Fall Coloring of Graphs

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Observe that, if G is Fall k-colorable than $\chi(G) \leq k \leq \delta(G) + 1$.

Sharp for $G = C_{6k}$.

Fall Coloring

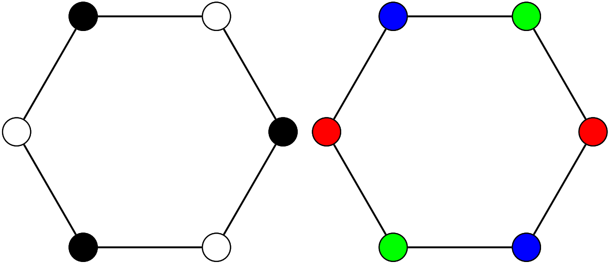

Figure: 2- and 3-fall-colouring of C_6

Fall Coloring

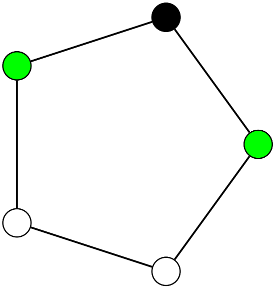

Figure: C_5 cannot be fall-coloured

Fall Coloring

$Fall(G)$ is the set of all k such that G has a Fall k-coloring.

- $Fall(C_n) \subseteq \{2,3\}$, and $2 \in Fall(C_n)$ iff $2|n$, $3 \in Fall(C_n)$ iff $3|n$.
- $Fall(K_n) = \{n\}$
- Complete k-partite graphs have only Fall k-colorings.
- k-Trees have Fall $(k+1)$-colorings.
- If G is $K_{m,m}-$ perfect matching then $Fall(G) = \{2, m\}$ [Cockayne, Hedetneimi, 1976].
- Fall(Petersen) is empty.

Fall Coloring

Introduced in this form by Dunbar, Hedetniemi, Hedetniemi, Jacobs, Knisely, Laskar and Rall (2000).

Related versions of the problem were studied by Berge (1960s), Cockayne, Hedetneimi (1976), Payan (1974), Erdős, Hobbs, Payan (1982), and others.
And now again since 2000.

Fall 2-colorings

Does $2 \in \mathit{Fall}(G)$?
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Fall 2-colorings

Does $2 \in \mathit{Fall}(G)$?
or, even more simply, Does G have two disjoint maximal independent sets?

- Berge and, independently, Payan(1974) conjectured that every regular graph has two disjoint maximal independent sets.
  Disproved by Payan (1977)
- Erdős, Hobbs, Payan (1982) improved results of Cockayne, Hedetneimi (1976) and others to show dense graphs ($\delta(G) > n - O(\sqrt{n})$) have this property.
- Henning, Lowenstein, Rautenbach (2009) showed the decision problem is NP-complete,
  even when restricted to graphs of max degree 4 [Raczek, Janczewski, Malafiejska, 2011]
- On the easier side, 2-chromatic graphs without isolated vertices are always Fall 2-colorable.

Constructions

Given G, what is $Fall(G)$?

This is hard to answer since $Fall(G)$ need not be an interval of numbers, can be empty, and may not relate to Fall sets of subgraphs.

Can we construct a family of graphs whose Fall set equals an arbitrary collection of integers?

Constructions

Dunbar et al., 2000

Let $S = \{s_1, s_2, \ldots, s_r\}$ be given, with $s_i \neq 1, \forall i$.
Let $G = K_{s_1} \times K_{s_2} \times \ldots \times K_{s_r}$. Then $S \subseteq \mathit{Fall}(G)$.

Each $(r-1)$-dimensional cardinal hyperplane is an independent dominating set.

Constructions

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Let $S = \{s_1, s_2, \ldots, s_k\}$, be a multiset with $s_i \neq 1, \forall i$.
Let $G = \square_{s \in S} K_s$.
A subset of V(G) is an independent dominating set iff it corresponds to s_i vertices which share the same coordinates, except on the ith position.

Constructions

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Let $|S| = 2$. Then the two previous constructions are identical. Moreover, $Fall(G) = S$.

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Let $|S| = 3$. $Fall(\overline{\square_{s \in S} K_s})$ is the set of all numbers which can be expressed as sums of exactly s_i summands with values in $S \setminus \{s_i\}$, for each i.

