List Coloring Cartesian Products of Graphs: Criticality and List Color Function

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Joint work with

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List Coloring

- List coloring was introduced independently by Vizing (1976) and Erdős, Rubin, and Taylor (1979), as a generalization of usual graph coloring.

- For graph G suppose each $v \in V(G)$ is assigned a list, $L(v)$, of colors. We refer to L as a list assignment. An acceptable L-coloring for G is a proper coloring, f, of G such that $f(v) \in L(v)$ for all $v \in V(G)$.

- When an acceptable L-coloring for G exists, we say that G is L-colorable or L-choosable.

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List Chromatic Number

- The list chromatic number of a graph G, written $\chi_\ell(G)$, is the smallest k such that G is L-colorable whenever $|L(v)| \geq k$ for each $v \in V(G)$.
- When $\chi_\ell(G) = k$ we say that G has list chromatic number k or that G is k-choosable.
- We immediately have that if $\chi(G)$ is the typical chromatic number of a graph G, then

$$\chi(G) \leq \chi_\ell(G).$$

But we know the gap between $\chi(G)$ and $\chi_\ell(G)$ can be arbitrarily large

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Two Types of of Questions

- When does $\chi(G) = \chi_\ell(G)$?
  A graph is chromatic choosable if $\chi(G) = \chi_\ell(G)$.
- How large can be the gap between $\chi(G)$ and $\chi_\ell(G)$?
- We will ask both these questions in the context of Cartesian Products of Graphs.

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Theorem (Ohba's Conjecture: Noel, Reed, Wu (2014))
If $\chi(G) \geq \frac{|V(G)|-1}{2}$ then, $\chi_\ell(G) = \chi(G)$.

Conjecture (List Coloring Conjecture)
If G is a line graph of some loopless multigraph, then $\chi_\ell(G) = \chi(G)$.

- Galvin (1995) showed that the List Coloring Conjecture holds for bipartite multigraphs.
- Total graphs and claw free graphs are also conjectured to be chromatic choosable.

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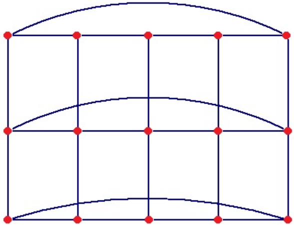
Cartesian Product of Graphs

- The Cartesian Product $G \Box H$ of graphs G and H is a graph with vertex set $V(G) \times V(H)$.
  Two vertices (u, v) and (u', v') are adjacent in $G \Box H$ if either $u = u'$ and $vv' \in E(H)$ or $uu' \in E(G)$ and $v = v'$.
- Here's $C_5 \Box P_3$:



- Every connected graph has a unique factorization under the Cartesian product (that can be found in linear time and space).
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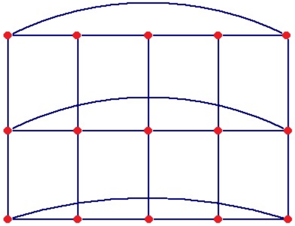

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List Coloring the Cartesian Product of Graphs

Theorem (Borowiecki, Jendrol, Kral, Miskuf (2006))
$\chi_\ell(G \square H) \leq \min\{\chi_\ell(G) + \mathrm{Col}(H), \mathrm{Col}(G) + \chi_\ell(H)\} - 1.$

- $\mathrm{Col}(G)$, the coloring number of a graph G, is the smallest integer d for which there exists an ordering, $v_1, v_2, \ldots, v_n$, of the elements in $V(G)$ such that each vertex v_i has at most $d - 1$ neighbors among $v_1, v_2, \ldots, v_{i-1}$.
- An easy inductive argument proves this theorem.
- Borowiecki et al. showed that their bound is tight for certain factors ($G = H = K_{k,(2k)^{k(k+k^k)}}$), but in general, by a result of Alon, is exponential in the list-chromatic number, and not necessarily exact.

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Motivating Questions - Type I

- **Question:** Can we characterize situations where $G \square H$ is chromatic choosable?
- **Question:** For what (chromatic choosable) graphs G is $G \square P_n$ chromatic choosable?
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Motivating Questions - Type II

- For fixed G, a:
  $\chi_\ell(G \square K_{a,b}) \le \chi_\ell(G) + \mathrm{Col}(K_{a,b}) - 1 = \chi_\ell(G) + a$

  Question: Does there always exist a b such that this upper bound is attained?

