# Breaking Symmetries in Graphs

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# Graphs and Colorings

Graphs model binary relationships.

In a graph $G = (V(G), E(G))$, the objects under study are represented by vertices included in V(G).

If two objects are "related" then their corresponding vertices, say $u$ and $v$ in $V(G)$, are joined by an edge that is represented as $uv$ in $E(G)$.

# Graphs and Colorings

For a university semester, we could define a ‘conflict’ graph on courses, where each course is a vertex, and edges occur between pairs of vertices corresponding to courses with overlapping time.

Then we could be interested in assigning rooms (colors) to the courses (vertices), such that a particular room is not assigned to two courses with overlapping times (vertices joined by an edge get different colors).

The least number of rooms (colors) that would get the job done is called the *chromatic number of the graph*, denoted $\chi(G)$.

# Graphs and Colorings

Let $G = (V(G), E(G))$ be a graph.

A proper $k$-coloring of $G$ is a labeling of $V(G)$ with $k$ labels such that adjacent vertices get distinct labels.

Chromatic Number, $\chi(G)$ , is the least $k$ such that $G$ has a proper $k$-coloring.

# Graphs and Colorings

Some examples of Graphs:

$K_n$ : Complete graph on $n$ vertices. Each of the $\binom{n}{2}$ pairs of vertices is joined by an edge. $\chi(K_n) = n$

$P_n$ : Path on $n$ vertices. $\chi(P_n) = 2$

$C_n$ : Cycle on $n$ vertices. $\chi(C_{2k}) = 2, \chi(C_{2k+1}) = 3$

$K_{n_1,n_2,\ldots,n_t}$ : Complete $t$-partite graph on $n_1 + n_2 + \ldots + n_t$ vertices. $\chi(K_{n_1,n_2,\ldots,n_t}) = t$

# Symmetries in a Graph

In $K_n$, it is impossible to distinguish between any two vertices, $u$ and $v$, because they are structurally identical.

More formally, there is an bijection on $V(K_n)$ that interchanges $u$ and $v$ without affecting the structure of $K_n$.

Such bijections are called automorphisms of $G$.

# Symmetries in a Graph

An automorphism of $G$ is $\rho : V(G) \to V(G)$,
a bijection that preserves edges and non-edges of $G$,
i.e., $uv \in E(G)$ iff $\rho(u)\rho(v) \in E(G)$.

$Aut(G)$ is the set (group) of all automorphisms of $G$.

$Aut(K_n) = S_n$, the Symmetric group formed by all the permutations on $n$ objects.

$Aut(C_n) = D_{2n}$, the Dihedral group formed by rotations and flips.

# Distinguishing Vertices

If we want to distinguish vertices in $K_n$, we have to give each of them a distinct name (label). So, $K_n$ needs $n$ labels.

