Equity in Public Transit Network Design

with a view towards

Ethics in Mathematical Modeling

Hemanshu Kaul

Illinois Institute of Technology

www.math.iit.edu/∼kaul

kaul@iit.edu

Includes joint work with

Adam Rumpf

Mathematical Models

Mathematical Models are ubiquitous in the Sciences, Engineering, Social Sciences, and beyond. They serve as conceptual representation of some observed phenomenon that we seek to understand.

- Classical Mechanics: trajectory of a projectile, motion of planets, etc.
- Population Growth: Malthusian model, logistic growth, etc.
- Epidemiology: SIS, SIR, SIRS, etc.
- Microeconomic models: Consumer Choice Theory, Supply and Demand, etc.
- Financial Markets: Derivative pricing, Risk and Portfolio management, etc.

Mathematical Models

Mathematical Models are ubiquitous in the Sciences, Engineering, Social Sciences, and beyond. They serve as conceptual representation of some observed phenomenon that we seek to understand.

- Classical Mechanics: trajectory of a projectile, motion of planets, etc.
- Population Growth: Malthusian model, logistic growth, etc.
- Epidemiology: SIS, SIR, SIRS, etc.
- Microeconomic models: Consumer Choice Theory, Supply and Demand, etc.
- Financial Markets: Derivative pricing, Risk and Portfolio management, etc.

A Simple Example

Growth of yeast

n	P_n
0	9.6
1	18.3
2	29.0
3	47.2
4	71.1
5	119.1
6	174.6
7	257.3

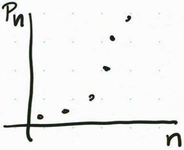
Observations from a lab

A Simple Example

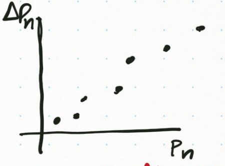
Growth of yeast

n	P_n	$\Delta P_n = P_{n+1} - P_n$
0	9.6	8.7
1	18.3	10.7
2	29.0	18.2
3	47.2	23.9
4	71.1	48.0
5	119.1	55.5
6	174.6	82.7
7	257.3	

Observations from a lab



not linear



roughly linear

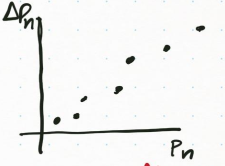
A Simple Example

Growth of yeast

n	P_n	$\Delta P_n = P_{n+1} - P_n$
0	9.6	8.7
1	18.3	10.7
2	29.0	18.2
3	47.2	23.9
4	71.1	48.0
5	119.1	55.5
6	174.6	82.7
7	257.3	

Observations from a lab

not linear



roughly linear

$$\frac{\Delta P_n}{P_n} \cong 0.6057 \quad \left(\text{avg. of } \frac{\Delta P_i}{P_i}\right)$$

i.e., $\Delta P_n = (0.6057) P_n$

i.e., $P_{n+1} - P_n = (0.6057) P_n$

i.e., $\boxed{P_{n+1} = (1.6057) P_n}$

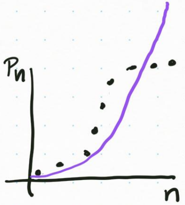
Does the model match reality?

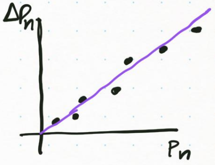
A Simple Example

Growth of yeast

n	P_n	$\Delta P_n = P_{n+1} - P_n$
0	9.6	8.7
1	18.3	10.7
2	29.0	18.2
3	47.2	23.9
4	71.1	48.0
5	119.1	55.5
6	174.6	82.7
7	257.3	

Observations from a lab



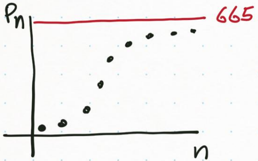


Model: $P_{n+1} = (1.6057)P_n$ Predicted value

How can we improve this model?

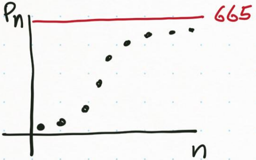
A fundamental implicit assumption

A Simple Example

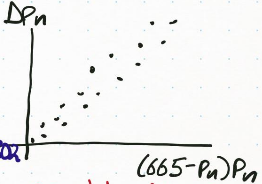


Limitation of resources (e.g. food) leads to limits on maximum population that can be supported.

A Simple Example



Limitation of resources (e.g. food) leads to limits on maximum population that can be supported.



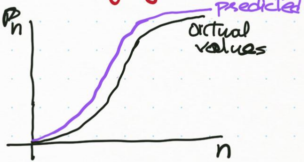
New model $\Delta P_n \propto (665-P_n)P_n$

avg. of $\frac{\Delta P_i}{(665-P_i)P_i}$ gives $\frac{\Delta P_n}{(665-P_n)P_n} \cong 0.000802$