For example, when $S = \{2, 3, 4\}$, $Fall(G) = \{6, 7, \cdots, 12\}$;
(like $6 = 3 + 3$, $7 = 4 + 3$, $8 = 4 + 4$, $9 = 3 + 2 + 2 + 2$, $10 = 3 + 3 + 2 + 2$, $11 = 3 + 3 + 3 + 2$, $12 = 4 + 4 + 4$.)

On the other hand, when $S = \{2, 3, 5\}$, $Fall(G) = \{6, 8, \cdots, 15\}$.

This form of summation also works for $|S| > 3$ to determine the max and min values in $Fall(G)$.

Constructions

Dunbar et al. (2000) asked a natural question:
Can the difference between $\chi(G)$ and min *Fall*(G) be made arbitrarily large?

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Let $k \geq 3$ and let $t > k$. Then, there exists a graph G with $\chi(G) = k$ and min *Fall*$(G) = t$.

We modify $\overline{K_k \square K_t}$ by removing the edges of an appropriately chosen induced $(t-1)$-star from it.

Unique Coloring

Observe that If G is uniquely k-colorable, then $k \in Fall(G)$.
Since there is a unique k-coloring, every vertex has a neighbor in each color class, other than its own.

Converse is not true: e.g. $K_k \times K_k$.

Bollobás (1978) showed that high minimum degree forces a k-colorable graph to be k-chromatic (if $\delta(G) > \frac{k-2}{k-1}n(G)$) and uniquely k-colorable (if $\delta(G) > \frac{3k-5}{3k-2}n(G)$).

Can we show something better for Fall coloring?

Unique Coloring

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Let G be a k-colorable graph on n vertices, for $2 \leq k \leq n$. If $\delta(G) > \frac{k-2}{k-1}n$, then every k-coloring of G is also a Fall k-coloring.

This is sharp.

Unique Coloring

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Let G be a k-colorable, $\frac{k-2}{k-1}n$-regular graph, for $2 \leq k \leq n$. Then every k-coloring of G is either a Fall k-coloring or can be converted to a $(k-1)$-fall-coloring, by merging two color classes.
Furthermore, there always exists some graph as described above, which is $(k-1)$-fall-colorable.

The graph which shows the sharpness of the previous result is none other than the Turán Graph, $T(n,(k-1))$.

Graph Products

Cartesian product of graphs is well-behaved under Fall coloring.

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Let $k \in \mathit{Fall}(G)$ and let H be a k-colorable graph.
Then $G \square H$ is k-fall-colorable.

Usual coloring works.

We have that $\mathit{Fall}(G \square H) \supseteq (\mathit{Fall}(G) \cup \mathit{Fall}(H)) \setminus [\chi(G \square H) - 1]$.
Regarding equality, consider that $C_5 \square C_5$ is 5-fall-colorable, even though C_5 is not.

Graph Products

Similarly, for categorical product of graphs:

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Let $k \in Fall(G)$ and let H have no trivial components. Then $k \in Fall(G \times H)$.

The above tells us that $Fall(G) \cup Fall(H) \subseteq Fall(G \times H)$.

Perfect Graphs

A graph G is said to be a threshold graph iff it can be constructed iteratively by adding a new vertex, at each step, whose neighborhood is determined by its type:

1. Type 0 vertices are not adjacent to any previously added vertices.
2. Type 1 vertices are adjacent to all previously added vertices.

Thus, a threshold graph on n vertices, can be uniquely identified by a binary string of length n, starting with a 0.

Perfect Graphs

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A threshold graph G is fall-colorable iff it can be described by a bit string of the form $0^{+}1^{*}$.
Moreover, if G is a threshold graph described by $0^{+}1^{k-1}$. Then, $Fall(G) = \{k\}$.

Perfect Graphs

A graph is a split graph if its vertices can be partitioned into a clique and an independent set.

Note that the threshold graphs are a subfamily of split graphs.

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Let G be a Split Graph, with independent set I and clique K, so that K is maximal.
G is fall-colorable iff each vertex in I has exactly one non-neighbor in K.
Moreover, in this situation, G is only k-fall-colorable, where $k = \delta(G) + 1 = |K|$.

Perfect Graphs

Note that in both these families, threshold graphs and split graphs,

$Fall(G) = \emptyset$, or
$Fall(G) = \{\omega(G)\} = \{\chi(G)\} = \{\delta(G) + 1\}$.

We conjecture that this is true for all perfect graphs.

Lyle, Drake, Laskar (2005) have shown this is true for strongly chordal graphs.