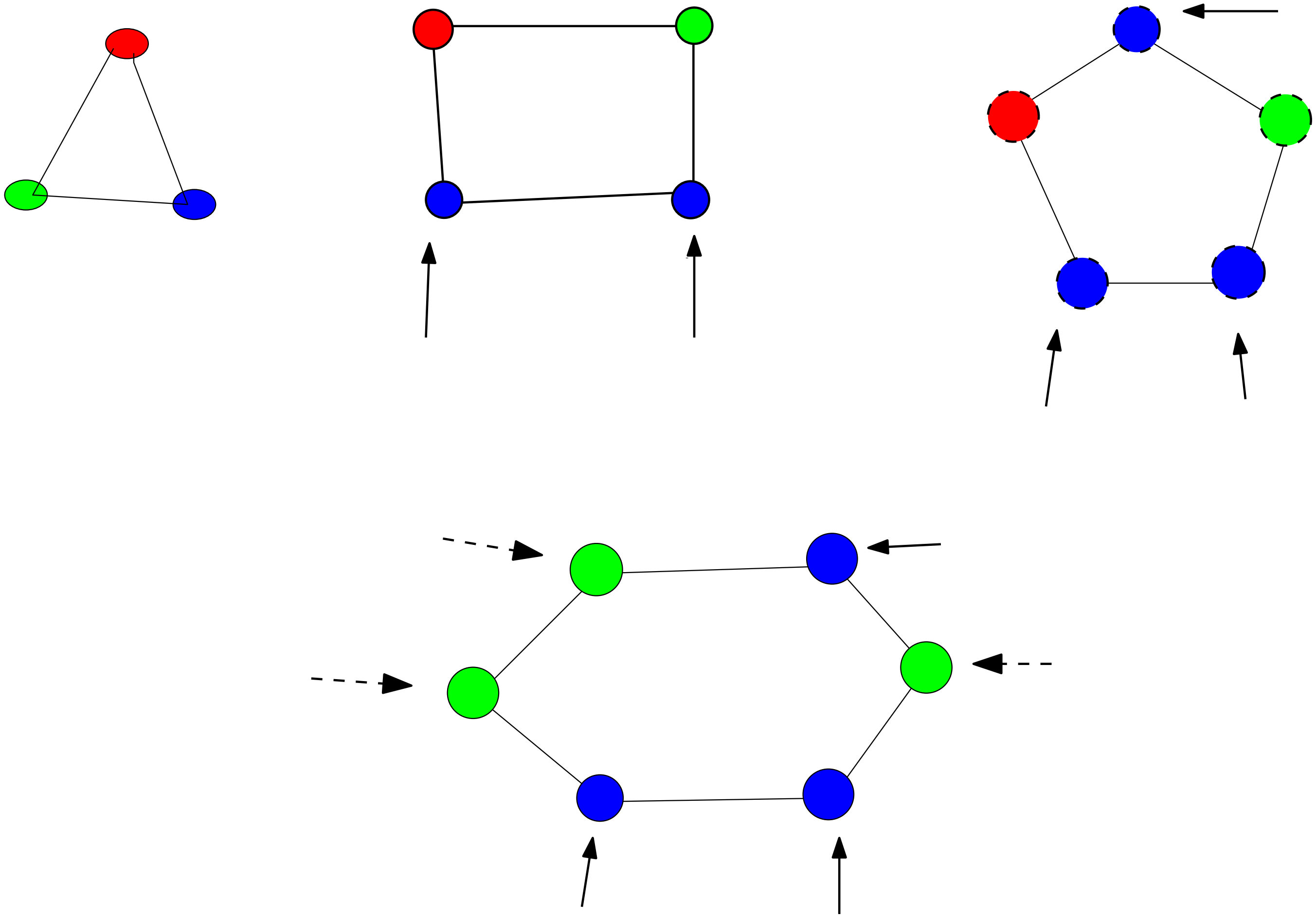
But, many times we can get away with using far less number of labels.

# Distinguishing Vertices

"Suppose you have a key ring with $n$ identical looking keys. You wish to label the handles of the keys in order to tell them apart. How many labels will you need?"

We want to figure out how many labels we need to distinguish vertices in $C_n$.

# Distinguishing Vertices



# Distinguishing Vertices

$C_3$, $C_4$, and $C_5$ need three labels.

When $n \geq 6$, $C_n$ needs only two labels !!

So we want to be able to ‘decode’ the ‘real identity’ of a vertex using only these (few) labels and the structure of the graph.

# Distinguishing Number

A distinguishing $k$-labeling of $G$ is a labeling of $V(G)$ with $k$ labels such that the only color-preserving automorphism of $G$ is the identity.

Distinguishing Number, $D(G)$ , is the least $k$ such that $G$ has a distinguishing $k$-labeling.

Introduced by Albertson and Collins in 1996.

Since then, a whole class of research literature combining graphs and group actions has arisen around this topic.

# Distinguishing Number

Some examples:

$D(G) = 1$ if and only if $Aut(G) = \{$identity$\}$

$D(K_n) = D(K_{1,n}) = n$. Both have $Aut(G) = S_n$.

It is possible to construct a graph $G$ with $Aut(G) = S_n$ and $D(G) = \sqrt{n}$.

