Graph Packing - Conjectures and Results

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Introduction

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We may assume $|V_1| = |V_2| = n$ by adding isolated vertices.

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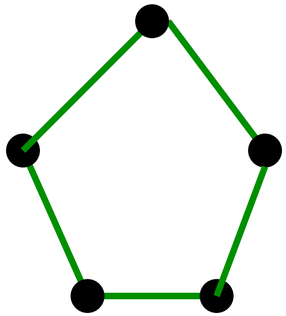
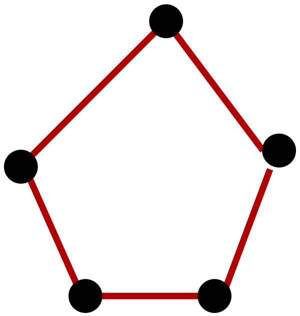
such that the images of the edge sets do not intersect.

- there exists a bijection $V_1 \leftrightarrow V_2$ such that $e \in E_1 \Rightarrow e \notin E_2$.
- G_1 is a subgraph of $\overline{G_2}$.

Examples and Non-Examples



C_5



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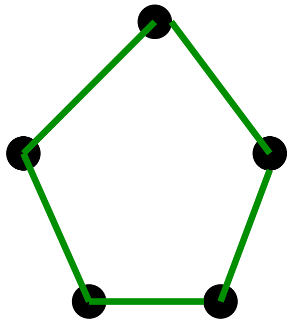

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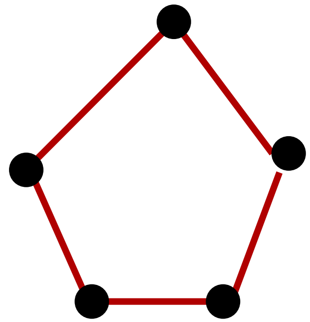

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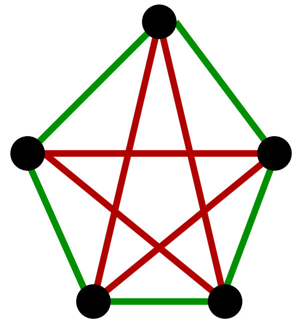
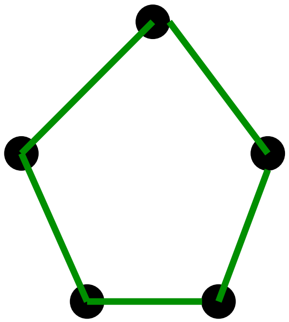


Packing

Examples and Non-Examples



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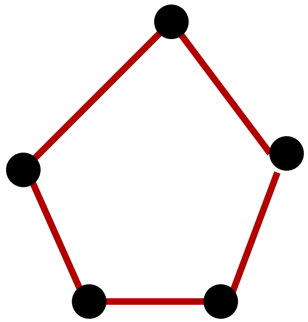

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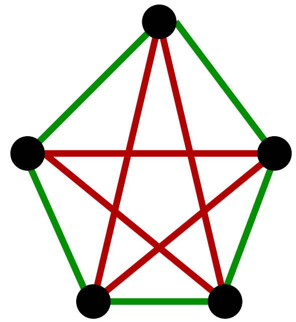

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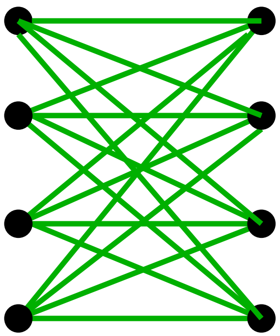
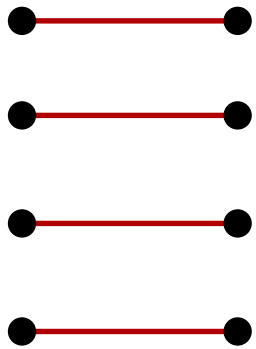


$K_{4,4}$



$4\ K_2$

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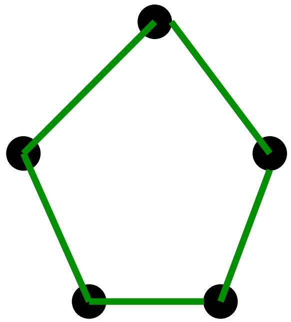

