Chromatic Polynomial and Counting List and DP Colorings of Graphs: Problems and Progress

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Graph Coloring

- Color vertices so that any vertices with an edge between them must get different colors.
- A proper m-coloring of a graph G is a labeling $f : V(G) \to [m]$, such that $f(u) \neq f(v)$ whenever u and v are adjacent in G.
- Minimum number of colors needed for such a coloring is called the chromatic number $\chi(G)$ of the graph G.
- Each vertex has the same list of colors $[m]$ available to it.

List Coloring

- List coloring was introduced independently by Vizing (1976) and Erdős, Rubin, and Taylor (1979), as a generalization of usual graph coloring.

- For graph G suppose each $v \in V(G)$ is assigned a list, $L(v)$, of colors. We refer to L as a list assignment. If all the lists associated with the list assignment L have size m, we say that L is an m-assignment.

- An L-coloring for G is a proper coloring, f, of G such that $f(v) \in L(v)$ for all $v \in V(G)$.

- When an L-coloring for G exists, we say that G is L-colorable or L-choosable.

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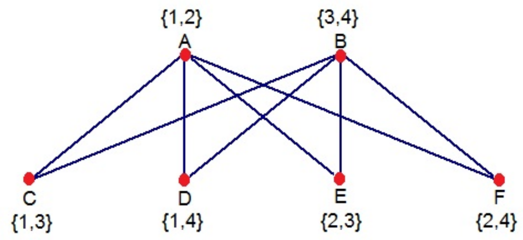
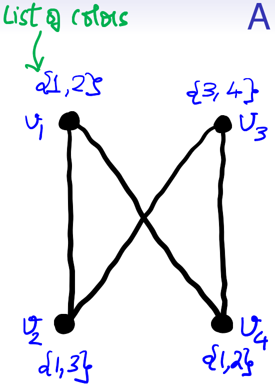
- For example, $2 = \chi(K_{2,4}) < \chi_\ell(K_{2,4}) = 3$.

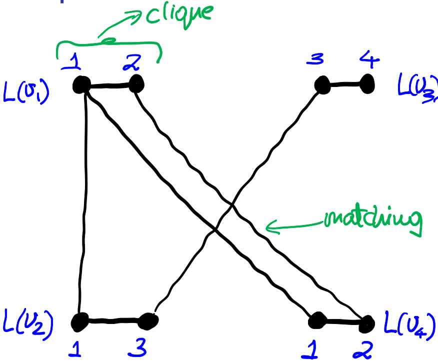


List Chromatic Number

- Since usual coloring corresponds to a constant list assignment,
$$\chi(G) \leq \chi_\ell(G).$$
- The gap between $\chi(G)$ and $\chi_\ell(G)$ can be arbitrarily large: $\chi_\ell(K_{k,t}) = k + 1$ iff $t \geq k^k$.

A Different Perspective





Proper L-coloring

$v_1 \to 2$

$v_2 \to 1$

$v_3 \to 3$

$v_4 \to 1$

corresponds to

Independent set of size 4 here.

$2 \in L(v_1)$, $1 \in L(v_2)$, $3 \in L(v_3)$, $1 \in L(v_4)$

DP-Coloring

- In 2015, Dvořák and Postle introduced DP-coloring (they called it correspondence coloring) of graphs.
- Intuitively, DP-coloring considers the worst-case scenario of how many colors we need in the lists if we no longer can identify the names of the colors. Each vertex still gets a list of colors but identification of which colors are different can vary from edge to edge.

- A (DP-)cover of G is a pair $\mathcal{H} = (L, H)$ consisting of a graph H and a function $L : V(G) \to \mathcal{P}(V(H))$ satisfying:

  (1) the set $\{L(u) : u \in V(G)\}$ is a partition of $V(H)$;
  (2) for every $u \in V(G)$, the graph $H[L(u)]$ is complete;
  (3) if $E_H(L(u), L(v))$ is nonempty, then $u = v$ or $uv \in E(G)$;
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- Intuition:
  Blow up each vertex u in G into a clique of size $|L(u)|$;
  Add a matching (possibly empty) between any two such cliques for vertices u and v if uv is an edge in G.