$D(K_{n,n}) = n + 1$.

$D(C_n)$ equals $3$ if $3 \leq n \leq 5$, and equals $2$ if $n \geq 6$.

$D(P_n) = 2$.

# Distinguishing Number

In general the value of the Distinguishing number is strongly influenced by the relevant Automorphism group, rather than the particular graph.

For a group $\Gamma$, $D(\Gamma) = \{D(G) \;:\; Aut(G) \cong \Gamma,\; G \text{ graph}\}$

**Theorem** [Albertson + Collins, 1996]
$D(D_{2n}) = \{2\}$ unless $n = 3, 4, 5, 6, 10$, in which case,
$D(D_{2n}) = \{2, 3\}$.

**Theorem** [Tymoczko, 2004]
$D(S_n) \subseteq \{2, 3, \ldots, n\}$.

**Conjecture** [Klavzar+ Wong + Zhu, 2005]
$D(S_n) = \{\lceil n^{1/k} \rceil : k \in \mathbb{Z}^+\}$.

# Distinguishing through proper colorings

Distinguishing numbers tend to be fixed numbers that depend more on the automorphism structure than the graph structure.

We want a proper coloring (not just an unrestricted labeling) that breaks all the symmetries of a graph, identifying each of its vertices uniquely.

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We want a proper coloring (not just an unrestricted labeling) that breaks all the symmetries of a graph, identifying each of its vertices uniquely.

Recall the conflict graph for courses. "Find a coloring of the conflict graph that uniquely identifies each course as well as specifying the room each would use."

We not only 'decode' the 'real identity' of a vertex using only these (few) labels and the structure of the graph, but get a useful partition of the vertices into 'conflict-free' subsets.

# Distinguishing Chromatic Number

A distinguishing proper $k$-coloring of $G$ is a proper $k$-coloring of $G$ such that the only color-preserving automorphism of $G$ is the identity.

Distinguishing Chromatic Number, $\chi_D(G)$ , is the least $k$ such that $G$ has a distinguishing proper $k$-coloring.

Introduced by Collins and Trenk in 2005.

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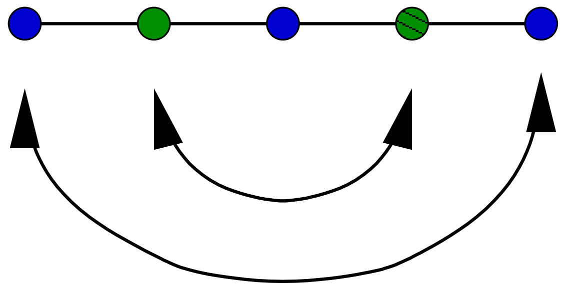
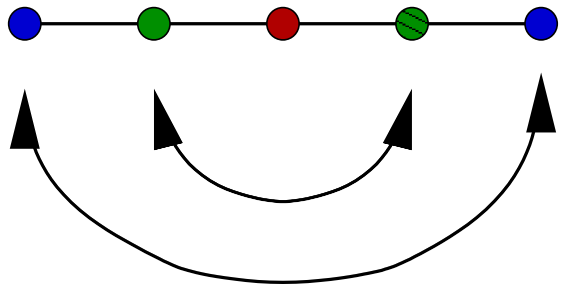
Note that the chromatic number, $\chi(G)$, is an immediate lower bound for $\chi_D(G)$.

# Examples



Not Distinguishing

# Examples



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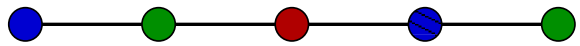

Distinguishing

$\chi_D(P_{2n+1}) = 3$ and $\chi_D(P_{2n}) = 2$

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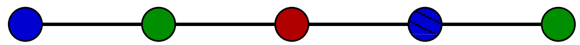