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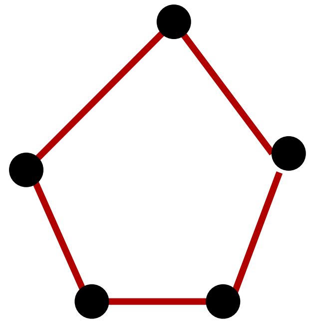

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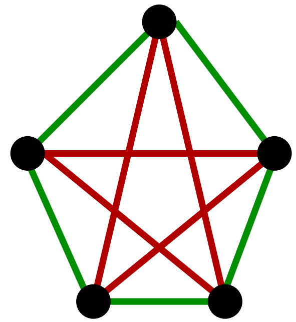

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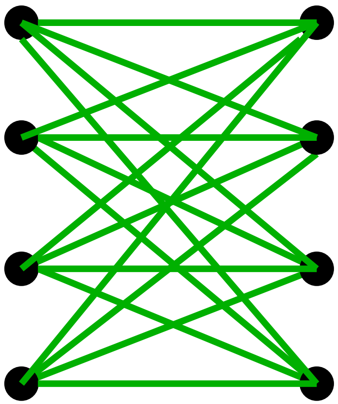

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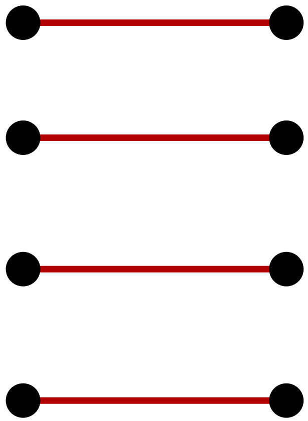

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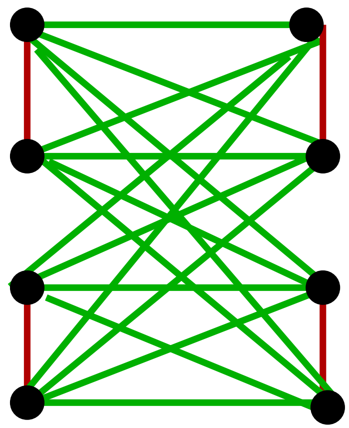

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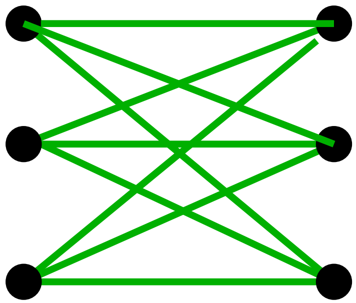

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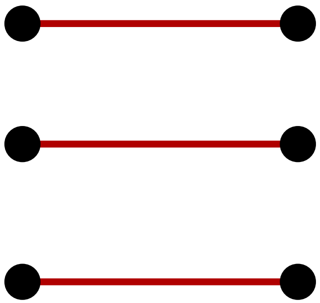

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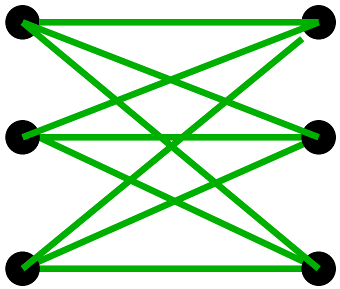

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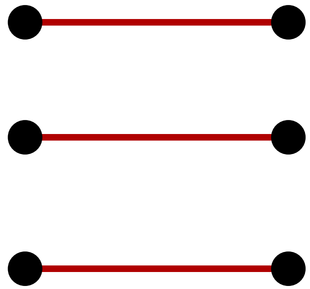

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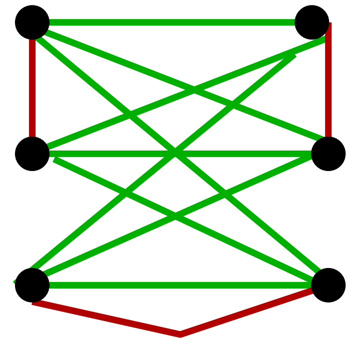

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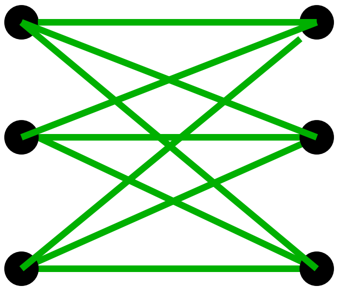

