Allocation of Resources through Coloring and Counting

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Conflict-free allocation of limited resources is a fundamental problem in applied mathematics that is naturally studied under Graph Theory, a field of expertise here at Illinois Tech.

Graphs are mathematical models of relationships like friendship, genetic ancestry, weblinks, transportation links, between entities such as humans, genomes, webpages, cities, and more. They are the mathematical structures underlying networks of all kinds, including transportation, social, communication, and epidemiological.

For example, suppose we want to assign frequencies (a limited resource) to radio stations (the entities) so that close-by stations get different frequencies to avoid interference (the conflict relationship). Such fundamental resource allocation problems are studied as “coloring problems” on graphs: colors (resources) should be assigned to vertices (entities) so that related vertices with an edge between them (conflict) get different colors.

Beyond this basic coloring problem, additional requirements in the resource allocation problems are modeled by introducing further mathematical constraints. Two prominent such generalizations are: List coloring, introduced independently by Vizing and by Erdős et al. in the 1970s, and Correspondence coloring (now called DP-coloring) introduced by Dvořák and Postle in 2015. In list coloring, each vertex has a restricted list of colors available to use on it. Such problems turn out to be harder than usual coloring problems as we have to consider all possible assignments of lists of colors to the vertices. Even harder are the DP-coloring problems which consider the worst-case scenario of how many colors we have to use in a list coloring if we no longer can identity the names of the colors: each vertex in the graph still gets a list of colors but identification of which colors are different can vary from edge to edge.

Thinking of these as optimization problems, the least number of colors (referred to as the appropriate chromatic number) needed for a graph under each notion, increases from ordinary coloring to list coloring to DP-coloring.

Starting with the famous Four Color Conjecture, coloring problems have been a driving force for the development of new ideas and methods in graph theory for the last 170 years. I will describe two important such algebraic ideas developed over 70 years apart and our contributions to them.

In 1912, Birkhoff introduced the chromatic polynomial to count the number of colorings of a graph using a fixed number of colors. Chromatic polynomial and its generalizations are a central object of study in algebraic combinatorics. Around 1990, Kostochka and Siderenko implicitly introduced the list color function, a list coloring analogue of the chromatic polynomial. Over the past 30 years, it has been discovered that the list color function can behave differently from the chromatic polynomial (in fact it need not even be a polynomial, hence the moniker of 'function'), but for large enough number of colors the list color function and the chromatic polynomial are the same.

With my former Ph.D. student, Jeff Mudrock, who is now a professor of mathematics at College of Lake County (CLC), we have a long-term project "Counting Colorings and Coloring by Counting" exploring these algebraic connections to coloring. We have shown that the knowledge of the list color function is not just meaningful for enumeration, but it is essential to understanding and bounding the list chromatic number of certain graphs, an unusual application of counting to optimal coloring. In fact, we show the list color function can encapsulate the threshold for when the list chromatic number equals the (ordinary) chromatic number, and when these two chromatic numbers are far apart. What about DP-colorings?

Mudrock and I recently introduced the notion of DP color function, the DP coloring analogue of the chromatic polynomial. We have shown that DP color function, unlike the list color function, need not equal the chromatic polynomial even if we allow use of a large number of colors. But this does not mean it is a useless idea. We, with graduate student Gunjan Sharma, show that DP color function is essential to finding sharp bounds on DP-chromatic numbers and the threshold of difference between the DP- and ordinary chromatic numbers. In fact, we conjecture that the DP-color function ultimately behaves like a polynomial, just not necessarily the chromatic polynomial, although such a polynomial will agree with the chromatic polynomial in the coefficients of its three highest degree monomials. With undergraduate students at CLC, who are or will be at different universities like Illinois Tech, University of Illinois Chicago, Dartmouth, University of Wisconsin at Madison, Rhode Island, etc., we have been directing research on this topic since summer 2020,which has led to several papers towards this conjecture, and more importantly to new tools for understanding DP color function and its applications. By Fall 2021, over twelve undergraduate students have participated in this research.

Coloring problems have another distinct connection with polynomials. The polynomial method, in the words of Terry Tao, encodes sets of objects in the zero set of a polynomial whose degree is bounded by some function of the number of objects, and then tools from algebra are used to understand the roots of this polynomial. This powerful method, now used all over mathematics, first arose in the 1980s as the Combinatorial Nullstellensatz (CN). Some of the powerful applications of CN are to difficult list coloring problems in the form of the Alon-Tarsi method. However, it is known that Alon-Tarsi is not applicable to DP coloring, removing a major tool for such coloring problems. Recently, Mudrock and I were able to derive new tools from CN based on polynomials over finite fields that are applicable to DP coloring.

The possibilities for these new polynomial and counting tools seem limitless as we continue to build our understanding of list and DP colorings of graphs. See https://www.math.iit.edu/~kaul/papers.html for the corresponding papers.