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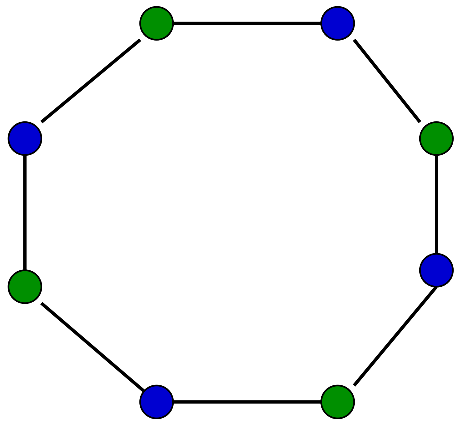

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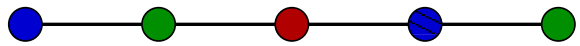

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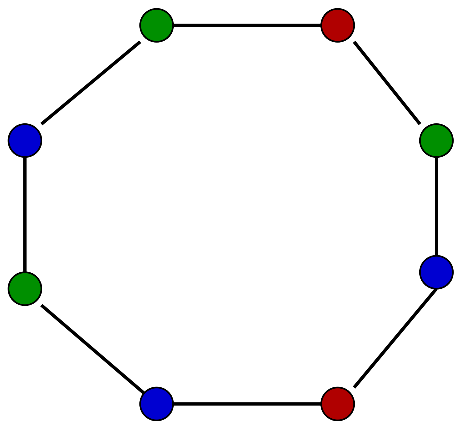

Distinguishing

$\chi_D(C_n) = 3$ except $\chi_D(C_4) = \chi_D(C_6) = 4$

# Motivating Question

When are $D(G)$ and $\chi_D(G)$ small?

Just one more than the minimum allowed?

# Motivating Question

When are $D(G)$ and $\chi_D(G)$ small?

Just one more than the minimum allowed?

Find a large general class of graphs for which

$D(G) \leq 1 + 1$

$\chi_D(G) \leq \chi(G) + 1$

Our answer will be in terms of Cartesian Product of Graphs.

# Cartesian Product of Graphs

Let $G = (V(G), E(G))$ and $H = (V(H), E(H))$ be two graphs.

$G \square H$ denotes the Cartesian product of $G$ and $H$.

$V(G \square H) = \{(u, v) | u \in V(G), v \in V(H)\}$.

vertex $(u, v)$ is adjacent to vertex $(w, z)$ if either $u = w$ and $vz \in E(H)$ or $v = z$ and $uw \in E(G)$.

# Cartesian Product of Graphs

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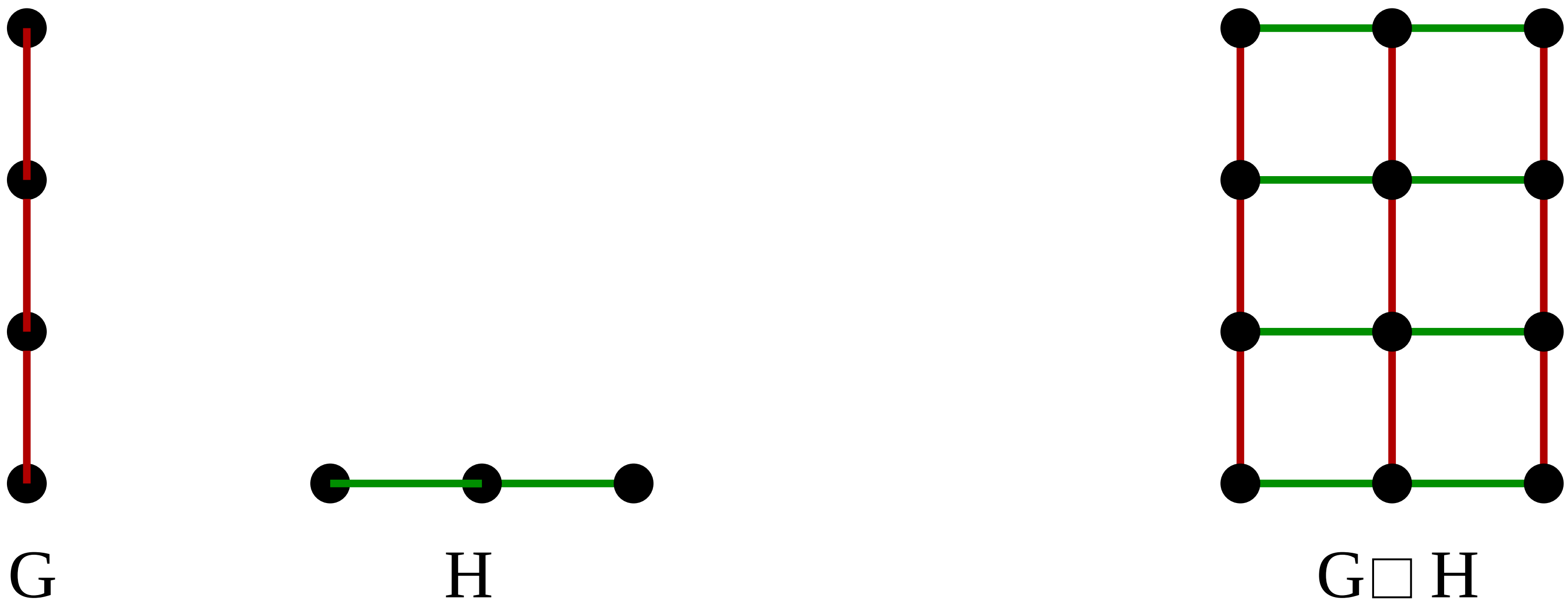
Extend this definition to $G_1 \square G_2 \square \ldots \square G_d$.