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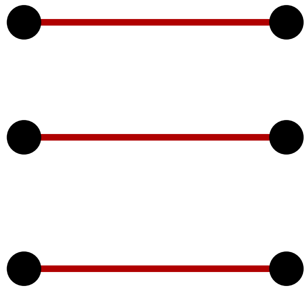

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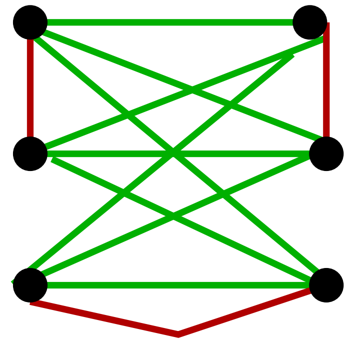

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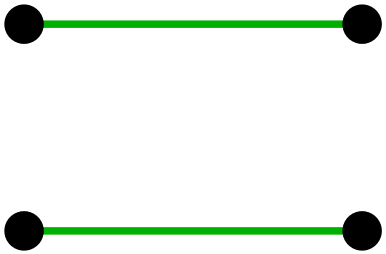
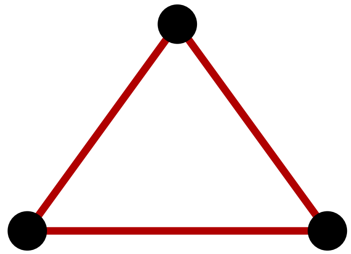


$2\ K_2$



K_3

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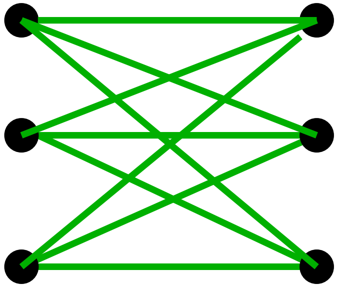

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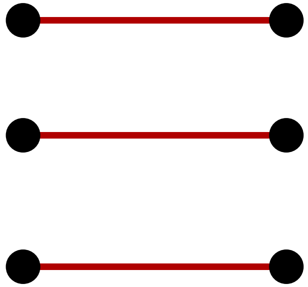

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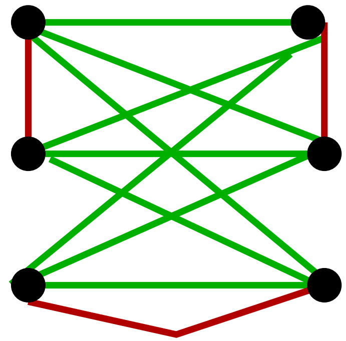

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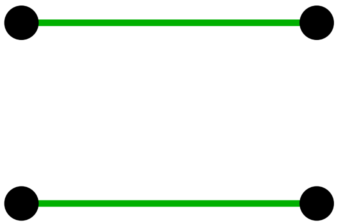

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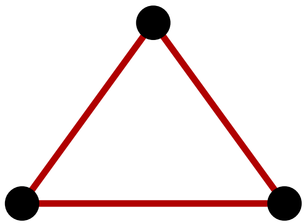

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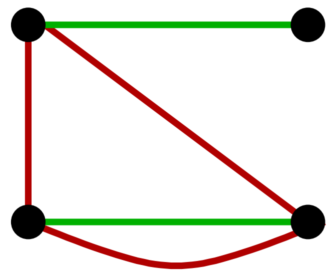

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A Common Generalization

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- Ramsey-type problems.
- “most” problems in Extremal Graph Theory.

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In subgraph problems, (usually) at least one of the two graphs is fixed.

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In packing problems, each member of a ‘large’ family of graphs contains each member of another ‘large’ family of graphs.

Theorem [Bollobas + Eldridge, 1978, & Teo + Yap, 1990]: If $\Delta_1, \Delta_2 < n-1$, and $e(G_1)+e(G_2) \leq 2n-2$, then G_1 and G_2 do not pack if and only if they are one of the thirteen specified pairs of graphs.

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In packing problems, each member of a 'large' family of graphs contains each member of another 'large' family of graphs.

Theorem: If $e(G_1)e(G_2) < \binom{n}{2}$, then G_1 and G_2 pack.

Proof. Pick a random bijection between $V(G_1)$ and $V(G_2)$, uniformly among the set of all $n!$ such bijections.