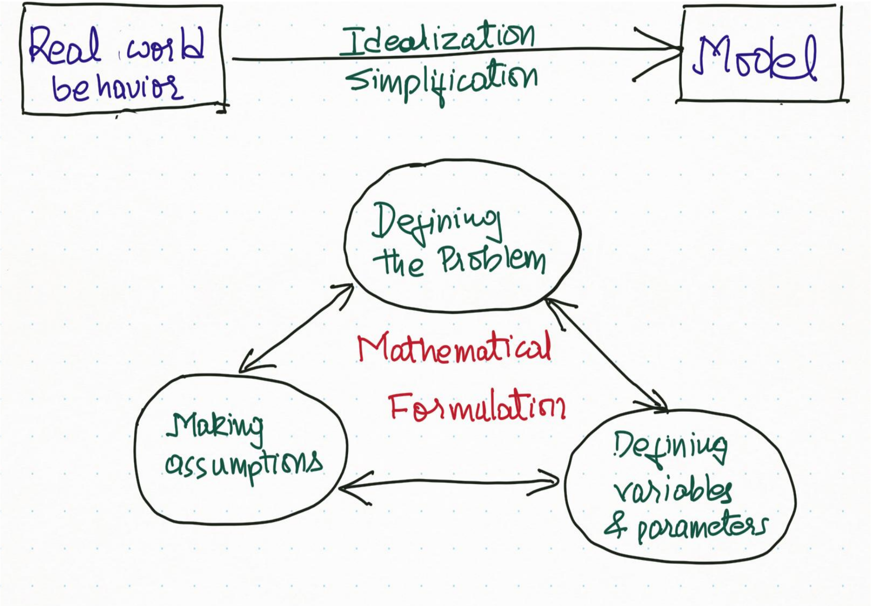
i.e. $P_{n+1} - P_n = (0.000802)(665-P_n)P_n$

i.e. $P_{n+1} = P_n + (0.000802)(665-P_n)P_n$ with $P_0 = 9.6$

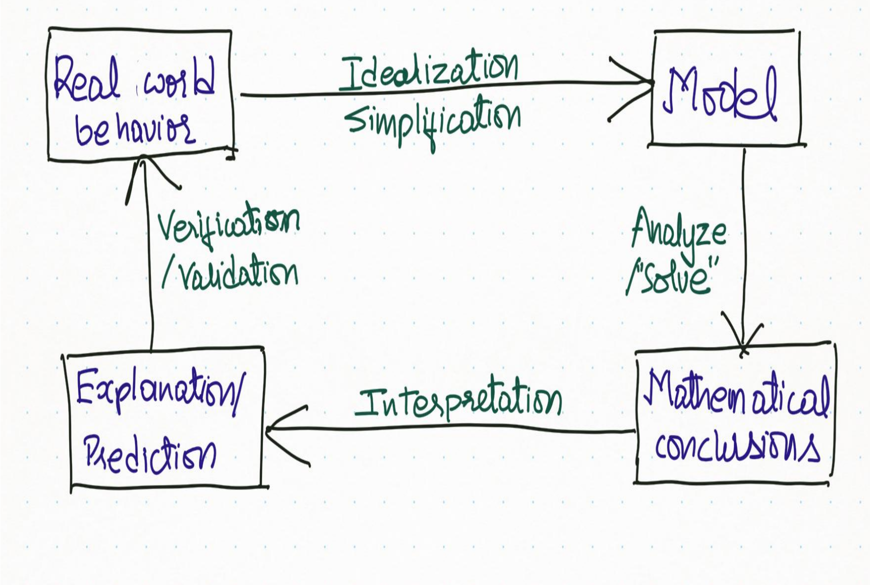
roughly linear



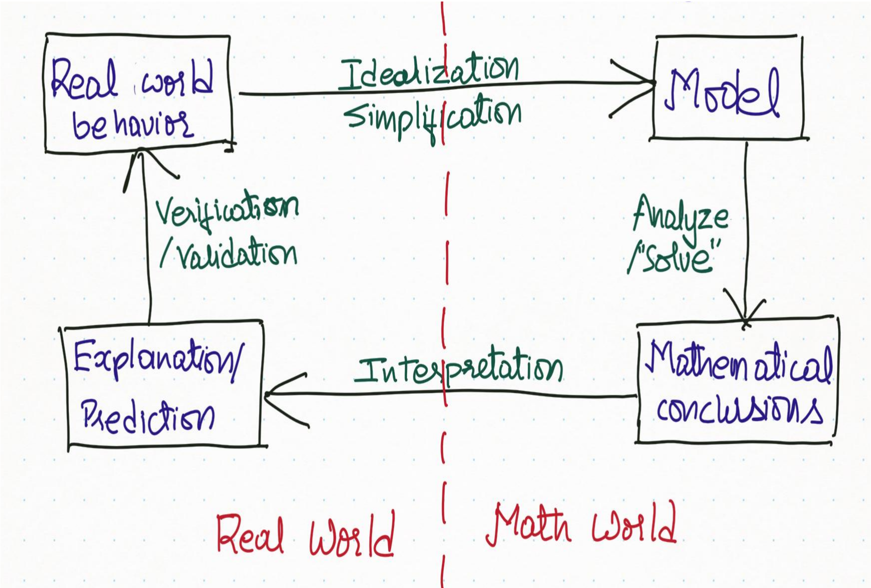
Mathematical Modeling Process



Mathematical Modeling Process



Mathematical Modeling Process



All Models are wrong, but some are useful!

- A good model reveals relationships that may not be apparent superficially.
- “Analysis/ Solution” builds strategies/ decisionmaking processes that are more sophisticated/ powerful than a naive approach.
- Allows for experimentation/ simulation when its impossible or too expensive in the real world.

All Models are wrong, but some are useful!

- A good model reveals relationships that may not be apparent superficially.
- “Analysis/ Solution” builds strategies/ decisionmaking processes that are more sophisticated/ powerful than a naive approach.
- Allows for experimentation/ simulation when its impossible or too expensive in the real world.