Theorem (Borowiecki, Jendrol, Kral, Miskuf (2006))
$\chi_\ell(G \square K_{a,b}) = \chi_\ell(G) + a$, *whenever* $b \ge (\chi_\ell(G) + a - 1)^{a|V(G)|}$

Question: Can we improve the lower bound on b?

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When $G = K_1$, $G \Box K_{a,b} = K_{a,b}$, and we know

Theorem (Folklore, 1970s) $\chi_\ell(K_{a,b}) = a + 1$ iff $b \geq a^a$

- When $b \geq a$, we know $\chi_\ell(K_{a,b}) \leq \mathrm{Col}(K_{a,b}) = a + 1$.
  So, for fixed a, this theorem tells us the smallest value of b such that $\chi_\ell(K_{a,b})$ is as large as possible (i.e., far from being chromatic-choosable).

  Question: For which G, can we have a similar result for $G \Box K_{a,b}$?

- We can construct a sequence of graphs with increasing list chromatic number starting from chromatic number 2:
  $\chi(K_{a,a^a}) = \chi(K_{1,1}) = 2 = \chi_\ell(K_{1,1}) < 3 = \chi_\ell(K_{2,4}) < 4 = \chi_\ell(K_{3,27}) < \ldots < a + 1 = \chi_\ell(K_{a,a^a})$

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How can we prove these?

Corollary (K. and Mudrock)

$$\chi_\ell(C_{2t+1}\square K_{1,s}) = \begin{cases} 3 & \textit{if } s < 2^{2t+1}-2 \\ 4 & \textit{if } s \geq 2^{2t+1}-2. \end{cases}$$

Corollary (K. and Mudrock)

$$\chi_\ell(K_n\square K_{1,s}) = \begin{cases} n & \textit{if } s < n! \\ n+1 & \textit{if } s \geq n!. \end{cases}$$

Corollary (K. and Mudrock)

$$\chi_\ell((K_n \vee C_{2t+1})\square K_{1,s}) = \begin{cases} n+3 & \textit{if } s < \frac{1}{3}(n+3)!(4^t-1) \\ n+4 & \textit{if } s \geq \frac{1}{3}(n+3)!(4^t-1). \end{cases}$$

Corollary (K. and Mudrock)

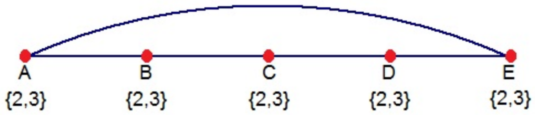
$\chi_\ell(C_{2t+1}\square K_{2,b}) = 5$ *if and only if* $b \geq 9(9^t-1)^2$.

Corollary (K. and Mudrock)

$\chi_\ell(K_n\square K_{a,b}) = n+a$ *if and only if* $b \geq \left(\frac{(n+a-1)!}{(a-1)!}\right)^a$

Strong Chromatic Choosability

- We introduce the notion of strong chromatic choosability:
- List assignment, L, for G is a bad k-assignment for G if G is not L-colorable and $|L(v)| = k$ for each $v \in V(G)$.
- List assignment, L, is constant if $L(v)$ is the same for each $v \in V(G)$.
- A constant (and bad) 2-assignment for a C_5:



- A graph G is said to be strong k-chromatic choosable if $\chi(G) = k$ and every bad $(k-1)$-assignment for G is constant.

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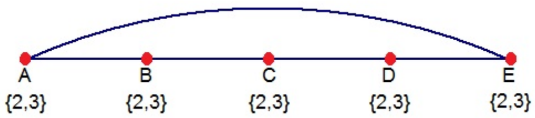

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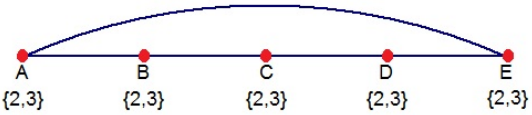

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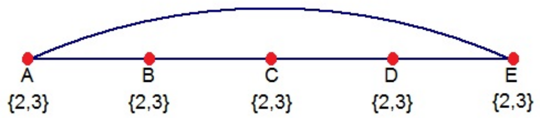

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Proposition (K. and Mudrock, 2021)

Let G be a strong k-chromatic choosable graph. Then
(i) $\chi(G) = k = \chi_\ell(G)$ (i.e. G is chromatic choosable),
(ii) $\chi(G - \{v\}) \le \chi_\ell(G - \{v\}) < k$ for any $v \in V(G)$,
(iii) $k = 2$ if and only if G is K_2,
(iv) $k = 3$ if and only if G is an odd cycle,
(v) $G \vee K_p$ is strong $(k+p)$-chromatic choosable for any $p \in \mathbb{N}$.