Sharp for $G_1 = S_{2m}$, star of order $2m$, and $G_2 = mK_2$, matching of size m, where $n = 2m$.

A Distinction

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Erdős-Sos Conjecture (1963) : Let G be a graph of order n and T be a tree of size k.
If $e(G) < \frac{1}{2}n(n-k)$ then T and G pack.

Each graph with more than $\frac{1}{2}n(k-1)$ edges contains every tree of size k.

This says average degree k guarantees every tree of size k. The corresponding minimum degree result is easy (induction on k).

Sharp, if true. Take disjoint copies of k-cliques.

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Tree Packing Conjecture (Gyarfas $\sim$ 1968) : Any family of trees $T_2, \ldots, T_n$, where T_i has order i, can be packed.

In other words, any family of trees $T_2, \ldots, T_n$ decomposes K_n.

Known only for special classes of trees, and for a sequence of $n/\sqrt{2}$ such trees (Bollobas, 1983).

Sauer and Spencer's Packing Theorem

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If $\delta(G) > \frac{(2k-1)(n-1)+1}{2k}$, then
G contains all graphs with maximum degree at most k.

Sauer and Spencer's Packing Theorem

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This is sharp.

For n even.

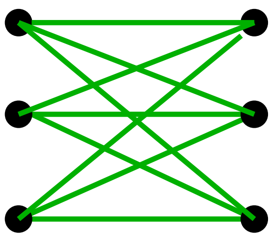
$G_1 = \frac{n}{2}K_2$, a perfect matching on n vertices.

$G_2 \supseteq K_{\frac{n}{2}+1}$, or

$G_2 = K_{\frac{n}{2},\frac{n}{2}}$ with $\frac{n}{2}$ odd.

Then, $2\Delta_1\Delta_2 = n$, and G_1 and G_2 do not pack.

Sauer and Spencer's Packing Theorem



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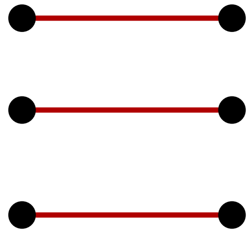

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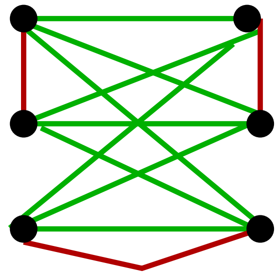
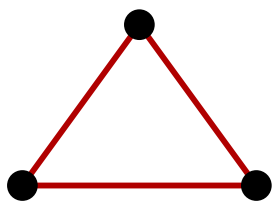


No Packing

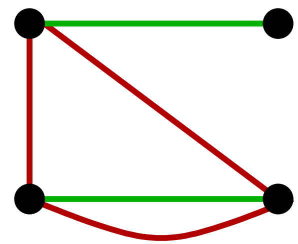
$G_1 = K_{\frac{n}{2},\frac{n}{2}}$ with $\frac{n}{2}$ odd

$G_2 = \frac{n}{2}K_2$

$2\ K_2$



K_3



No Packing

$G_1 = \frac{n}{2}K_2$

$G_2 \supseteq K_{\frac{n}{2}+1}$

Extending the Sauer-Spencer Theorem

Theorem 1 [Kaul + Kostochka, *CPC 2007*]:
If $2\Delta_1\Delta_2 \leq n$, then
G_1 and G_2 do not pack if and only if
one of G_1 and G_2 is a perfect matching and the other
either is $K_{\frac{n}{2},\frac{n}{2}}$ with $\frac{n}{2}$ odd or contains $K_{\frac{n}{2}+1}$.

This result characterizes the extremal graphs for the Sauer-Spencer Theorem.

To appear in *Combinatorics, Probability and Computing*.

Extending the Sauer-Spencer Theorem

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$\Delta_1\Delta_2 \le \frac{1}{2}n$ is sharp exactly when one of Δ_1, Δ_2 is small.

Can we improve the bound on $\Delta_1\Delta_2$, if both Δ_1 and Δ_2 are large ?

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Bollobás-Eldridge Graph Packing Conjecture :
If $(\Delta_1+1)(\Delta_2+1) \leq n+1$ then G_1 and G_2 pack.

Theorem 1 can be thought of as a small step towards this longstanding conjecture.