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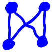
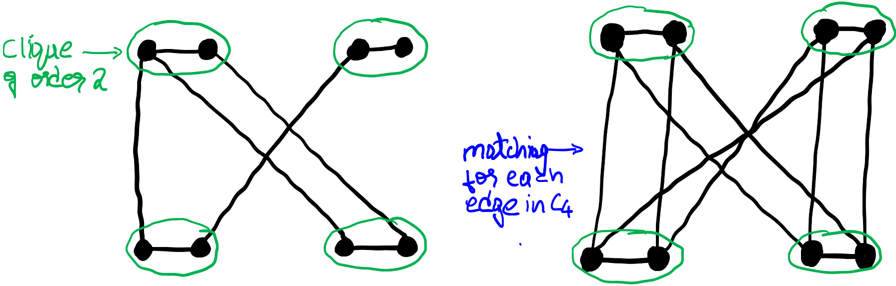
  Blow up each vertex u in G into a clique of size $|L(u)|$;
  Add a matching (possibly empty) between any two such cliques for vertices u and v if uv is an edge in G.
- A cover $\mathcal{H} = (L, H)$ is called m-fold if $|L(u)| = m$ for all u.
- Two 2-fold covers of C_4: 



DP-Chromatic Number of a Graph

- Given $\mathcal{H} = (L, H)$, a cover of G, an $\mathcal{H}$-coloring of G is an independent set in H of size $|V(G)|$. Equivalently, an independent transversal in $\mathcal{H}$.
- The DP-chromatic number of a graph G, $\chi_{DP}(G)$, is the smallest m such that G admits an $\mathcal{H}$-coloring for every m-fold cover $\mathcal{H}$ of G.

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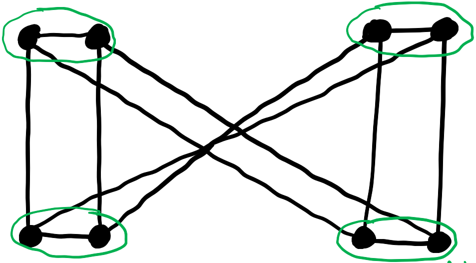
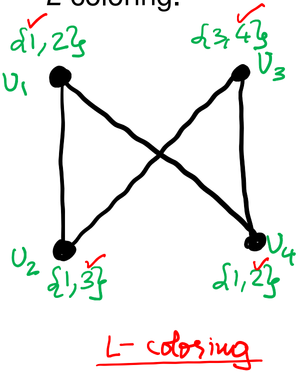
- The DP-chromatic number of a graph G, $\chi_{DP}(G)$, is the smallest m such that G admits an $\mathcal{H}$-coloring for every m-fold cover $\mathcal{H}$ of G.
- $\chi_{DP}(C_4) > 2 = \chi_\ell(C_4)$:

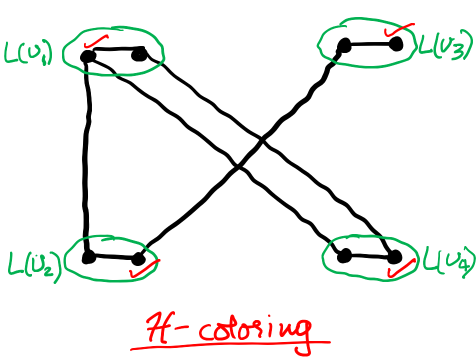


No independent set of size 4 in this 2-fold cover of C_4

DP-Coloring and List Coloring

- Given an m-assignment, L, for a graph G, it is easy to construct an m-fold cover $\mathcal{H}$ of G such that:
  G has an $\mathcal{H}$-coloring if and only if G has a proper L-coloring.





- $\chi(G) \leq \chi_\ell(G) \leq \chi_{DP}(G)$.

The Chromatic Polynomial

- Birkhoff 1912: For $m \in \mathbb{N}$, let $P(G, m)$ denote the number of proper colorings of G where the colors used come from $\{1, \ldots, m\}$.

- $P(G, m)$ is a polynomial in m of degree $|V(G)|$. We call $P(G, m)$ the chromatic polynomial of G.

- $P(K_n, m) = m(m-1)\cdots(m-(n-1))$ and $P(\bar{K}_n, m) = m^n$.
- For any tree T on n vertices, $P(T, m) = m(m-1)^{n-1}$.
- $P(C_n, m) = (m-1)^n + (-1)^n(m-1)$.

The List Color Function

- $P(G, L)$ be the number of proper L-colorings of G.
- Kostochka and Sidorenko 1990: The list color function $P_\ell(G, m)$ is the minimum value of $P(G, L)$ over all possible m-assignments L for G.
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- In general, $P_\ell(G, m) \leq P(G, m)$.
- $P(K_{2,4}, 2) = 2$, and yet $P_\ell(K_{2,4}, 2) = 0$.
- $P_\ell(K_{3,26}, 3) \leq 3^8 2^{12} < 3^1 2^{26} \leq P(K_{3,26}, 3)$.