All Models are wrong, but some are useful!

The “wrong” above is also the source of the power of the modeling process: the first step where we make assumptions/ idealization of the phenomenon under study.

The art of modeling lies in making the right choices in our setup. These choices act as axioms for our model and drive the whole modeling process from ‘solution’ to ‘interpretation’ and ‘validation’

- Classical Mechanics: assumes mass is ‘fixed’ & acts like a point, forces act instantaneously, Newton’s laws of motion, etc. (later: a inertial frame of reference, non-relativistic, non-quantum.)
- Population Growth: Change in population is proportional to current population, Environmental limit to population, etc.

All Models are wrong, but some are useful!

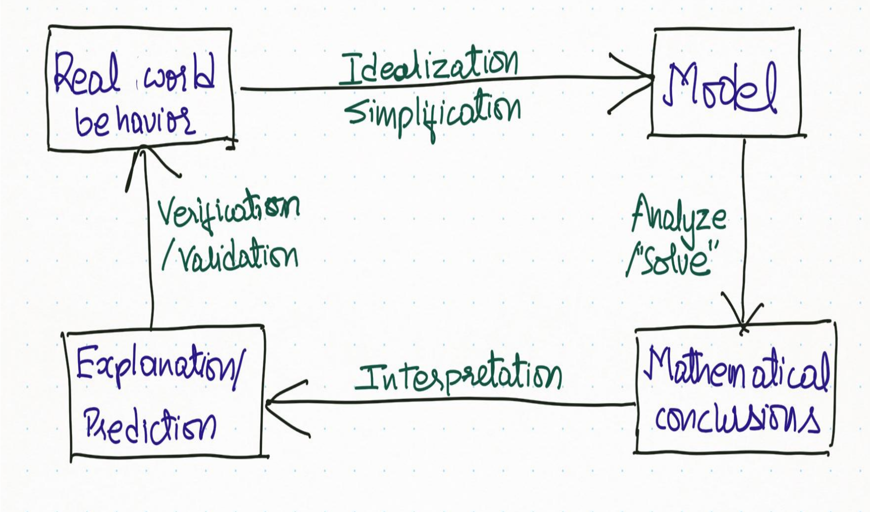
The “wrong” above is also the source of the power of the modeling process: the first step where we make assumptions/ idealization of the phenomenon under study.

The art of modeling lies in making the right choices in our setup. These choices act as axioms for our model and drive the whole modeling process from ‘solution’ to ‘interpretation’ and ‘validation’

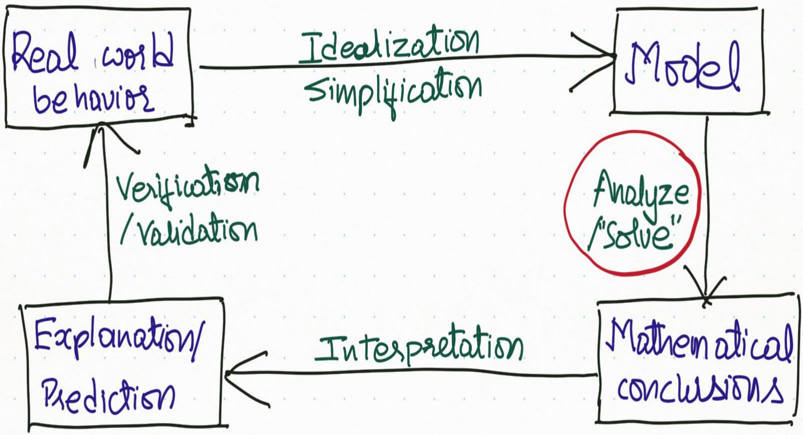
- Classical Mechanics: assumes mass is ‘fixed’ & acts like a point, forces act instantaneously, Newton’s laws of motion, etc. (later: a inertial frame of reference, non-relativistic, non-quantum.)
- Population Growth: Change in population is proportional to current population, Environmental limit to population, etc.

Mathematical Modeling Process

Where are ethical concerns important below?