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If $\delta(G) > \frac{kn-1}{k+1}$, then
G contains all graphs with maximum degree at most k.

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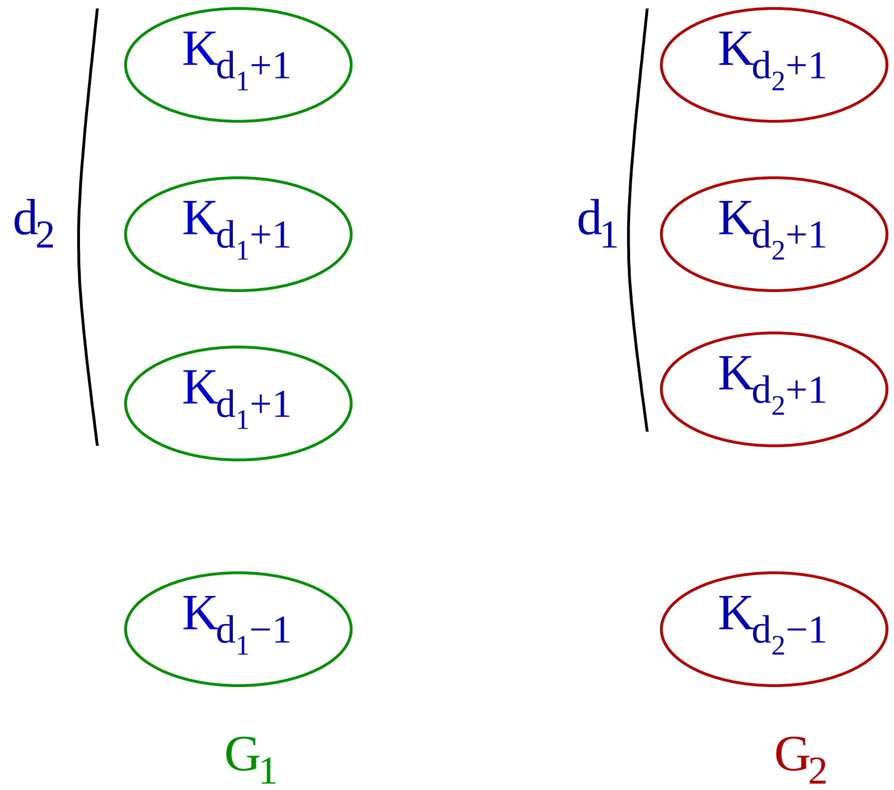
If true, this conjecture would be sharp.



$n = (d_1 + 1)(d_2 + 1) - 2$, $\Delta_1 = d_1$, $\Delta_2 = d_2$.

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If $(\Delta_1 + 1)(\Delta_2 + 1) \leq n + 1$ then G_1 and G_2 pack.

If true, this conjecture would be a considerable extension of the Hajnal-Szemerédi theorem on equitable colorings :
Every graph G has an equitable k-coloring for $k \geq \Delta(G) + 1$.

Equitable colorings of graphs have been used to

- extend Chernoff-Hoeffding concentration bounds to dependent random variables (Pemmaraju, 2003)
- extend Arnold-Groeneveld order statistics bounds to dependent random variables (Kaul + Jacobson, 2006)

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The conjecture has only been proved when

$\Delta_1 \leq 2$ [Aigner + Brandt (1993), and Alon + Fischer (1996)],

$\Delta_1 = 3$ and n is huge [Csaba + Shokoufandeh + Szemerédi (2003)].

Near-packing of degree 1 [Eaton (2000)].

Reformulating the Conjecture

Let us consider a refinement of the Bollobás-Eldridge Conjecture.

Conjecture : For a fixed $0 \leq \epsilon \leq 1$.
If $(\Delta_1 + 1)(\Delta_2 + 1) \leq \frac{n}{2}(1 + \epsilon) + 1$, then G_1 and G_2 pack.

For $\epsilon = 0$, this is essentially the Sauer-Spencer Theorem, while $\epsilon = 1$ gives the Bollobás-Eldridge conjecture.

For any $\epsilon > 0$ this would improve the Sauer-Spencer result (in a different way than Theorem 1).