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Theorem (Kostochka, Sidorenko (1990); Kirov, Naimi (2016); K., Mudrock (2021))

1) $P_\ell(G, m) = P(G, m)$ for all m, if G is chordal.
2) $P_\ell(C_n, m) = P(C_n, m) = (m-1)^n + (-1)^n(m-1)$ for all m.
3) $P_\ell(C_n \vee K_k, m) = P(C_n \vee K_k, m)$ for all m.

The List Color Function

- $P_\ell(G, m) \leq P(G, m)$. And for some G, $P_\ell(G, m) < P(G, m)$
- $P_\ell(G, m)$ need not be a polynomial, but it will equal the chromatic polynomial ultimately.

Theorem (Wang, Qian, Yan (2017); improving Thomassen (2009), Donner (1992), question of Kostochka & Sidorenko (1990))

For any connected graph G with t edges,
$P_\ell(G, m) = P(G, m)$ for $m > \frac{t-1}{ln(1+\sqrt{2})} \approx 1.135(t-1)$.

The DP Color Function

- For $\mathcal{H} = (L, H)$, a cover of graph G, $P_{DP}(G, \mathcal{H})$ be the number of $\mathcal{H}$-colorings of G.

- K. and Mudrock 2021: The DP color function, $P_{DP}(G, m)$, is the minimum value of $P_{DP}(G, \mathcal{H})$ where the minimum is taken over all possible m-fold covers $\mathcal{H}$ of G.

- $P(C_4, 2) = P_\ell(C_4, 2) = 2$, and yet $P_{DP}(C_4, 2) = 0$.

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How is DP Color Function useful?

- Guaranteed number of DP-colorings regardless of the cover being used.

How is DP Color Function useful?

- Lower bound on both $P_\ell(G, m)$ and $P(G, m)$.

Theorem (Bernshteyn, Brazelton, Cao, Kang (2021+))

For any triangle-free graph G with n vertices, t edges, $\Delta(G)$ large enough, and $m > (1 + o(1))\Delta(G)/\log \Delta(G)$, $P_{DP}(G, m) \geq (1 - \delta)^n (1 - \frac{1}{m})^t m^n$.

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Close to being sharp modulo the $(1-\delta)^n$ error term.

Proposition (K., Mudrock (2021))

For any graph G, $P_{DP}(G, m) \leq (1-\frac{1}{m})^{|E(G)|} m^{|V(G)|}$, for all m.

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Proposition (K., Mudrock (2021))

For any graph G, $P_{DP}(G,m) \leq (1-\frac{1}{m})^{|E(G)|} m^{|V(G)|}$, for all m.

This upper bound is the same as the lower bound on $P(G,m)$ when G is bipartite, as claimed by the well-known Sidorenko's conjecture on counting homomorphisms from bipartite graphs.

Corollary (K., Mudrock (2021))

For any connected graph G, $P_{DP}(G,m) = (1-\frac{1}{m})^{|E(G)|} m^{|V(G)|}$ for all m if and only if G is a tree.

How is DP Color Function useful?

- It can capture the behavior of extremal values:

How is DP Color Function useful?

Theorem (K., Mudrock, Sharma, Stratton (2021+))

For any graphs G and H,

- $\chi_{DP}(G \square H) \leq \min\{\chi_{DP}(G) + col(H), \chi_{DP}(H) + col(G)\} - 1.$
- $\chi_{DP}(G \square K_{k,t}) = \chi_{DP}(G) + k$ *when* $t \geq (P_{DP}(G, \chi_{DP}(G) + k - 1))^k.$
- $\chi_{DP}(C_{2m+1} \square K_{k,t}) = k + 3$ *when* $t \geq \left(\frac{2k \ln(k+2)}{(k+1)!}\right)(P_{DP}(C_{2m+1}, k+2))^k.$
- $\chi_{DP}(C_{2m+1} \square K_{1,t}) = 4$ *iff* $t \geq \frac{P_{DP}(C_{2m+1},3)}{3} = \frac{2^{2m+1}-2}{3}.$
- $\chi_{DP}(C_{2m+2} \square K_{k,t}) = k + 3$ *when* $t \geq \left(\frac{2 \ln(k+2)}{\lfloor (k+2)/2 \rfloor (k-1)!}\right)(P_{DP}(C_{2m+2}, k+2))^k.$
- $\chi_{DP}(C_{2m+2} \square K_{1,t}) = 4$ *iff* $t \geq P_{DP}(C_{2m+2}, 3) = 2^{2m+2} - 1.$

A Natural Question

We know:

Theorem (Wang, Qian, Yan (2017); improving Thomassen (2009), Donner (1992))

For any connected graph G with t edges,
$P_\ell(G, m) = P(G, m)$ *for* $m > \frac{t-1}{\ln(1+\sqrt{2})} \approx 1.135(t-1)$.