- This family contains strong k-critical graphs, studied by Steibitz, Tuza, and Voigt (2008), for color criticality in the context of list coloring.
- Strong k-critical graph is k-critical and every bad $(k-1)$-assignment is constant.
- We are essentially relaxing edge-criticality in strong k-critical graphs to vertex-criticality.

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- This family contains strong k-critical graphs, studied by Steibitz, Tuza, and Voigt (2008), for color criticality in the context of list coloring.
- Strong k-critical graph is k-critical and every bad $(k-1)$-assignment is constant.
- We are essentially relaxing edge-criticality in strong k-critical graphs to vertex-criticality.

Strong Chromatic Choosability

There are many infinite families of graphs that satisfy these notions.

Are there strongly chromatic choosable graphs which are not strongly critical?

Yes, we can construct examples of strong k-chromatic choosable graphs which are not strongly k-critical for each $k \geq 4$.

Lemma (K. and Mudrock, 2021)

Let G be a strong k-chromatic choosable graph. Let $A, B \subseteq V(G)$ such that $A \cup B = V(G)$ and $C = A \cap B$ with $|A|, |B| > |C|$, $0 < |C| \leq 3$ when k is even and $0 < |C| \leq 4$ when k is odd. Form G' by adding vertices u and s to G, and edges so that u is adjacent to every vertex in A and s is adjacent to every vertex in B. If $\chi(G') > k$, then G' is strong $(k+1)$-chromatic choosable.

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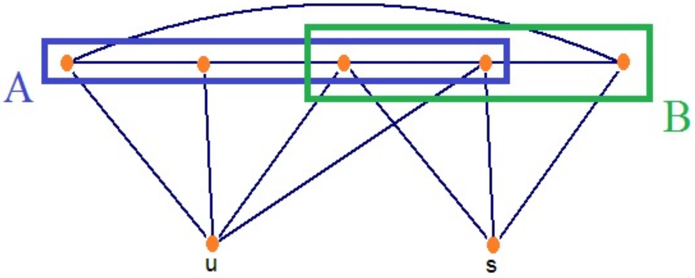
Strong Chromatic Choosability

The graph below is strong 4-chromatic choosable, but it is not strong 4-critical.



Unique List Colorabliltiy

Theorem (Akbari, Mirrokni, Sadjad (2006))

Let G be a graph with n vertices and m edges and $f : V(G) \to \mathbb{N}$ be a function such that $\sum_{v \in V(G)} f(v) = m + n$. If there is a list assignment, L, for G such that $|L(v)| = f(v)$ for each $v \in V(G)$ and G has a unique L-coloring, then G is f-choosable.

- We say that G is f-choosable if G is L-colorable whenever $|L(v)| = f(v)$ for each $v \in V(G)$.
- This result helps us to prove something about f-choosability by finding one list assignment with the needed properties (Tough!).

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Important Implication of the Akbari et al.

From the Akbari et al. (2006) result, we may deduce the following lemma which is a key ingredient in the proof of our main result.

Lemma

Suppose G is a strong k-chromatic choosable graph with n vertices and m edges.
Let L be a list assignment with $|L(v)| \geq k-1$ for all v and L is a not a constant $(k-1)$-assignment for G.
If $m \leq n(k-2)$ then G has at least two L-colorings.