Towards the Bollobás-Eldridge Conjecture

Theorem 2 [Kaul + Kostochka + Yu, *Combinatorica 2008+*]:
For $\epsilon = 0.2$, and $\Delta_1, \Delta_2 \geq 300$,
If $(\Delta_1 + 1)(\Delta_2 + 1) \leq \frac{n}{2}(1 + \epsilon) + 1$, then G_1 and G_2 pack.

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Theorem 2 [Kaul + Kostochka + Yu, *Combinatorica 2008+*]:
For $\epsilon = 0.2$, and $\Delta_1, \Delta_2 \geq 300$,
If $(\Delta_1 + 1)(\Delta_2 + 1) \leq \frac{n}{2}(1 + \epsilon) + 1$, then G_1 and G_2 pack.

In other words,

Theorem 2 [Kaul + Kostochka + Yu, *Combinatorica 2008+*]:
For $\Delta_1, \Delta_2 \geq 300$,
If $(\Delta_1 + 1)(\Delta_2 + 1) \leq (0.6)n + 1$, then G_1 and G_2 pack.

Some Ideas for the Proofs

We have to analyze the 'minimal' graphs that do not pack (under the given condition on Δ_1 and Δ_2).

(G_1, G_2) is a *critical pair* if G_1 and G_2 do not pack, but for each $e_1 \in E(G_1)$, $G_1 - e_1$ and G_2 pack, and for each $e_2 \in E(G_2)$, G_1 and $G_2 - e_2$ pack.

G_1 and G_2 do not pack, but removing one edge from either G_1 or G_2 allows them to pack.

Some Ideas for the Proofs

Each bijection $f : V_1 \rightarrow V_2$ generates a (multi)graph G_f, with

$$\mathbf{V(G_f) = \{(u, f(u)) \ : \ u \in V_1\}}$$

$$\mathbf{(u, f(u)) \leftrightarrow (u', f(u')) \Leftrightarrow uu' \in E_1} \text{ or } \mathbf{f(u)f(u') \in E_2}$$

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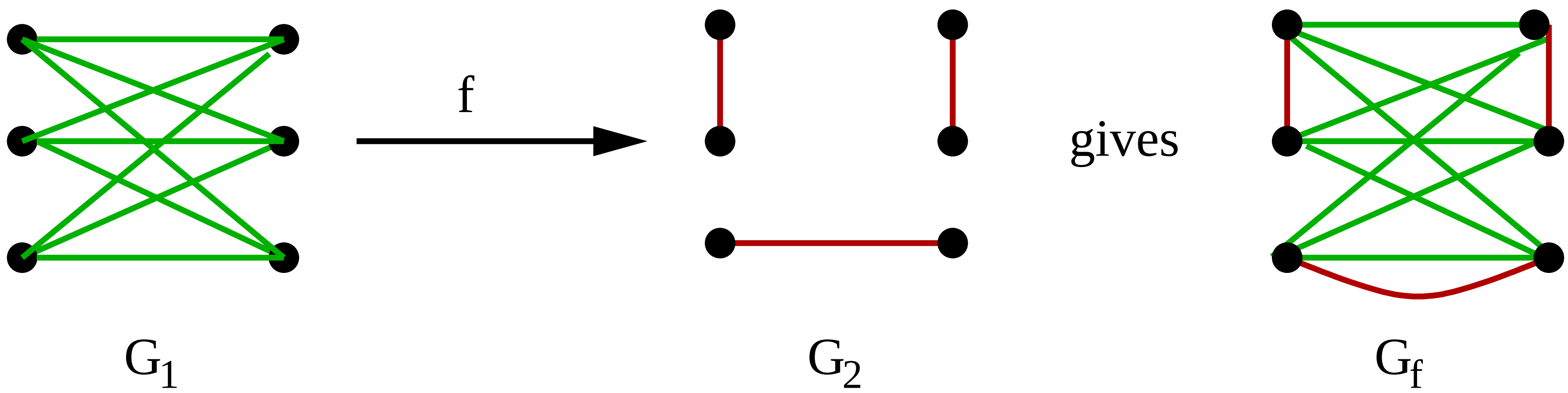

Some Ideas for the Proofs

$(u_1, \ldots, u_k)$-*switch* means replace f by f', with

$$f'(u) = \begin{cases} f(u) & , \quad u \neq u_1, u_2, \ldots, u_k \\ f(u_{i+1}) & , \quad u = u_i\,,\ 1 \leq i \leq k-1 \\ f(u_1) & , \quad u = u_k \end{cases}$$