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- For every graph G, does $P_{DP}(G, m) = P(G, m)$ for sufficiently large m?

DP Color Function is different

Theorem (K., Mudrock (2021))

If G is a graph with girth that is even, then there is an N such that $P_{DP}(G, m) < P(G, m)$ whenever $m \geq N$.

Furthermore, for any integer $g \geq 3$ there exists a graph G with girth g and an N such that $P_{DP}(G, m) < P(G, m)$ whenever $m \geq N$.

Theorem (Dong, Yang (2022))

If G contains an edge e such that the length of a shortest cycle containing e in G is even, then there exists $N \in \mathbb{N}$ such that $P_{DP}(M, m) < P(M, m)$ whenever $m \geq N$.

Second Natural Question

- For which graphs G does $P_{DP}(G, m) = P(G, m)$ for all m?
- For which graphs G does there exist N such that $P_{DP}(G, m) = P(G, m)$ for all $m \geq N$?

Theorem (K., Mudrock (2021))

If G is chordal, then $P_{DP}(G, m) = P(G, m)$ for every m.

- a straightforward application of perfect elimination ordering.

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- a straightforward application of perfect elimination ordering.

Unicyclic Graphs

- A unicyclic graph is a connected graph containing exactly one cycle.
- If G is a unicyclic graph on n vertices that contains a cycle on t vertices, then $P(G, m) = (m-1)^n + (-1)^t (m-1)^{n-t+1}$

Theorem (K., Mudrock (2021))

Suppose G is a unicyclic graph on n vertices.

(1) If G contains a cycle on $2k+1$ vertices, then $P_{DP}(G, m) = P(G, m)$ for all m.

(2) If G contains a cycle on $2k+2$ vertices, then $P_{DP}(G, m) = (m-1)^n - (m-1)^{n-2k-2}$ for all $m \geq 2$.

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Theta Graphs

- A Generalized Theta graph $\Theta(l_1, \ldots, l_k)$ consists of a pair of end vertices joined by k internally disjoint paths of lengths $l_1, \ldots, l_k$. $\Theta(l_1, l_2, l_3)$ is simply called a Theta graph.
- $P(\Theta(l_1, \ldots, l_k), m) = \frac{\prod_{i=1}^{k}((m-1)^{l_i+1}+(-1)^{l_i+1}(m-1))}{(m(m-1))^{k-1}} + \frac{\prod_{i=1}^{k}((m-1)^{l_i}+(-1)^{l_i}(m-1))}{m^{k-1}}$.
- Widely studied for many graph theoretic problems and are the main subject of two classical papers on the chromatic polynomial by Sokal, which include the celebrated result that the zeros of the chromatic polynomials of the Generalized Theta graphs are dense in the whole complex plane with the possible exception of the unit disc around the origin (by including the join of Generalized Theta graphs with K_2 this extends to all of the complex plane).

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Theta Graphs

Extending results of K. and Mudrock (2021),

Theorem (Halberg, K., Liu, Mudrock, Shin, Thomason (2021+))

Let $G = \Theta(l_1, l_2, l_3)$ and $2 \leq l_1 \leq l_2 \leq l_3$.

(1) If the parity of l_1 is different from both l_2 and l_3, then $P_{DP}(G, m) = P(G, m)$ for all m.

(2) If the parity of l_1 is the same as l_2 and different from l_3, then for $m \geq 2$: $P_{DP}(G, m) =$

$$\frac{1}{m}\left[(m-1)^{l_1+l_2+l_3} + (m-1)^{l_1} - (m-1)^{l_2+1} - (m-1)^{l_3} + (-1)^{l_3+1}(m-2)\right].$$

(3) If the parity of l_1 is the same as l_3 and different from l_2, then for $m \geq 2$: $P_{DP}(G, m) =$

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(4) If l_1, l_2 and l_3 all have the same parity, then for $m \geq 3$: $P_{DP}(G, m) =$

$$\frac{1}{m}\left[(m-1)^{l_1+l_2+l_3} - (m-1)^{l_1} - (m-1)^{l_2} - (m-1)^{l_3} + 2(-1)^{l_1+l_2+l_3}\right].$$

Two Fundamental Questions

- For which graphs G does there exist N such that $P_{DP}(G, m) = P(G, m)$ for all $m \geq N$?

