operator H_0, a.i. $R_0(\lambda) = (H_0 - \lambda)^{-1}$ $(\lambda \in \rho(H_0))$. For simplicity of notation we will denote by $Q(\lambda)$ the following operator valued function

$$Q(\lambda) = SR_0(\lambda)R \quad (\lambda \in \mathbb{C}\backslash[-2,2]).$$

Let $\widetilde{Q}(z) = Q(a(z))$ $(z \in \mathbb{T}_{\_})$. Put $\mu(\lambda) = z$ in relation (2). Since $a(\mu(\lambda)) - \lambda = 0$, we have $a(z) = \lambda$ and multiplying (2) by S and R to the right and to the left respectively we obtain

$$\widetilde{Q}(z) = -zS(V-z)^{-1}(V^*-z)^{-1}R \quad (z \in \mathbb{T}_{\_}). \tag{3}$$

Lemma 1. *There exists $\delta \in (0,1)$, such that the function $\widetilde{Q}(z)$ has holomorphic extension from $\mathbb{T}_{\_}$ to $W_\delta = \{z \in \mathbb{C} \mid |z| > \delta\}$.*

Proof. From equality (3) it follows that

$$(\widetilde{Q}(z)f,g) = -(\sum_{n=0}^{\infty} \frac{V^{*^n}Rf}{z^n}, \sum_{k=0}^{\infty} \frac{V^{*^k}Sg}{\bar{z}^{\ k+1}}), \tag{4}$$

where $z \in \mathbb{T}_{\_},\ f,g \in \mathfrak{H}$.

Let $r_1 = \overline{\lim\limits_{n\to\infty}} \sqrt[n]{\|SV^n\|}$, $r_2 = \overline{\lim\limits_{n\to\infty}} \sqrt[n]{\|RV^n\|}$ and $\delta = \max\{r_1, r_2\}$. By condition (ii) of Theorem 1, we have $\delta \in (0,1)$. So, the series $\sum\limits_{n=0}^{\infty} z^{-n}V^{*^n}Rf$, $\sum\limits_{n=0}^{\infty} z^{-n}V^{*^n}Sf$ converge on the domain W_δ, for each $f \in \mathfrak{H}$. Moreover,

$$|(\sum_{n=0}^{\infty} \frac{V^{*^n}Rf}{z^n}, \sum_{k=0}^{\infty} \frac{V^{*^k}Sg}{\bar{z}^{\ k+1}})| \leqslant \|\sum_{n=0}^{\infty} \frac{V^{*n}Rf}{z^n}\| \, \|\sum_{k=0}^{\infty} \frac{V^{*k}Sg}{\bar{z}^{\ k+1}}\| \leqslant$$

$$\leqslant \|f\|\|g\| \sum_{n=0}^{\infty} \frac{\|V^{*n}R\|}{|z|^n} \sum_{k=0}^{\infty} \frac{\|V^{*k}S\|}{|\bar{z}|^{k+1}} \leqslant c\|f\|\|g\|,$$

where $z \in W_\delta,\ f,g \in \mathfrak{H}$, $c = c(\delta) \in \mathbb{R}_+$.

Thus, the right-hand of the equality (4) represents a bounded bilineare functional on $\mathfrak{H} \times \mathfrak{H}$ for each $z \in W_\delta$. According to Riesz's theorem, this bilineare functional generates an operator $Q_1(z) \in \mathbb{B}(\mathfrak{H})$ $(z \in W_\delta)$.

Since

$$(Q_1(z)f,g) = -\sum_{n,k=0}^{\infty} \frac{1}{z^{n+k+1}}(V^{*n}Rf, V^{*k}Sg) \quad (z \in W_\delta; f,g \in \mathfrak{H}), \tag{5}$$

we conclude that the operator valued function $Q_1(z)$ is weakly analytic on W_δ and as weak analyticity coincides with strong analyticity (see for instance [13]), the function $Q_1(z)$ is analytic on W_δ $(0<\delta<1)$. By equality (4), the function $Q_1(z)$ coincides with $\widetilde{Q}(z)$ on $\mathbb{T}_-$, and so, we have that $Q_1(z)$ is a holomorphic extension of the function $\widetilde{Q}(z)$ from $\mathbb{T}_-$ to W_δ for an $\delta \in (0,1)$. Lemma 1 is proved.