Denote $G^d = \square_{i=1}^{d} G$.

A very special but important case, $K_2^d$ denoted by $Q_d$, is the $d$-dimensional hypercube.

# Cartesian Product of Graphs



# Cartesian Product of Graphs

A graph $G$ is said to be a prime graph if whenever $G = G_1 \Box G_2$, then either $G_1$ or $G_2$ is a singleton vertex.

Prime Decomposition Theorem [Sabidussi(1960) and Vizing(1963)] Let $G$ be a connected graph, then $G \cong G_1^{p_1} \Box G_2^{p_2} \Box \ldots \Box G_d^{p_d}$, where $G_i$ and $G_j$ are distinct prime graphs for $i \neq j$, and $p_i$ are constants.

Theorem [Imrich(1969) and Miller(1970)]
All automorphisms of a cartesian product of graphs are induced by the automorphisms of the factors and by transpositions of isomorphic factors.

# Cartesian Product of Graphs

Fact: Let $G = \Box_{i=1}^{d} G_i$. Then $\chi(G) = \max_{i=1,\ldots,d} \{\chi(G_i)\}$

Let $f_i$ be an optimal proper coloring of $G_i$, $i = 1, \ldots, d$.

Canonical Coloring $f^d : V(G) \to \{0, 1, \ldots, t-1\}$ as

$$f^d(v_1, v_2, \ldots, v_d) = \sum_{i=1}^{d} f_i(v_i) \bmod t \ , \qquad t = \max_i \{\chi(G_i)\}$$

There is an edge between $v$ and $v'$ in $G$ if and only if they differ in only one coordinate, say $v_i$ and $v_i'$. So, $v_i v_i'$ is an edge in $G_i$. Then $f^d(v) - f^d(v') = f_i(v_i) - f_i(v_i')$ which is nonzero modulo $t$ because $f_i$ is a proper coloring. Thus, $f^d$ gives a proper coloring of $\Box_{i=1}^{d} G_i$.

# Small Distinguishing Number

**Theorem** [Bogstad + Cowen, 2004]
$D(Q_d) = 2$, for $d \geq 4$, and $D(Q_2) = D(Q_3) = 3$
where $Q_d$ is the $d$-dimensional hypercube.

**Theorem** [Albertson, 2005]
$D(G^d) = 2$, for $d \geq 4$, if $G$ is a prime graph.

**Theorem** [Klavzar + Zhu , 2006]
$D(G^d) = 2$, for $d \geq 3$.

Follows from $D(K_n^d) = 2$, proved using a probabilistic argument (when automorphisms of $G$ have few fixed points then $D(G)$ is large).