- Given a graph G does there always exist an $N \in \mathbb{N}$ and a polynomial $p(m)$ such that $P_{DP}(G, m) = p(m)$ whenever $m \geq N$?

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Generalized Theta Graphs

Theorem (Halberg, K., Liu, Mudrock, Shin, Thomason (2021+))
Let $G = \Theta(l_1, \ldots, l_k)$ where $k \geq 2$, $l_1 \leq \cdots \leq l_k$, and $l_2 \geq 2$.

(i) If there is a $j \in \{2, \ldots, k\}$ such that l_1 and l_j have the same parity, then there is an $N \in \mathbb{N}$ such that $P_{DP}(G, m) < P(G, m)$ for all $m \geq N$.

(ii) If l_1 and l_j have different parity for each $j \in \{2, \ldots, k\}$, then there is an $N \in \mathbb{N}$ such that $P_{DP}(G, m) = P(G, m)$ for all $m \geq N$.

- Statement (i) does not answer the question of whether $P_{DP}(G, m)$ equals a polynomial for sufficiently large m. To answer that question, we study the DP color function of a class of graphs that contains all Generalized Theta graphs.

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Graphs with a Feedback Vertex Set of Order One

- A feedback vertex set of a graph is a subset of vertices whose removal makes the resulting induced subgraph acyclic. Clearly, a Generalized Theta graph has a feedback vertex set of size one.

Theorem (Halberg, K., Liu, Mudrock, Shin, Thomason (2021+))

Let G be a graph with a feedback vertex set of order one. Then there exists N and a polynomial $p(m)$ such that $P_{DP}(G, m) = p(m)$ for all $m \geq N$.

- We consider a decomposition G into a star G_1 and a spanning forest G_0, and then carefully count the number of $\mathcal{H}_0$-colorings of G_0 that are not $\mathcal{H}$-colorings of G, where $\mathcal{H}_0$ is the m-fold cover of G_0 induced by a given m-fold cover $\mathcal{H}$ of G.

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What is the polynomial?

Theorem (Halberg, K., Liu, Mudrock, Shin, Thomason (2021+))

Let G be a graph with a feedback vertex set of order one. Then there exists N and a polynomial $p(m)$ s.t. $P_{DP}(G, m) = p(m)$ for all $m \geq N$.

- There is no explicit formula for the polynomial $p(m)$ but we know its three highest degree terms are the same as $P(G, m)$.
- By extension of results of and answering a question of K. and Mudrock (2021),

Theorem (Mudrock, Thomason (2021))

For any graph G, $P(G, m) - P_{DP}(G, m) = O(m^{n-3})$ as $m \to \infty$.

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When does List Color Ftn equal Chromatic Poly?

- Given any graph G, the list color function number of G, denoted $\nu_\ell(G)$, is the smallest $m \geq \chi(G)$ such that $P_\ell(G, m) = P(G, m)$.

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- By Donner's 1992 result, we know that both $\nu_\ell(G)$ and $\tau_\ell(G)$ are finite for any graph G. Furthermore, $\chi(G) \leq \chi_\ell(G) \leq \nu_\ell(G) \leq \tau_\ell(G)$.

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Theorem (Thomassen (2009))

$\tau_\ell(G) \leq |V(G)|^{10} + 1.$

Theorem (Wang, Qian, Yan (2017))

$\tau_\ell(G) \leq (|E(G)| - 1)/\ln(1 + \sqrt{2}) + 1.$

When does List Color Ftn equal Chromatic Poly?

- Two well-known open questions on the list color function can be stated as:
  - Kirov and Naimi 2016: For every graph G, is it the case that $\nu_\ell(G) = \tau_\ell(G)$?
  - Thomassen 2009: Is there a universal constant μ such that for any graph G, $\tau_\ell(G) - \chi_\ell(G) \leq \mu$?

When does List Color Ftn equal Chromatic Poly?

- Kirov and Naimi 2016: For every graph G, is it the case that $\nu_\ell(G) = \tau_\ell(G)$?

  A question of stickiness: Do the list color function and the corresponding chromatic polynomial of a graph stay the same after the first point at which they are both nonzero and equal?

  Still Open. But corresponding DP color function question has been answered negatively.

When does List Color Ftn equal Chromatic Poly?

- Thomassen 2009: Is there a universal constant μ such that for any graph G, $\tau_\ell(G) \leq \chi_\ell(G) + \mu$?

  The answer is no in a very strong sense.