Further we preserve the notation $\widetilde{Q}(z)$ for holomorphic extension of function $\widetilde{Q}(z)$ on W_δ. By Lemma 1, the operator valued function $\widetilde{Q}(z)$ is continuous on $\overline{\mathbb{T}}_-$, and so, there exists the limit $s-\lim\limits_{\varepsilon\searrow 0} Q(\lambda \pm i\varepsilon)$ for every $\lambda \in [-2,2]$. In the sequel, we denote by $Q_\pm(\lambda)$ the operator valued function defined on $\overline{\Pi}_\pm$, which is equal to $Q(\lambda)$ if $\lambda \in \Pi_\pm = \{z \in \mathbb{C} \mid \pm Imz > 0\}$ and to $Q_\pm(\lambda)$ if $\lambda \in \mathbb{R}$.

For the proof of Theorem 1, we will use the fact that if $\lambda \in \sigma_p(H)$ then $Ker(I + Q_\pm(\lambda)T) \neq 0$, i.e.,

$$(I + Q_\pm(\lambda)T)f = 0, \qquad (f \in \mathfrak{H}, f \neq 0). \tag{6}$$

Essentially, the proof of this result can be done using the method from papers Kuroda [14] and M.Schehter [15], where the selfadjoint operator H is considered.

In the case when $\lambda \in \rho(H_0)$, one can see at once that the operator $H-\lambda I$ is invertible if and only if the operator $I+R_0(\lambda)RTS$ is invertible which is equivalent to the invertibility of the operator $I + SR_0(\lambda)RT$ (see for instance [16]). If $\lambda \in \sigma(H_0)$, the proof, in essentially, coincides with the proof given in [16].

Due to the previous statements it is sufficient to prove that $Ker(I+\widetilde{Q}(z)T) \neq 0$ for a finite number of values z from $\overline{\mathbb{T}}_-$. Since the operator valued function $\widetilde{Q}(z)T$ is holomorphic on W_δ $(0<\delta<1)$, takes values in $\mathbb{B}_\infty$ $(\mathfrak{H})$ and $\|\widetilde{Q}(z)T\| \to 0$ $(|z| \to 0)$ then by the theorem about the operator valued function which is holomorphic on a domain [9, 10], it follows that $Ker(I+\widetilde{Q}(z)T) \neq 0$ for a finite set of values $z_k \in \overline{\mathbb{T}}_-$ $(k=1,\dots,n)$. Moreover, $dim\ Ker(I+\widetilde{Q}(z_k)T) < \infty$ $(k=1,\dots,n)$.

Thus Theorem 1 is proved.

2. Let $\mathfrak{H}$ be a Hilbert space and V be an one-sided translation in $\mathfrak{H}$, i.e. $V^*V = I$ and there exists a subspace (called the wandering subspace of V) $\mathfrak{L}, \mathfrak{L} \subset \mathfrak{H}$, such that

$$V^n\mathfrak{L} \perp \mathfrak{L} \quad (n \in \mathbb{N}), \quad \mathfrak{H} = \sum_{n=0}^{\infty} \oplus V^n\mathfrak{L}. \tag{7}$$

It should be noted that every $h \in \mathfrak{H}$ can be represented in the form

$$h = \sum_{n=0}^{\infty} V^n g_n, \tag{8}$$

where $g_n \in \mathfrak{L}$ $(n=0,1,\dots)$. Moreover, $\|h\|^2 = \sum_0^\infty \|g_n\|^2$. Let us consider the operator $H = V + V^* + B$ acting in the space $\mathfrak{H}$, where $B \in \mathbb{B}$ $(\mathfrak{H})$. We assume $S = R = S_\delta$, where $S_\delta h = \sum_{n=0}^\infty e^{-\delta n} V^n g_n$, h is given by (8) and $\delta \in \mathbb{R}$. For $\delta > 0$, the operator S_δ is invertible, the inverse operator is not bounded and it is equal to $S_{-\delta}$. Here we consider the situation when the operator $S_{-\delta}BS_{-\delta}$ is densely defined and bounded in $\mathfrak{H}$ for some $\delta > 0$. Its extension will be denoted by T.