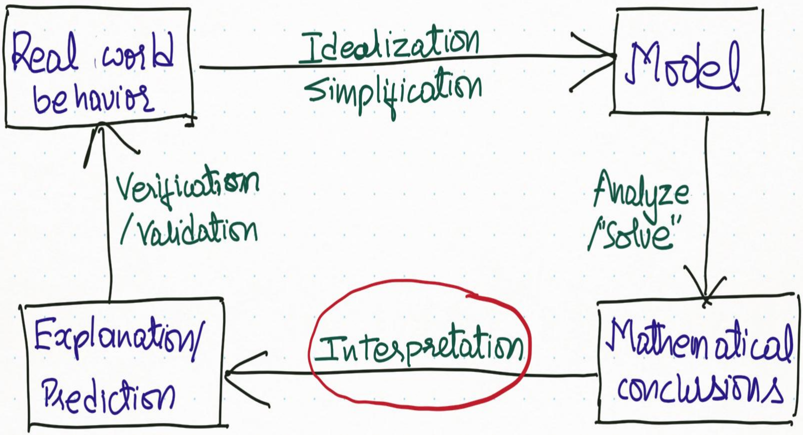


Mathematical Modeling Process



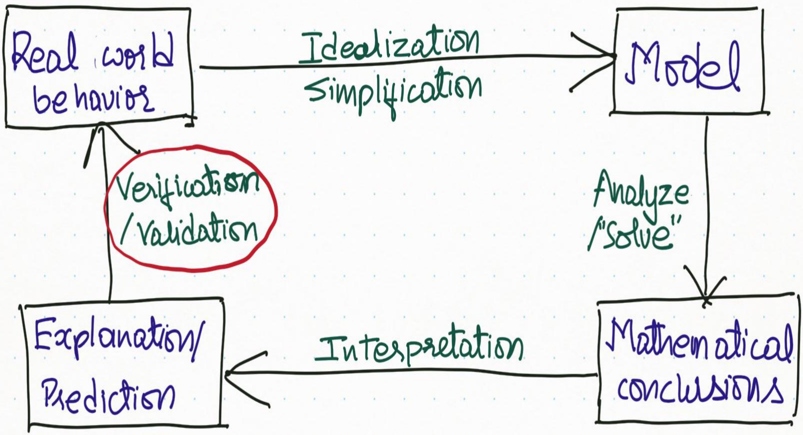
e.g. Algorithmic bias.

Mathematical Modeling Process



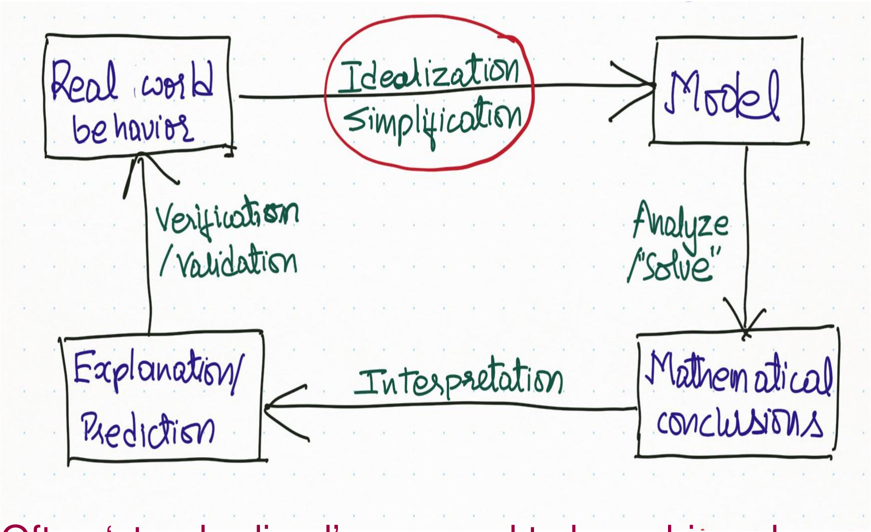
e.g. Bioethics.

Mathematical Modeling Process



e.g. Data bias/Statistical ethics.

Mathematical Modeling Process



Often 'standardized', assumed to be unbiased, or simply ignored.

Inbuilt Bias?

Are our (mathematical/ scientific/ social) assumptions creating an inbuilt bias that can not be overcome even if we do everything else perfectly (avoid any bias in our algorithmic computations, in the statistical/ data collection/ analysis, and in the real-world recommendations)?

How can we understand which choices are ethical?

Inbuilt Bias?

Are our (mathematical/ scientific/ social) assumptions creating an inbuilt bias that can not be overcome even if we do everything else perfectly (avoid any bias in our algorithmic computations, in the statistical/ data collection/ analysis, and in the real-world recommendations)?

How can we understand which choices are ethical?

Ethical Decision Making

Some perspectives for Ethical standards:

- Utilitarian: Which choice will create the most good and cause least harm?
- Rights: Which choice best respects the rights of those who have a stake?
- Fairness/Justice: Which choice treats all people fairly and equitably?
- Common Good: Which choice best serves the community as a whole?
- Virtue: Which choice leads me to act as a person I want to be (encompassing the ideals above and more)?

Advertisement#1: Look up ‘Markkula Center for Applied Ethics’.

Ethical Decision Making

Some perspectives for Ethical standards:

- Utilitarian: Which choice will create the most good and cause least harm?
- Rights: Which choice best respects the rights of those who have a stake?
- Fairness/Justice: Which choice treats all people fairly and equitably?
- Common Good: Which choice best serves the community as a whole?
- Virtue: Which choice leads me to act as a person I want to be (encompassing the ideals above and more)?

Advertisement#1: Look up ‘Markkula Center for Applied Ethics’.

Ethical Decision Making

Some perspectives for Ethical standards:

- Utilitarian: Which choice will create the most good and cause least harm?
- Rights: Which choice best respects the rights of those who have a stake?
- Fairness/Justice: Which choice treats all people fairly and equitably?
- Common Good: Which choice best serves the community as a whole?
- Virtue: Which choice leads me to act as a person I want to be (encompassing the ideals above and more)?

Advertisement#1: Look up ‘Markkula Center for Applied Ethics’.

Two Examples

Are our (mathematical/ scientific/ social) assumptions creating an inbuilt bias that can not be overcome even if we do everything else perfectly (avoid any bias in our algorithmic computations, in the statistical/ data collection/ analysis, and in the real-world recommendations)?

First, a historical example to illustrate this bias.

Second, an example of our attempt at improving a mathematical model from an ethical perspective.

Love Canal Controversy (1970s-1980s)

Love Canal, a suburban town in NY close to Niagara Falls.