Theorem (K., Kumar, Mudrock, Rewers, Shin, To (2022+))

There is a constant C such that for each $l \geq 16$,
$\tau_\ell(K_{2,l}) - \chi_\ell(K_{2,l}) = \tau_\ell(K_{2,l}) - 3 \geq C\sqrt{l}$.

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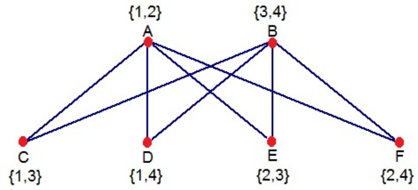
- To prove a lower bound on $\tau(G)$, we need an upper bound on $P_\ell(G, m)$ that is smaller than $P(G, m)$ for some m.



We generalize this folklore 'bad' list assignment and count the number of such list colorings to get an upper bound on $P_\ell(K_{n,n^n t}, m)$.

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- For $G = K_{2,l}$, with bipartition $\{x_1, x_2\}, \{y_1, \ldots, y_l\}$.
  We consider the m-assignment L for G:
  $L(x_1) = [m]$ and $L(x_2) = [m-2] \cup \{m+1, m+2\}$.
  Let $z_1 = |\{j \in [l] : L(y_j) = [m-2] \cup \{m-1, m+1\}\}|$,
  $z_2 = |\{j \in [l] : L(y_j) = [m-2] \cup \{m-1, m+2\}\}|$,
  $z_3 = |\{j \in [l] : L(y_j) = [m-2] \cup \{m, m+1\}\}|$, and
  $z_4 = |\{j \in [l] : L(y_j) = [m-2] \cup \{m, m+2\}\}|$.
  We say L is balanced if $\sum_{i=1}^{4} z_i = l$ and $|z_j - z_i| \leq 1$ for all $i, j \in [4]$.
  We use these balanced list assignments to inductively build the generalized 'bad' list assignments for $K_{2,l}$.

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- Threshold Extremal functions:
  $\delta_{max}(t) = \max\{\tau_\ell(G) - \chi_\ell(G) : |E(G)| \leq t\}$
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Theorem (Wang et al. (2017) and K. et al. (2022+))

$C_1\sqrt{t} \leq \delta_{max}(t) \leq C_2 t$ *for large enough t*
$C_3\sqrt{t} \leq \tau_{max}(t) \leq C_2 t$ *for large enough t*

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$C_3\sqrt{t} \leq \tau_{max}(t) \leq C_2 t$ for large enough t

- What is the asymptotic behavior of $\delta_{max}(t)$?
  What is the asymptotic behavior of $\tau_{max}(t)$? In particular, is $\tau_{max}(t) = \omega(\sqrt{t})$?

  Since $\chi_\ell(G) = O(\sqrt{|E(G)|})$ as $|E(G)| \to \infty$,
  if $\tau_{max}(t) = \omega(\sqrt{t})$ as $t \to \infty$, then $\delta_{max}(t) \sim \tau_{max}(t)$ as $t \to \infty$.

When does DP Color Ftn equal Chromatic Poly?

- Given any graph G, the DP color function number of G, denoted $\nu_{DP}(G)$, is the smallest $m \geq \chi(G)$ such that $P_{DP}(G, m) = P(G, m)$.
  If $P(G, m) - P_{DP}(G, m) > 0$ for all m, we let $\nu_{DP}(G) = \infty$.

- The DP color function threshold of G, denoted $\tau_{DP}(G)$, is the smallest $k \geq \chi(G)$ such that $P_{DP}(G, m) = P(G, m)$ whenever $m \geq k$.
  If $P(G, m) - P_{DP}(G, m) > 0$ for infinitely many m, we let $\tau_{DP}(G) = \infty$.

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  If $P(G, m) - P_{DP}(G, m) > 0$ for infinitely many m, we let $\tau_{DP}(G) = \infty$.

- $\chi(G) \leq \chi_\ell(G) \leq \chi_{DP}(G) \leq \nu_{DP}(G) \leq \tau_{DP}(G)$.

When does DP Color Ftn equal Chromatic Poly?

- We can now ask two natural questions about the DP color function:
  - For every graph G, is it the case that $\nu_{DP}(G) = \tau_{DP}(G)$?
  - When is $\tau_{DP}(G)$ finite?
    Find any universal bounds on τ_{DP}.

When does DP Color Ftn equal Chromatic Poly?

- Kirov and Naimi 2016: For every graph G, is it the case that $\nu_\ell(G) = \tau_\ell(G)$?