Theorem 2. *If the operator B is such that the operator T is compact for some $\delta > 0$, then the operator H has at most a finite set of eigenvalues. Every eigenvalue has finite multiplicity.*

Proof. The operator H satisfies the conditions of Theorem 1. It is clear that condition (i) is satisfied and since

$$\|SV^n h\|^2 = \|S(\sum_{k=0}^{\infty} V^{n+k} g_k)\|^2 = \|\sum_{k=0}^{\infty} e^{-\delta(n+k)} V^{n+k} g_k\|^2 =$$

$$= e^{-2\delta n} \|\sum_{k=0}^{\infty} e^{-\delta k} V^{n+k} g_k\|^2 \leqslant e^{-2\delta n} \sum_{k=0}^{\infty} e^{-\delta k} \|V^{n+k} g_k\|^2 \leqslant$$

$$\leqslant e^{-2\delta n} \sum_{k=0}^{\infty} \|g_k\|^2 = e^{-2\delta n} \|h\|^2 \quad (h \in \mathfrak{H}; n \in \mathfrak{N}),$$

we have $\|SV^n\| < e^{-\delta n}$ $(\delta > 0)$. Thus, the condition (ii) of Theorem 1 is checked. Theorem 2 is proved.

In particular, let $\mathfrak{H} = l_2$ and V be the canonical translation in $\mathfrak{H}$, i.e. $Vx = (0, x_1, x_2, \dots, x_{n-1}, \dots)$ $(x = (x_n) \in l_2)$. Consider the perturbation of operator $H_0 = V + V^*$ with operator $B = [b_{jk}]_{j,k=1}^{\infty} \in \mathbb{B}$ $(\mathfrak{H})$ (b_{jk} are complex numbers). From Theorem 2 follows immediately the next

Theorem 3. *If the operator generated by the matrix $[e^{\delta(n+k)} b_{nk}]_{n,k=0}^{\infty}$ is bounded for some $\delta > 0$, then the operator $H = V + V^* + B$ has a finite set of eigenvalues, each of them being of finite multiplicity.*

3. In the space l_2 we consider the operator H generated by Jacobi matrix, i.e. $H = [a_{jk}]_{j,k=1}^{\infty}$, $a_{jk} \in \mathbb{C}$, $a_{jk} = 0$ if $|j - k| > 1$ $(j, k = 1, 2, \dots)$.

Suppose that $a_{j,j+k} \to a_k$ $(k = -1, 0, 1)$, $a_1 = \overline{a}_{-1}$, $a_0 \in \mathbb{R}$. Let us denote $\alpha_{jk} = a_{j,j+k} - a_k$, $(k = -1, 0, 1)$. From Theorem 3 one can see at once the following

Theorem 4. *If the sequences $(e^{\delta n} \alpha_{nk})_{n=1}^{\infty}$ $(k = -1, 0, 1)$ converge for some $\delta > 0$, then the point spectrum of the operator H is at most a finite set. Furthermore, the possible eigenvalue has finite multiplicity.*

4. It should be noted that Theorem 1 can be applied to different cases. For example, we consider the operator V generated in the space $L_2(\mathbb{R})$ by the expression

$$(Vf)(x) = \begin{cases} f(x-1), & x \leq 1 \\ 0, & 0 < x < 1. \end{cases}$$

As a perturbation of the operator $H_0 = V + V^*$ consider the integral operator

$$(Bf)(x) = \int_{\mathbb{R}_+} b(x,y) f(y) dy \quad (f \in L_2(\mathbb{R}_+)).$$

Suppose that the kernel $b(\cdot,\cdot)$ is such that the operator B (generally speaking non-selfadjoint) is bounded in $L_2(\mathbb{R}_+)$. Let S and T be operators in $L_2(\mathbb{R}_+)$ of the form

$$(Sf)(x) = e^{-\delta x} f(x), \ (Tf)(x) = \int\limits_{\mathbb{R}_+} e^{\delta(x+y)} b(x,y) f(y) dy,$$

where $\delta > 0$. Then $B = STS$. An easy checking of the conditions of Theorem 1 gives the following

Theorem 5. *If the integral operator with the kernel $e^{\delta(x+y)}b(x,y)$ is bounded on $L_2(\mathbb{R}_+)$ for some $\delta > 0$, then the operator H has a finite set of eigenvalues, each of them being of finite multiplicity.*

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Moldova State University
A. Mateevici str., 60, MD-2009 Chişinău, Moldova

E-mail: cialenco@usm.md
E-mail: pco@usm.md

Primit la redacţie la 20.03.98