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green-neighbors of $u_i \longrightarrow$ green-neighbors of u_{i-1}

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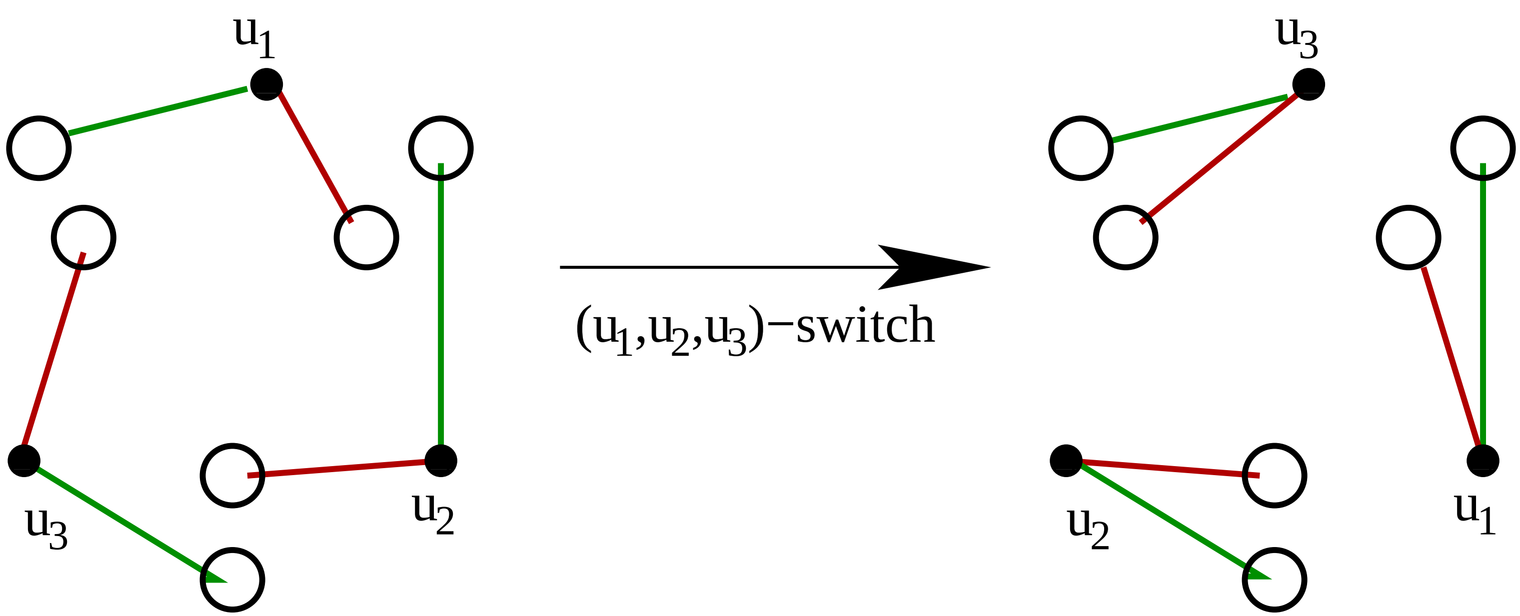

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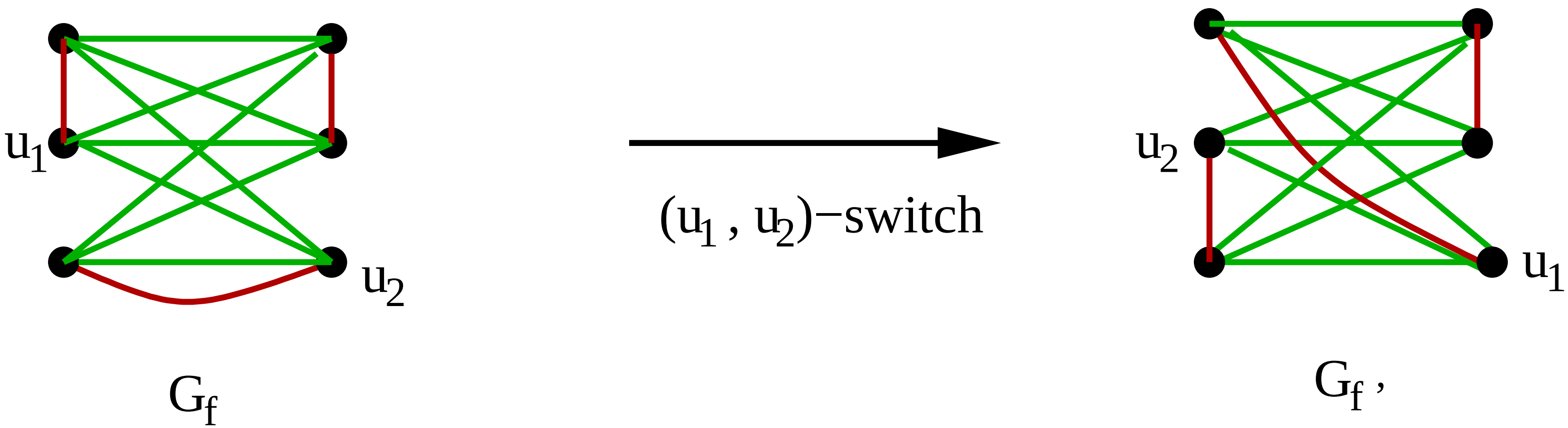