# Large Distinguishing Chromatic Number

Recall, $\chi(G) \leq \chi_D(G)$

In general, $\chi_D(G)$ might need many more colors than $\chi(G)$.

Theorem [Collins + Trenk, 2006]
$\chi_D(G) = n(G) \Leftrightarrow G$ is a complete multipartite graph.

$\chi_D(K_{n_1,n_2,\ldots,n_t}) = \sum_{i=1}^{t} n_i$ while $\chi(K_{n_1,n_2,\ldots,n_t}) = t$, arbitrarily far apart.

Making our task more difficult.

# Hamming Graphs and Hypercubes

Theorem [Choi + Hartke + Kaul, 2006+]

Given $t_i \geq 2$, $\chi_D(\square_{i=1}^{d} K_{t_i}) \leq \max_i\{t_i\} + 1$ , for $d \geq 5$.

# Hamming Graphs and Hypercubes

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Given $t_i \geq 2$, $\chi_D(\square_{i=1}^d K_{t_i}) \leq \max_i\{t_i\} + 1$ , for $d \geq 5$.

Corollary : Given $t \geq 2$, $\chi_D(K_t^d) \leq t + 1$ , for $d \geq 5$.

Both these upper bounds are $1$ more than their respective lower bounds.

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Both these upper bounds are $1$ more than their respective lower bounds.

These results allow us to exactly determine the distinguishing chromatic number of hypercubes.

Corollary : $\chi_D(Q_d) = 3$ , for $d \geq 5$.

# Main Theorem

**Theorem** [Choi + Hartke + Kaul, 2006+]
Let $G$ be a graph. Then there exists an integer $d_G$ such that for all $d \geq d_G$ , $\chi_D(G^d) \leq \chi(G) + 1$.

By the Prime Decomposition Theorem for Graphs, $G = G_1^{p_1} \square G_2^{p_2} \square \ldots \square G_k^{p_k}$, where $G_i$ are distinct prime graphs. (This prime decomposition can be found in polynomial time)

Then, $$d_G = \max_{i=1,\ldots,k} \{\frac{\lg n(G_i)}{p_i}\} + 5$$

Note, $n(G) = (n(G_1))^{p_1} * (n(G_2))^{p_2} * \cdots * (n(G_k))^{p_k}$

At worst, $d_G = \lg n(G) + 5$ suffices.

# Main Theorem

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when, $n(G) = (n(G_1))^{p_1} * (n(G_2))^{p_2} * \cdots * (n(G_k))^{p_k}$

$d_G$ is unlikely to be a constant, as the example of Complete Multipartite Graphs indicates –

pushing $\chi_D(K_{n_1,n_2,\ldots,n_t})$ down from $n(G)$ to $t+1$ can not happen with only a fixed number of products.

# Proof Idea for Main Theorem

Fix an optimal proper coloring of $G$.

Embed $G$ in a complete multipartite graph $H$.

Form $H$ by adding all the missing edges between the color classes of $G$.

Now work with $H$.

BUT $\quad G \subseteq H \not\Rightarrow \chi_D(G) \leq \chi_D(H)$ !

# Proof Idea for Main Theorem

Fix an optimal proper coloring of $G$.

Embed $G$ in a complete multipartite graph $H$.

Form $H$ by adding all the missing edges between the color classes of $G$.

Then construct a distinguishing proper coloring of $H^d$ that is also a distinguishing proper coloring of $G^d$.

Study Distinguishing Chromatic Number of Cartesian Products of Complete Multipartite Graphs.

# Complete Multipartite Graphs

Theorem [Choi + Hartke + Kaul, 2006+]
Let $H$ be a complete multipartite graph. Then
$\chi_D(H^d) \leq \chi(H) + 1$ , for $d \geq \lg n(H) + 5$ .

This is already enough to prove Theorem 1 for prime graphs.

# Complete Multipartite Graphs

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This is already enough to prove Theorem 1 for prime graphs.