Historical timeline:

- 1890s A canal built by W.T. Love for a hydroelectric plant (never built).
- 1905 Hooker Electrochemicals (HE) factory established.
- 1942-1952 HE allowed to dispose 22000 tons of chemical waste in the canal in fiber and metal barrels.
- 1953 Canal filled so it was covered by soil and grass grew.
- 1956 School for 400 children built and soon a town grew.

Love Canal Controversy (1970s-1980s)

Historical timeline (contd.):

- 1970s Chemical odors noticeable.
- 1978 Newspaper Articles and NY Dept of Health investigation. 80+ chemicals including 10 carcinogens found in soil.
- 1978 Govt. offers to buy 239 houses closest to the site and reclocate those residents. Waste site sealed.
- 1978 NYHD does health inspections and concludes rest of Love Canal far from the waste site is safe to live in.

Love Canal Controversy (1970s-1980s)

Historical timeline (contd.):

- 1978 Lois Gibbs, a homemaker organized a residents association. She sat down with a map of Love Canal and put a pin on every home with a registered health problem. She noticed a pattern of narrow paths corresponding to filled in old streams and swales - the “ wet homes”.
- 1978 Lois Gibbs and the residents association did a thorough survey of all residents and with help of Beverly Paigen (cancer researcher from nearby research institute) found indisputable evidence that residents in “wet homes” were 3 times more likely to have miscarriages, birth defects, asthma, UTIs, etc. than those in “dry homes”.

Love Canal Controversy (1970s-1980s)

Historical timeline (contd.):

- 1978 Gibbs and Paigen argued that the evacuations should prioritize residents from wet homes and ultimately all residents should be evacuated due to the pattern of widespread contamination. As opposed NYDH's decision to buy and evacuate only houses close to the waste site.
- 1980-81 With media coverage, Gibbs-Paigen prevailed and most of Love Canal was vacated by 1981.
- 1980s EPA Superfund sites were created to handle such sites of environmental disaster.

Difference Between the Two Approaches

- Different Models

  NYDH Scientists built a precise mathematical model based on the assumption that toxins spread more or less homogeneously radially outward from the waste site. This is traditional scientific practice going back to Galileo, we start with a simple model and add complexities to it as needed.

  Paigen based her analysis on Gibbs hypothesis that toxins spread faster through the dried streams and swales.

Difference Between the Two Approaches

- "Burden of Proof?"

  NYDH scientists: Need strong evidence that Love Canal is unsafe. "we are objective, we only deal with numbers"

  Paigen Need strong evidence to conclude Love Canal is safe since a mistake could result in damage to human life, in context of its history and available data.

Precautionary Principle

Paigen's position is an application of what has become known as the "Precautionary principle".

Originated in 1970s (or, even earlier) and formalized in 1992 under the Rio Declaration by representatives of 178 nations.

"Where there are threats of serious or irreversible damage, lack of full scientific certainty shall not be used as a reason for postponing cost-effective measures to prevent environmental degradation."

Some Guidelines

- When doing applied math, the practitioners are not "outside the system", scientific objectivity is tricky,
- "Standardized" models should not be used blindly in all situations. Uncertainty of assumptions and evidence should be considered in the context of "burden of proof".
- Multidisciplinary teams with a variety of relevant specializations should work together.[1]
- Local stakeholders should be involved in the process and should be encouraged to be critical.

---

[1] Incident of Cesium in the sheep of Cumbria via radioactive cloud from Chernobyl: soil as a physical transport vs chemical transport.

How is a public transit system designed?[2]

The urban transit network design problem (UTNDP) has been well-studied over the past many decades.
The types of problems/ decisions studied include:

- Strategic: Building new streets/ transit lines; Designing transit routes; etc.
- Tactical: Allocating exclusive bus lanes; Determining Transit line frequencies; etc.
- Operational: Scheduling traffic lights; Scheduling of repairs; Determining transit schedule; etc.

[2] *A Public Transit Network Optimization Model for Equitable Access to Social Services*, (with Adam Rumpf), Inaugural ACM Conference on Equity and Access in Algorithms, Mechanisms, and Optimization (EAAMO '21).
Advertisement#2: Look up 'Mechanism Design for Social Good (MD4SG)'.

How is a public transit system designed?

The UTNDP is a bi-level problem.
The upper level corresponds to the transit policy under consideration which is studied through design decisions with objectives like (operator and user) costs, and socio-economic constraints.
The lower level corresponds to the user behavior through the transit network and the travel choices they make based on the policy decisions in the upper level.

Conceptually, these are two games tied-in together: the one between upper and lower levels, and the other within the lower level. Together they can be expressed as a mathematical program with equilibrium constraints.

How is a public transit system designed?

The design and decision process in UTNDP is thus modeled as an optimization problem with underlying game-theoretic assumptions about user behavior (Wardrop Equilibrium of a non-cooperative Nash game).

The most common objectives consist of optimizing some combination of

- Operator costs: fleet size, vehicle maintenance, travel distance, number of stops, profit, etc.
- User costs: travel time, waiting time, walking distance, number of transfers, vehicle crowding, etc.

These are subject to standard constraints of transportation network flow and design, as well as any budgetary bounds.

How should a public transit system be designed?

Almost all studies under UTNDP framework only consider factors corresponding to travel times (congestion), user demand, or operational costs. A major review of urban transportation network design problems (Farahani et al. EJOR 2013) identified only one paper where environmental/ health concerns (CO_2 emissions) were explicitly included.

What about community access to social services?