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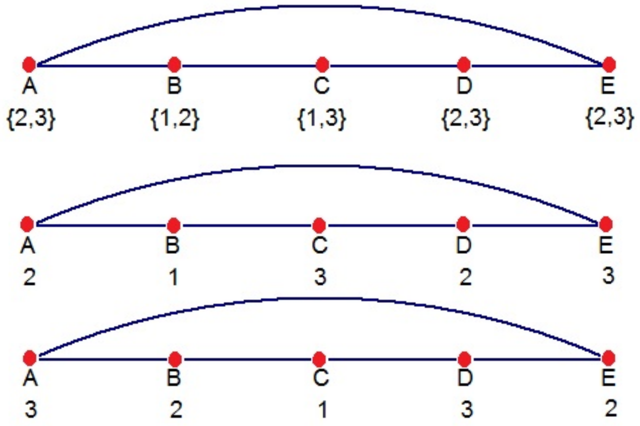

The Edge Condition

We say a strong k-chromatic choosable graph with n vertices and m edges satisfies the edge condition if $m \leq n(k-2)$.

- **All** strongly chromatic choosable graphs we have encountered thus far satisfy the edge condition.
- Moreover, any strongly chromatic choosable graph which satisfies the edge condition will remain a strongly chromatic choosable graph satisfying the edge condition when joined to a complete graph.

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Unfortunately, we suspect there exist strong k-chromatic choosable graphs which do not satisfy the edge condition for each $k \geq 4$. We have constructed examples in the cases of $k = 4, 5, 6, 7$.

On the bright side, we have the following result.

Lemma (K. and Mudrock, 2021)

Suppose that G is a strong k-chromatic choosable graph which does not satisfy the edge condition. Then, for sufficiently large p, $G \vee K_p$ is a strong $(k+p)$-chromatic choosable graph which satisfies the edge condition.

- Above, p needs to satisfy $p(2k + p - 3)/2 \geq m - n(k - 2)$.

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Main Results - Part 1

Theorem (K. and Mudrock, 2021)

Let M be a strong k-chromatic choosable graph which satisfies the edge condition,
and let H be a graph which contains a Hamilton path, $w_1, w_2, \ldots, w_m$, such that w_i has at most ρ neighbors among $w_1, \ldots, w_{i-1}$. Then,

$$\chi_\ell(M \square H) \leq k + \rho - 1.$$

This result improves upon the Borowiecki et al. bound when H is a path, grid, power of a path, cycle, complete graph, etc.

Proof of Main Result

We always use fixed colors for the first $m-2$ copies of M. We then use a loaded inductive hypothesis to possibly modify how we will color the $(m-1)$st copy of M in $M\square H$.

The loaded induction:

Lemma (K. and Mudrock, 2021)

Let M and H satisfy the hypotheses of the theorem, and let $G = M\square H$.
Let L be an arbitrary $(k+\rho-1)$-assignment for G. Then, there exist two L-colorings of G, c_1 and c_2, with the property that there exists a vertex, v, in the m^{th} copy of M in G such that $c_1(v) \neq c_2(v)$, and for any u not in the m^{th} copy of M in G, $c_1(u) = c_2(u)$.

Chromatic Choosable Grid-like Graphs

Corollary (K. and Mudrock, 2021)

Suppose that M is a strong k-chromatic choosable graph which satisfies the edge condition. Then, $M \square P_n$ is chromatic choosable.

This Corollary shows that the bound in our main result is tight. Note that this is where $\rho = 1$.

What about $\rho > 1$?

The List Color Function

- For $k \in \mathbb{N}$, let $P(G, k)$ denote the number of proper colorings of G with colors from $\{1, \ldots, k\}$.

- It is known that $P(G, k)$ is a polynomial in k of degree $|V(G)|$. We call $P(G, k)$ the chromatic polynomial of G.

- The list color function of G, $P_\ell(G, k)$, is the minimum number of k-list colorings of G where the minimum is taken over all k-list assignments for G.

- Recall, $P(K_{2,4}, 2) = 2$, and yet $P_\ell(K_{2,4}, 2) = 0$
- For every graph G and each $k \in \mathbb{N}$, $P_\ell(G, k) \leq P(G, k)$.

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Theorem (Kostochka and Sidorenko (1990))

If G is a chordal graph (i.e. all cycles contained in G with 4 or more vertices have a chord), then $P_\ell(G,k) = P(G,k)$ for each $k \in \mathbb{N}$.

$P_\ell(G,k)$ need not be a polynomial.

Theorem (Thomassen (2009))

For any graph G, $P_\ell(G,k) = P(G,k)$ provided $k > |V(G)|^{10}$.