  Still Open. But corresponding DP color function question has been answered negatively.

- For every graph G, is it the case that $\nu_{DP}(G) = \tau_{DP}(G)$? No!

Theorem (K., Maxfield, Mudrock, Thomason (2022+))

If G is $\Theta(2,3,3,3,2)$ or $\Theta(2,3,3,3,3,3,2,2)$, then $P_{DP}(G,3) = P(G,3)$ and there is an N such that $P_{DP}(G,m) < P(G,m)$ for all $m \geq N$.

When does DP Color Ftn equal Chromatic Poly?

- Find any universal bounds on τ_{DP} (when finite).
  This problem is wide open with very little progress.

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- Dong and Yang (2022) imply that $\tau_{DP}(K_p \vee G) < \infty$. Can we say more?

When does DP Color Ftn equal Chromatic Poly?

Theorem (Becker, Hewitt, K., Maxfield, Mudrock, Spivey, Thomason, Wagstrom (2021+))

- *For any graph G,* $\tau_{DP}(K_{p+1} \vee G) \leq \tau_{DP}(K_p \vee G) + 1$.
- *For any p and* $n \geq 3$, $\tau_{DP}(K_p \vee C_n) = 3 + p$.
- *Let* $M = K_1 \vee G$, *where G is the disjoint union of cycles* C_{k_i} *for* $i \in [n]$, *with each* $k_i \geq 3$.

$$\tau_{DP}(M) = \begin{cases} 5 & \text{if } \exists \text{ distinct } i, j \in [n] \text{ such that } k_i = k_j = 4 \\ 4 & \text{otherwise.} \end{cases}$$

Thank You!

Questions?

- For which graphs G does $\exists N$ such that $P_{DP}(G, m) = P(G, m)$ for all $m \geq N$? That is, when is $\tau_{DP}(G)$ finite?
- Given a graph G does there always exist an $N \in \mathbb{N}$ and a polynomial $p(m)$ such that $P_{DP}(G, m) = p(m)$ whenever $m \geq N$?
- Given a graph G and $p \in \mathbb{N}$, what is the value of $\tau_{DP}(K_p \vee G)$?
- What is the asymptotic behavior of $\delta_{max}(t)$ and $\tau_{max}(t)$? In particular, is $\tau_{max}(t) = \omega(\sqrt{t})$?
- For fixed n what is the asymptotic behavior of $\tau_\ell(K_{n,l})$ as $l \to \infty$?
- Kirov and Naimi 2016: For every graph G, is it the case that $\nu_\ell(G) = \tau_\ell(G)$? That is, if $P_\ell(G, m) = P(G, m)$ for some $m \geq \chi(G)$, does it follow that $P_\ell(G, m+1) = P(G, m+1)$?

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Tools for DP Color Function - I

Classic tools like:

Lemma (from Whitney's Broken Circuit Theorem (1932))

G be a connected graph on n vertices and s edges with girth g. Suppose $P(G, m) = \sum_{i=0}^{n} (-1)^i a_i m^{n-i}$. Then, for $i = 0, 1, \ldots, g-2$
$a_i = \binom{s}{i}$ and $a_{g-1} = \binom{s}{g-1} - t$,
where t is the number of cycles of length g contained in G.

- Inclusion-Exclusion type arguments.
- AM-GM inequality, and its generalization the Rearrangement Inequality.
- Probabilistic arguments/ Random constructions.

Tools for DP Color Function - II

Proposition (K., Mudrock (2021))

$P_{DP}(G, m) \leq \frac{m^n(m-1)^{|E(G)|}}{m^{|E(G)|}}$ for all m.

- Expected number of independent transversals in a random m-fold cover.

Tools for DP Color Function - II

Proposition (K., Mudrock (2021))

$P_{DP}(G, m) \leq \frac{m^n(m-1)^{|E(G)|}}{m^{|E(G)|}}$ for all m.

- This upper bound is the same as the lower bound on $P(G, m)$ when G is bipartite, as claimed by the well-known Sidorenko's conjecture on counting homomorphisms from bipartite graphs.

Corollary (K., Mudrock (2021))

For any connected graph G, $P_{DP}(G, m) = \frac{m^{|V(G)|}(m-1)^{|E(G)|}}{m^{|E(G)|}}$ for all m if and only if G is a tree.

Tools for DP Color Function - II

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Lemma (K., Mudrock (2021))

Let G be a graph with $e \in E(G)$.
If $m \geq 2$ and $P(G - \{e\}, m) < \frac{m}{m-1} P(G, m)$,
then $P_{DP}(G, m) < P(G, m)$.

