<u>How we discover and create Mathematics: An example</u>

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Creating mathematics is like discovering new exotic patterns and creating a language for describing them. You go exploring into an unknown realm hoping to discover a pattern never seen before and to describe it in a precise language.

Let me give a simple example that leads to interesting mathematical insights.

After attending many parties, a mathematician notices that in any group of six or more people she can always find three people who are mutual acquaintances or three people who are mutual strangers. However, she wants to convince herself that this is really true and not just an artifact of the few parties she has been attending. So she describes it in a precise language (mathematical, naturally) and gives a logical argument (a proof) for why it must always be true, no matter which group of six people are observed, living/dead/yet-to-be-born.[*]

OK, that's neat, but that's just a puzzle, a curiosity. We could stop here and never think about this puzzle again. However, our curious mathematician wonders, 'Does this observation have something deeper underlying it?'. Is there a deeper pattern of relationships of which this fact is just the tip of an iceberg. So the mathematician investigates further, tries to formulate the right "path" of exploring this "world" of relationships - acquaintance vs. stranger. And she goes on to discover that this is truly a widespread phenomenon, a pattern occurring in many seemingly unrelated situations.

It turns out that there is nothing special about "three" when we want to find groups of three mutual acquaintances or three mutual strangers. Lets say you are looking for a pattern of "k mutual acquaintances or k mutual strangers", where k is your favorite positive integer. Could we guarantee that, by going to a party with some large number of people in attendance, say N, we will always find k mutual acquaintances or k mutual strangers there? This is no longer obvious. But the answer turns out to be yes. For example, for $k = 4$, in any group of 18 people we can guarantee there will always be 4 mutual acquaintances or 4 mutual strangers. By going from $k = 3$ in our original puzzle to $k = 4$, this is already a much harder problem[†].

When a problem is difficult, its often helpful to make it more general or abstract (and seemingly, even more difficult). Our mathematician, by thinking about the problem in

[*]How we do this is not important here. Graph theory, the mathematical language for modeling pairwise relationships such as in social networks, is used for this purpose. This puzzle is a popular first week HW problem in an introductory course on graph theory. Look up this Quanta article that illustrates how to do it.

[†]one that no instructor would assign as a HW problem even in an advanced course.

general, is now able to prove that: For any number $k \geq 3$, it can be guaranteed that there is a number N such that ANY collection of N or more people will always contain the pattern "k mutual acquaintances or k mutual strangers"[‡]. No matter how a group of people are related to each other as pairs of acquaintances or strangers, it can not prevent the existence of a large relationship pattern (a subgroup of people who are all mutual acquaintances or all mutual strangers) within any such group.

The beauty of this discovery is that it applies to not just this gimmicky example of people, but to any ensemble of related entities that are arbitrarily split into smaller groups. We are always guaranteed to find nicely ordered substructures (patterns corresponding to the particular ensemble of entities being studied) within any large disordered system.

You could try to apply this idea metaphorically to stars and finding "constellations" as patterns within them[§]. Or to be more precise, to points (observable stars) in the plane (sky as observed from Earth) with patterns (constellations) being described as convex k-gons. Convex k-gons are regular geometric shapes with k corners, such as triangles are convex 3-gons, rectangles are convex 4-gons, and so on. Then understanding the above phenomenon in this context leads us to discover that, if the number of observable stars is large enough then we can find any such constellation that we are looking for. That is, within any large enough collection of points in the plane, no matter how they are arranged, we can find whichever convex k-gon we are looking for[¶].

What I wrote above can be thought of as an attempt by a mathematician to describe the notion that "complete disorder is impossible". In any large enough ensemble, no matter how disordered, we can always find one of a few possible fixed patterns. We mathematicians have our own way of discovering (or, creating) what that could possibly mean. Like an artist we want to have a tangible outcome of an idea that exists only in our mind. Our tool is language, but it's not enough to give a vague metaphorical description. Our language is molded by logic and we build precise structures within this idealized universe. Although some might complain mathematical structures are not tangible, we would argue otherwise. Mathematical constructs are eternal patterns that exist forever in our mind's eye.

[‡]For each specific k, computing the exact value of the corresponding N is extremely hard. In fact, Paul Erdős, who re-discovered this theorem as a young man in 1933 after it had already been proved a few years earlier by Frank Ramsey, in whose honor this field of study is called Ramsey theory, used to tell a story to illustrate this point. If aliens attacked earth and demanded to know the exact value of N corresponding to $k = 5$ then Erdős recommended that all of humanity should pool their computational resources to find the exact answer. However, if the aliens demanded the answer corresponding to $k = 6$, we should straightaway attack the aliens with all our might. By focusing on the existence of a solution for all k, rather than the exact value for each specific k, we made the problem more general and abstract, but also relatively easier to answer. And easier to find a pattern that transcends specific k. Look up Ramsey's theorem on wikipedia.

[§]These ideas are not just abstract broodings, but are applied to concrete problems in mathematics and computer science. Look up 'Ramsey Theory Applications', a survey article by Vera Rosta.

[¶]Look up Erdős' 'Happy Ending Problem' on wikipedia.