Theorem (Wang, Qian, Yan (2017))

For any connected graph G with m edges, $P_\ell(G,k) = P(G,k)$ provided $k > \frac{m-1}{ln(1+\sqrt{2})} \approx 1.135(m-1)$.

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Theorem (K. and Mudrock, 2020)

$\chi_\ell(G \square K_{a,b}) = \chi_\ell(G) + a$, whenever $b \geq (P_\ell(G, \chi_\ell(G) + a - 1))^a$

- If G has at least one edge, then $P_\ell(G, \chi_\ell(G) + a - 1) < (\chi_\ell(G) + a - 1)^{|V(G)|}$; giving a (significant) improvement over the Borowiecki et al. bound.
- We can in fact prove:

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*Suppose H is a bipartite graph with partite sets A and B where $|A| = a$ and $|B| = b$. Let $\delta = \min_{v \in B} d_H(v)$.
If $b \geq (P_\ell(G, \chi_\ell(G) + \delta - 1))^a$, then $\chi_\ell(G \square H) \geq \chi_\ell(G) + \delta$.*

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If G *is a strong* k*-chromatic choosable graph and* $k \geq a + 1$*, then* $\chi_\ell(G \square K_{a,b}) = \chi_\ell(G) + a$ *if and only if* $b \geq (P_\ell(G, \chi_\ell(G) + a - 1))^a$.

The proof idea is:

If L is a $(\chi_\ell(G) + a - 1)$-assignment for $G \square K_{a,b}$, there is at most one proper L-coloring of the copies of G corresponding to the partite set of size a that leads to a bad assignment for a given "bottom" copy of G.

We show if two such colorings existed, we could obtain a proper a-coloring of G.

A simple counting argument completes the proof that there exists a proper L-coloring of $G \square K_{a,b}$ when

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$\chi_\ell(C_{2t+1} \square K_{2,b}) = 5$ if and only if $b \geq (P_\ell(C_{2t+1}, 4))^2 = (3^{2t+1} - 3)^2 = 9(9^t - 1)^2$.

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For $n \geq a+1$, $\chi_\ell(K_n \square K_{a,b}) = n + a$ if and only if $b \geq (P_\ell(K_n, n+a-1))^a = \left(\frac{(n+a-1)!}{(a-1)!}\right)^a$

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This corollary shows the bound in the Theorem is sharp for all a.

- We can construct an arbitrarily long sequence of graphs with increasing list chromatic number starting from chromatic number n: $\chi(K_n \square K_{a,b}) = \chi(K_n) = n = \chi_\ell(K_n \square K_{0,1}) < n+1 = \chi_\ell(K_n \square K_{1,n!}) < n+2 = \chi_\ell(K_n \square K_{2,((n+1)!)^2}) < \cdots$

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Suppose that G is a strong k-chromatic choosable graph with $k \geq 2$. Then, $\chi_\ell(G \square K_{1,s}) = \begin{cases} k & \text{if } s < P_\ell(G,k) \\ k+1 & \text{if } s \geq P_\ell(G,k). \end{cases}$

- Note we can think of a star in the second factor as a graph that is far from having a Hamilton path. Recall: Theorem (K. and Mudrock, 2021) Let M be a strong k-chromatic choosable graph which satisfies the edge condition, and let H be a graph which contains a Hamilton path, Then $\chi_\ell(M \square H) \leq k + \rho - 1$.
- This shows the exact transition from chromatic choosability for $G \square K_{1,s}$.
- The theorem is particularly useful (fun!) when we know $P_\ell(G,k)$.

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Counting List Colorings

- We want know (or provide bounds on) $P_\ell(G, k)$ for a strong k-chromatic choosable graph G.

Theorem (K. and Mudrock, 2021)

If G is a strong k-chromatic-choosable graph, then

$$P_\ell(G, m) \geq m \max_{v \in V(G)} P_\ell(G - \{v\}, m-1) \geq m$$

whenever $m \geq k$.

Counting List Colorings

- Armed with the fact that $P_\ell(C_{2t+1}, k) = P(C_{2t+1}, k)$ (can be proved using an elementary counting argument), the following Corollary is now easy to prove.