Tools for DP Color Function - III

- Let $\mathcal{H} = (L, H)$ be an m-fold cover of G. We say that $\mathcal{H}$ has a canonical labeling if it is possible to name the vertices of H so that $L(u) = \{(u, j) : j \in [m]\}$ and $(u, j)(v, j) \in E(H)$ for each $j \in [m]$ whenever $uv \in E(G)$.
- When $\mathcal{H}$ has a canonical labeling, G has an $\mathcal{H}$-coloring if and only if G has a proper m-coloring.
- Trees have a canonical labeling.
- Using canonical labeling, we can develop tools to handle graphs that are close to being a forest.

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Tools for DP Color Function - III

- A sharp bound when removing an edge gives us a canonical labeling.

Lemma (K., Mudrock (2021))

Let $\mathcal{H} = (L, H)$ be an m-fold cover of G with $m \geq 2$. Suppose $e = uv \in E(G)$. Let $H' = H - E_H(L(u), L(v))$ so that $\mathcal{H}' = (L, H')$ is an m-fold cover of $G - \{e\}$. If $\mathcal{H}'$ has a canonical labeling, then

$$P_{DP}(G, \mathcal{H}) \geq P(G - e, m) - \max\left\{P(G - e, m) - P(G, m), \frac{P(G,m)}{m-1}\right\}$$

Moreover, there exists an m-fold cover of G, $\mathcal{H}^* = (L, H^*)$, s.t.

$$P_{DP}(G, \mathcal{H}^*) = P(G - e, m) - \max\left\{P(G - e, m) - P(G, m), \frac{P(G,m)}{m-1}\right\}.$$

- Next, a sharp bound when removing an induced P_3.

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Tools for DP Color Function - III

Lemma (K., Mudrock (2021))

Let $\mathcal{H} = (L, H)$ be an m-fold cover of G with $m \geq 3$. Let e_1, e_2 be the edges of an induced path P of length two. Let $G_0 = G - \{e_1, e_2\}$, $G_1 = G - e_1$, $G_2 = G - e_2$, and G^* be the graph obtained by making P into K_3. Suppose $\mathcal{H}'$, the m-fold cover of G_0 induced by $\mathcal{H}$, has a canonical labeling. Let

$$A_1 = P(G_0, m) - P(G, m), A_2 = P(G_0, m) - P(G_2, m) + \frac{1}{m-1}P(G, m),$$

$$A_3 = P(G_0, m) - P(G_1, m) + \frac{1}{m-1}P(G, m),$$

$$A_4 = \frac{1}{m-1}\left(P(G_1, m) + P(G_2, m) + P(G^*, m) - P(G, m)\right), \text{ and}$$

$$A_5 = \frac{1}{m-1}\left(P(G_1, m) + P(G_2, m) - \frac{1}{m-2}P(G^*, m)\right).$$

Then, $P_{DP}(G, \mathcal{H}) \geq P(G_0, m) - \max\{A_1, A_2, A_3, A_4, A_5\}$.
Moreover, there exists an m-fold cover of G that achieves the equality.

Tools for DP Color Function - IV

- Clique-gluing and the closely related clique-sum are fundamental graph operations which have been used to give a structural characterization of many families of graphs.
- A simple example is that chordal graphs are precisely the graphs that can be formed by clique-gluings of cliques. While the most famous example would be Robertson and Seymour's seminal Graph Minor Structure Theorem characterizing minor-free families of graphs.

Tools for DP Color Function - IV

- We build a toolbox for studying K_p-gluings of graphs: Choose a copy of K_p contained in each G_i and form a new graph G ($\in \bigoplus_{i=1}^{n}(G_i, p)$), called a K_p-gluing of $G_1, \ldots, G_n$, from the union of $G_1, \ldots, G_n$ by arbitrarily identifying the chosen copies of K_p.

Tools for DP Color Function - IV

- Given vertex disjoint graphs $G_1, \ldots, G_n$, we define amalgamated cover, a natural analogue of "gluing" m-fold covers of each G_i together so that we get an m-fold cover for $G \in \bigoplus_{i=1}^{n}(G_i, p)$.

- We define separated covers, a natural analogue of "splitting" an m-fold cover of $G \in \bigoplus_{i=1}^{n}(G_i, p)$ into separate m-fold covers for each G_i.

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- We apply these ideas together with other tools to build a theory of DP Color Function of Clique-gluings of graphs and how the DP Color Function of such graphs compares with the corresponding chromatic polynomial.
  But that’s another talk.