Do all communities have equitable access to health care centers, city offices, parks, museums, etc.?

How can this accessibility be improved and made more equitable?

How should a public transit system be designed?

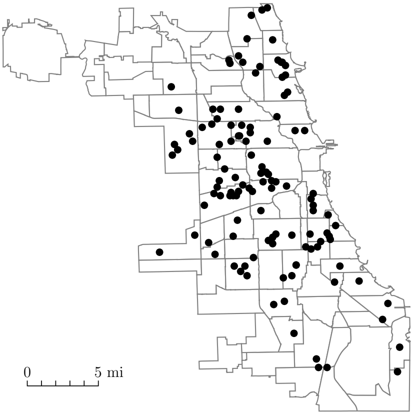
Almost all studies under UTNDP framework only consider factors corresponding to travel times (congestion), user demand, or operational costs. A major review of urban transportation network design problems (Farahani et al. EJOR 2013) identified only one paper where environmental/ health concerns (CO_2 emissions) were explicitly included.

What about community access to social services?

Do all communities have equitable access to health care centers, city offices, parks, museums, etc.?

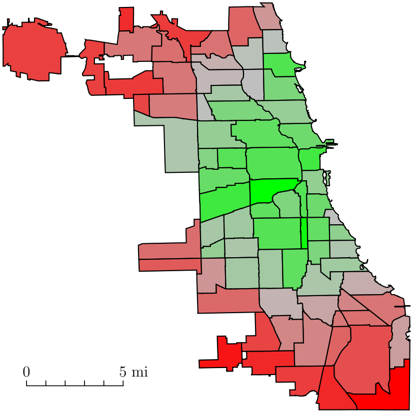
How can this accessibility be improved and made more equitable?

Primary Healthcare Accessibility in Chicago



Primary Healthcare Center Locations in Chicago

Primary Healthcare Accessibility in Chicago



Primary Healthcare Centers Accessibility Metric via CTA
in each of the 77 community areas

Community access to social services

Improvement of access to facilities is typically studied in form of Facility Location Problems.

- Build more primary care centers and place them optimally
- Build more CTA subway lines
- Add more CTA bus lines
- Add more CTA buses

Slow!
Needs major financial investments!

Community access to social services

Improvement of access to facilities is typically studied in form of Facility Location Problems.

- Build more primary care centers and place them optimally
- Build more CTA subway lines
- Add more CTA bus lines
- Add more CTA buses

Slow!
Needs major financial investments!

Community access to social services

We look at the problem as an inverse of the facility location problem.

Rather than changing (or adding) facility locations in a fixed transit network, we consider changing the transit network containing fixed facility locations.

Is it possible to do this without changes that are disruptive or expensive?

Our Modeling Principles

1. The model should be flexible, allowing the planner to choose any desired social access objective and any assumptions regarding user behavior.
2. The model must produce solutions that remain at or near the system's current cost and performance level while attempting to optimize the social access objective.
3. The design decisions should consist of measures which are low-cost, immediate, and easily-implemented.
4. We will assume that travel related to our social access goal makes up a relatively small proportion of the day-to-day public transit travel volume. In particular we will assume that the capacity of the facilities in question is a much more significant limiting factor than the public transit service capacity.

The *social access maximization problem* (SAMP)

$$\max_{\mathbf{y}} \quad \text{Access}(\mathbf{y}) \tag{1}$$

$$\text{s.t.} \quad \text{OperatorCost}(\mathbf{y}, \mathbf{x}) \leq B_{\text{operator}} \tag{2}$$

$$\text{UserCost}(\mathbf{x}) \leq B_{\text{user}} \tag{3}$$

$$\mathbf{x} = \text{TransitAssignment}(\mathbf{y}) \tag{4}$$

$$\sum_{l \in L^z} y_l \leq \sum_{l \in L^z} y_l^* \qquad \forall z \in Z \tag{5}$$

$$y_l^{\text{min}} \leq y_l \leq y_l^{\text{max}} \qquad \forall l \in L \tag{6}$$

$$y_l \in \mathbb{Z} \qquad \forall l \in L \tag{7}$$

Design decision variables are y_l, the number of vehicles that service line $l \in L$, in the set of transit lines under consideration. Essentially, we are reallocating (changing bus frequency of) existing buses inbetween existing lines and with some new express lines.

The *social access maximization problem* (SAMP)

- The overall objective (1) is to maximize $\mathrm{Access}(\mathbf{y})$.

$$\mathrm{Access}(\mathbf{y}) := \sum_{\substack{\mathcal{K} \text{ minimum elements} \\ \text{of } \{A_i(\mathbf{y})\}}} A_i(\mathbf{y})$$

where $\mathcal{K}$ is a model parameter, and $A_i(\mathbf{y})$ is a primary care accessibility metric of community i.

The *social access maximization problem* (SAMP)

- The overall objective (1) is to maximize $\mathrm{Access}(\mathbf{y})$.

$$\mathrm{Access}(\mathbf{y}) := \sum_{\substack{\mathcal{K} \text{ minimum elements} \\ \text{of } \{A_i(\mathbf{y})\}}} A_i(\mathbf{y})$$

  where $\mathcal{K}$ is a model parameter, and $A_i(\mathbf{y})$ is a primary care accessibility metric of community i.

- This objective refers to the $\mathcal{K}$ *current* least accessibility metrics.
  This generalizes the ideas of maximizing only the current minimum accessibility metric (the special case of $\mathcal{K} = 1$), and of maximizing the total (or equivalently the average) of all accessibility metrics.