Theorem [Choi + Hartke + Kaul, 2006+]
Let $H = \square_{i=1}^{k} H_i^{p_i}$, where $H_i$ are distinct complete multipartite graphs. Then

$$\chi_D(H^d) \leq \chi(H) + 1,$$

for $d \geq \max_{i=1,\ldots,k} \{\frac{\lg n_i}{p_i}\} + 5$, where $n_i = n(H_i)$.

# Outline of the Proof for Hamming Graphs

Start with the canonical proper coloring $f^d$ of cartesian products of graphs, $f^d : V(K_t^d) \to \{0, 1, \ldots, t-1\}$ with

$$f^d(v) = \sum_{i=1}^{d} f(v_i) \mod t,$$

where $f(v_i) = i$ is an optimal proper coloring of $K_t$.

# Outline of the Proof for Hamming Graphs

Derive $f^*$ from $f^d$ by changing the color of the following vertices from $f^d(v)$ to $*$ :

Origin : $0000\ldots000$ .

Group 1 : $A = \bigcup_{i=1}^{\lfloor \frac{d}{2} \rfloor} A_i$ , where $A_i = \{e^1_{i,j} \mid 1+i \le j \le d+1-i\}$

$v^*$ : the vertex with all coordinates equal to $1$

except for the $\lceil \frac{d+1}{2} \rceil$th coordinate which equals $0$.

$e^1_{i,j}$ is the vertex with all coordinates equal to $0$ except for the $i$th and $j$th coordinates which equal $1$.

# Outline of the Proof for Hamming Graphs

Uniquely identify each vertex of $K_t^d$ by reconstructing its original vector representation by using only the colors of the vertices and the structure of the graph.

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Uniquely identify each vertex of $K_t^d$ by reconstructing its original vector representation by using only the colors of the vertices and the structure of the graph.

*Step 1.* Distinguish $v^*$ from the Origin and the Group 1 by counting their distance two neighbors in the color class $*$.

In $Q_6$, $v^*$ has no vertices with color $*$ within distance two of it.

# Outline of the Proof for Hamming Graphs

Uniquely identify each vertex of $K_t^d$ by reconstructing its original vector representation by using only the colors of the vertices and the structure of the graph.

*Step 2.* Distinguish the Origin by counting the distance two neighbors in color class $*$.

In $Q_6$, Origin is the only vertex with color $*$ that is within distance two of every other vertex of color $*$.

# Outline of the Proof for Hamming Graphs

Uniquely identify each vertex of $K_t^d$ by reconstructing its original vector representation by using only the colors of the vertices and the structure of the graph.

*Step 3.* Assign the vector representations of weight one, with $1$ as the non-zero coordinate, to the correct vertices.

In $Q_6$, vertex $100000$ has $5$ neighbors in Group 1,
$010000$ has $4$,
$001000$ has $3$ (and is at distance $6$ from $v^*$),
$000100$ has $3$ (and is at distance $4$ from $v^*$),
$000010$ has $2$, and
$000001$ has $1$ such neighbors.

# Outline of the Proof for Hamming Graphs

Uniquely identify each vertex of $K_t^d$ by reconstructing its original vector representation by using only the colors of the vertices and the structure of the graph.

*Step 4.* Assign the vector representations of weight one, with $k > 1$ as the non-zero coordinate, to the correct vertices, by recovering the original canonical colors of all the vertices.

# Outline of the Proof for Hamming Graphs

Uniquely identify each vertex of $K_t^d$ by reconstructing its original vector representation by using only the colors of the vertices and the structure of the graph.

*Step 5.* Assign the vector representations of weight greater than one to the correct vertices.

Let $x$ be a vertex with weight $\omega \geq 2$. Then $x$ is the unique neighbor of the vertices, $y_1, y_2, \ldots, y_\omega$, formed by changing exactly one non-zero coordinate of $x$ to zero that is not the Origin.

For example, in $Q_6$, $110000$ is the unique vertex with $100000$ and $010000$ as its only weight one neighbors, and so on.