Corollary (K. and Mudrock, 2021)
For any $n, k, t \in \mathbb{N}$,

$$P_\ell(K_n \vee C_{2t+1}, k) = P(K_n \vee C_{2t+1}, k) = P(K_n, k)P(C_{2t+1}, k-n).$$

Moreover,
$$P_\ell(C_{2t+1} \vee K_n, k) = P(C_{2t+1} \vee K_n, k)$$
$$= \left[(k-n-1)^{2t+1} - (k-n-1)\right] \prod_{i=0}^{n-1} (k-i).$$

- We will now apply our main result to strongly chromatic choosable graphs of which we have a good understanding of the list color function.

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Corollary (K. and Mudrock, 2021)

For any $t \in \mathbb{N}$, we have that:

$$\chi_\ell(C_{2t+1} \square K_{1,s}) = \begin{cases} 3 & \text{if } s < 2^{2t+1} - 2 \\ 4 & \text{if } s \geq 2^{2t+1} - 2. \end{cases}$$

- Borowiecki et al. only gives us that $\chi_\ell(C_{2t+1} \square K_{1,s}) = 4$ when $s \geq 3^{2t+1}$.

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For any $t, n \in \mathbb{N}$, we have that:

$$\chi_\ell((K_n \vee C_{2t+1}) \square K_{1,s}) = \begin{cases} n+3 & \text{if } s < \frac{1}{3}(n+3)!(4^t-1) \\ n+4 & \text{if } s \geq \frac{1}{3}(n+3)!(4^t-1). \end{cases}$$

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Generalizing from Stars

Theorem (K. and Mudrock, 2021)

Let G be a strong k-chromatic choosable graph satisfying the edge condition, and
B be a subdivision of $K_{1,s}$ with $s < P_\ell(G, k)$.
Then, $\chi_\ell(G \square B) = k$.

Theorem (K. and Mudrock, 2021)

Let G is a strong k-chromatic choosable graph with $k \geq 2$. Let T be a rooted tree with root v_0.
Suppose that v_0 has at most $P_\ell(G, k) - 1$ children, and each $v \in V(T) - \{v_0\}$ has at most $P'_\ell(G, k-1) - 1$ children
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Main Results - Part 3

Theorem (K. and Mudrock, 2021)

Let M be a strong k-chromatic choosable graph which satisfies the edge condition. Then, for any $\rho \in \mathbb{N}$, there exists H such that $\chi_\ell(M \square H) = k + \rho - 1$.

- For each $\rho \in \mathbb{N}$ we define the $H = S_{M,B',\rho}$ graph inductively. Let $S_{M,B',1} = B'$, a subdivision of $K_{1,P_\ell(M,k)-1}$. Then, for $\rho \geq 2$ we construct $S_{M,B',\rho}$ as: Take $P_\ell(M, k + \rho - 2)$ disjoint copies of $S_{M,B',\rho-1}$ and join a single vertex to these copies.

- This in fact gives the sharpness for the following generalization of the first main theorem:

Theorem (K. and Mudrock, 2021)

Let M be a strong k-chromatic choosable graph which satisfies the edge condition, and H be a (M, ρ)-Cartesian accommodating graph. Then $\chi_\ell(M \square H) \leq k + \rho - 1$.

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Thank You!

Questions?

- For what graphs G, is $G \square P_n$ chromatic choosable for all $n \in \mathbb{N}$?
- Determine when $G \square H$ will be chromatic choosable based upon some property of the factors.
- Define $f_a(G)$ as the smallest b s.t. $\chi_\ell(G \square K_{a,b}) = \chi_\ell(G) + a$.
- For what graphs does $f_a(G) = (P_\ell(G, \chi_\ell(G) + a - 1))^a$?
- Does there exist a strongly chromatic-choosable graph M such that $f_a(M) < (P_\ell(M, \chi_\ell(M) + a - 1))^a$?
- Is it the case that $f_a(K_n) = \left(\frac{(n+a-1)!}{(a-1)!}\right)^a$ for each n, a?
- Is it always the case that $P_\ell(G, k) = P(G, k)$ when G is strong chromatic choosable?
- (Thomassen 2009) Does there exist a graph G and a natural number $k > 2$ such that $P_\ell(G, k) = 1$?
- (Mohar 2001) Let G be a $\Delta(G) + 1$-edge-critical graph. Then prove that $L(G)$ is strong $(\Delta(G) + 1)$-chromatic choosable.

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