The *social access maximization problem* (SAMP)

- Our specific choice of accessibility metric $A_i(\mathbf{y})$ is a gravity-based metric:

$$A_i(\mathbf{y}) := \sum_j \frac{S_j d_{ij}^{-\beta}(\mathbf{y})}{F_j(\mathbf{y})}$$

where S_j is the quality (patient capacity) of facility j, $d_{ij}(\mathbf{y})$ is the travel time from i to j, $\beta > 0$ is a gravitational decay model parameter, and $F_j(\mathbf{y})$ is a competition-related facility metric defined by

$$F_j(\mathbf{y}) := \sum_k P_k d_{kj}^{-\beta}(\mathbf{y})$$

where P_k is the population of community k seeking service.

$F_j(\mathbf{y})$ can be interpreted as a measure of how crowded facility j is, and is greater if the facility is close to many populous communities.

The *social access maximization problem* (SAMP)

- The overall objective (1) is to maximize $\mathrm{Access}(\mathbf{y})$.

$$\mathrm{Access}(\mathbf{y}) := \sum_{\substack{\mathcal{K}\text{ minimum elements} \\ \text{of } \{A_i(\mathbf{y})\}}} A_i(\mathbf{y})$$

The accessibility metric $A_i(\mathbf{y})$ rewards a community for being close to many facilities with high quality and low overcrowding.

The metric, itself, has no direct interpretation in isolation and is only meant for use in comparing communities across space or time.

The *social access maximization problem* (SAMP)

- Constraints (2) and (3) give explicit upper bounds of B_{operator} and B_{user} to the operator and user costs, respectively.

$$\text{OperatorCost}(\mathbf{y}, \mathbf{x}) \leq B_{\text{operator}} = (1+\epsilon) \text{ (current operator cost)}$$

$$\text{UserCost}(\mathbf{x}) \leq B_{\text{user}} = (1+\epsilon) \text{ (current user cost)}$$

$\epsilon \geq 0$ is a model parameter which specifies the allowed margin of increase in costs.

The *social access maximization problem* (SAMP)

- Constraints (4) implicitly define the user flow vector **x** as a result of the decision vector's effect on the transit assignment function.

$$\mathbf{x} = \text{TransitAssignment}(\mathbf{y})$$

It is based on a conical congestion function which increases the costs of arcs as they become more crowded, serving to discourage flows from exceeding line capacities. The output is the user-optimal flow (Wardrop Equilibrium) which corresponds to all users choosing a strategy for which their expected travel time cannot be improved with a unilateral change.

The *social access maximization problem* (SAMP)

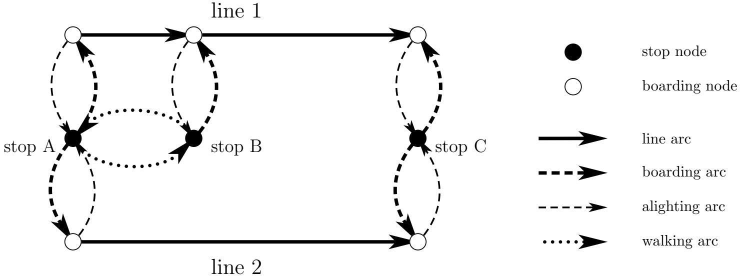
- Constraints (5) ensure that no new vehicles of any type are added:

$$\sum_{l \in L^z} y_l \leq \sum_{l \in L^z} y_l^* \qquad \forall z \in Z.$$

Constraints (6) enforce fleet size bounds for each line:

$$y_l^{\min} \leq y_l \leq y_l^{\max} \qquad \forall l \in L.$$

The underlying Network



Total 3852 Nodes that correspond to origin-destination pairs, public transit stops (CTA bus and subway); communities; primary healthcare centers.

Total 17522 Arcs that correspond to walking routes, and all CTA bus & subway lines.

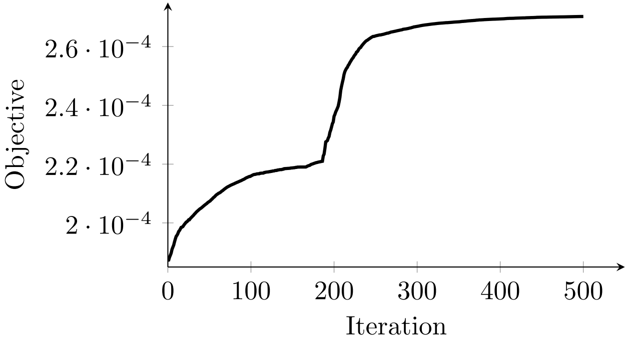
Data sources: 2010 Census; CTA General Transit Feed Specification (GTFS) files; City of Chicago healthcare data.

The SAMP Algorithm

We propose a hybrid tabu search/simulated annealing solution algorithm for the SAMP.

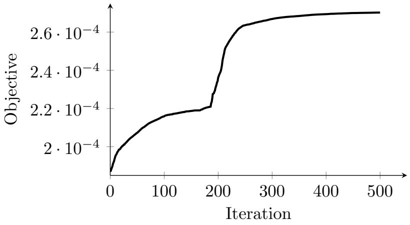
- A natural choice of initial feasible solution (the initial fleet vector) as well as a natural definition of local moves, consisting of adding, dropping, and swapping individual vehicles between compatible lines.
- Use an Simulated Annealing acceptance probability when considering whether to make a suboptimal move.
- Tabu rules stored in Short-Term-Memory prevent undoing recent additions or subtractions from each fleet, and attractive solutions stored in Long-Term-Memory consist of a combination of the second-best solutions from neighborhood searches which were not chosen, and of suboptimal moves that were previously denied by the SA criterion.

Computational Results



Objective value over 500 iterations of the SAMP algorithm

Computational Results



- The first 166 iterations consisted entirely of exchanging vehicles between non-express routes.
- The first express route was added during iteration 167, and the second was added during iteration 186, after which many more vehicles were added to express routes.
- A total of 71 vehicles were diverted from existing lines to express lines (4.25% of the 1668 total buses in service), with 14 of the 65 available express routes receiving at least one vehicle.

Computational Results

	Initial	Final	Rel. Diff.
Mean	$4.18299 \cdot 10^{-5}$	$4.31441 \cdot 10^{-5}$	3.14170%
Std. Dev.	$1.12720 \cdot 10^{-5}$	$0.74786 \cdot 10^{-5}$	-33.65312%
Median	$4.29677 \cdot 10^{-5}$	$4.17583 \cdot 10^{-5}$	-2.81456%
Max	$6.71548 \cdot 10^{-5}$	$6.47152 \cdot 10^{-5}$	-3.63287%
Min	$1.82806 \cdot 10^{-5}$	$2.65753 \cdot 10^{-5}$	$+45.37372\%$

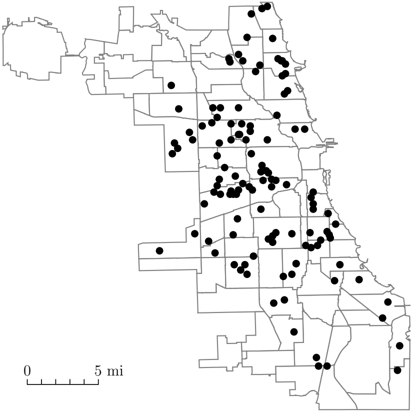
Table: Summary statistics for the 77 Chicago community area accessibility metrics before and after running the SAMP algorithm for 500 iterations.

Computational Results

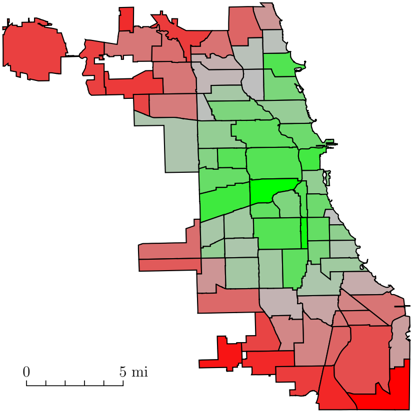
- The overall distribution of accessibility metrics within the city remained relatively balanced, with approximately half improving and half worsening while the mean and median remained roughly the same, both changing by less than 3.2%.
- The spread of accessibility levels also narrowed significantly, with the standard deviation decreasing by more than 33.6%, the maximum decreasing by less than 3.7%, and the minimum increasing by more than 45.3%.

Overall, Success!

Primary Healthcare Center Locations

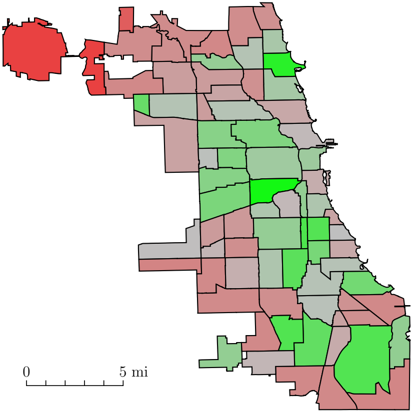


Computational Results



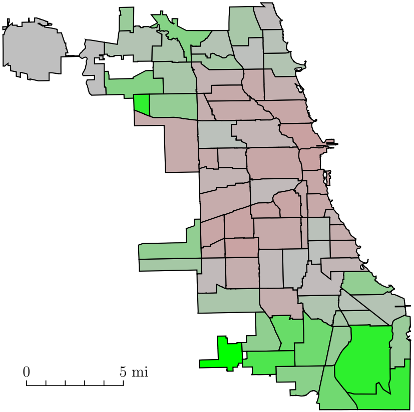
Initial Accessibility Metrics for each of the 77 community areas

Computational Results



Accessibility Metrics after 500 iterations of the SAMP Algorithm

Computational Results



Relative change in Accessibility Metrics after 500 iterations of the SAMP Algorithm

Sensitivity to Parameter Values

In the Chicago trials we used $\epsilon = 0.01$, $\mathcal{K} = 8$, and $\beta = 1.0$.
We generated artificial geographical networks to test the sensitivity of the three parameters.

- User Cost Increase Parameter ϵ
  $\epsilon > 0$ produced similar results with some variations in the solutions.
- Community Inclusion Parameter $\mathcal{K}$
  $\mathcal{K} = 1$ resulted in no change.
  $\mathcal{K} \in [6, 24]$ produced similar trends.
  $\mathcal{K} = 30$ produced a small decrease for the communities below the median and a significant increase for the communities above the median.
- Gravitational Decay Parameter β
  $\beta \in [1, 2]$ produced similar results with some variations in the solutions.

Thank You!

Any Questions? Thoughts?