Some Ideas for the Proofs

An important structure that we utilize in our proof is -

$(u_1, u_2; 1, 2)$-*link* is a path of length two (in G_f) from u_1 to u_2 whose first edge is in E_1 and the second edge is in E_2.

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A green-red path of length two from u_1 to u_2.

For $e \in E_1$, an e-*packing* (*quasi-packing*) of (G_1, G_2) is a bijection f between V_1 and V_2 such that e is the only edge in E_1 that shares its incident vertices with an edge from E_2.

Such a packing exists for every edge e in a critical pair.

Outline of the Proof of Theorem 2

Key Lemma : Let $u_1, \ldots, u_k$ be vertices of G. If

- for any i, there is no red-green path from u_i to u_{i+1}, and
- for $1 \leq i < j \leq k$, if $u_i u_j$ is a red edge, then $u_{i+1} u_{j+1}$ is either a red edge or is not an edge.

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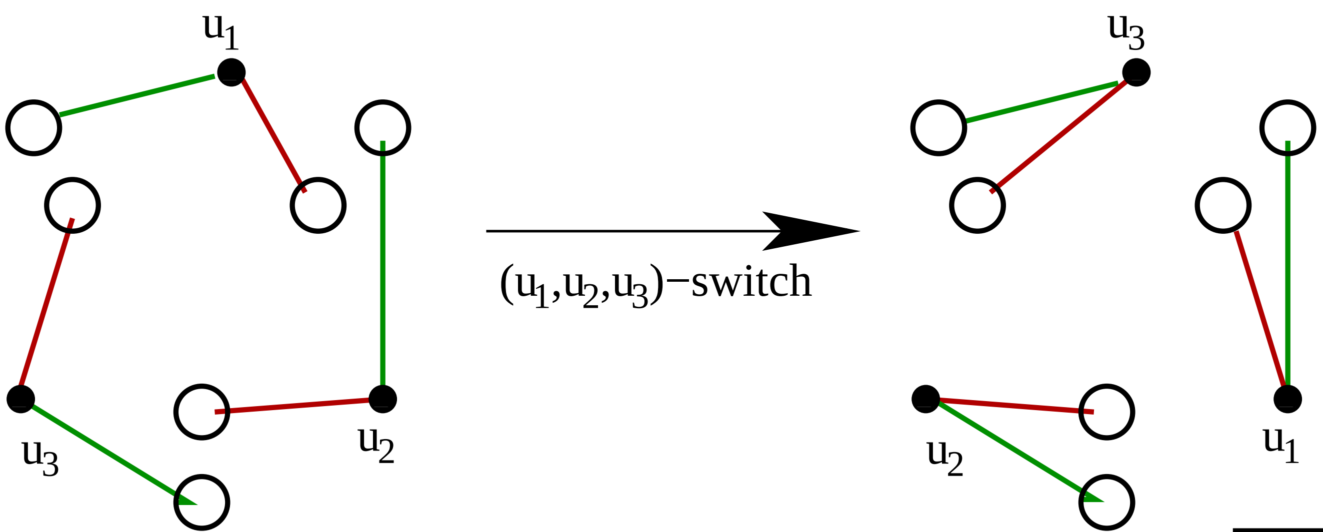

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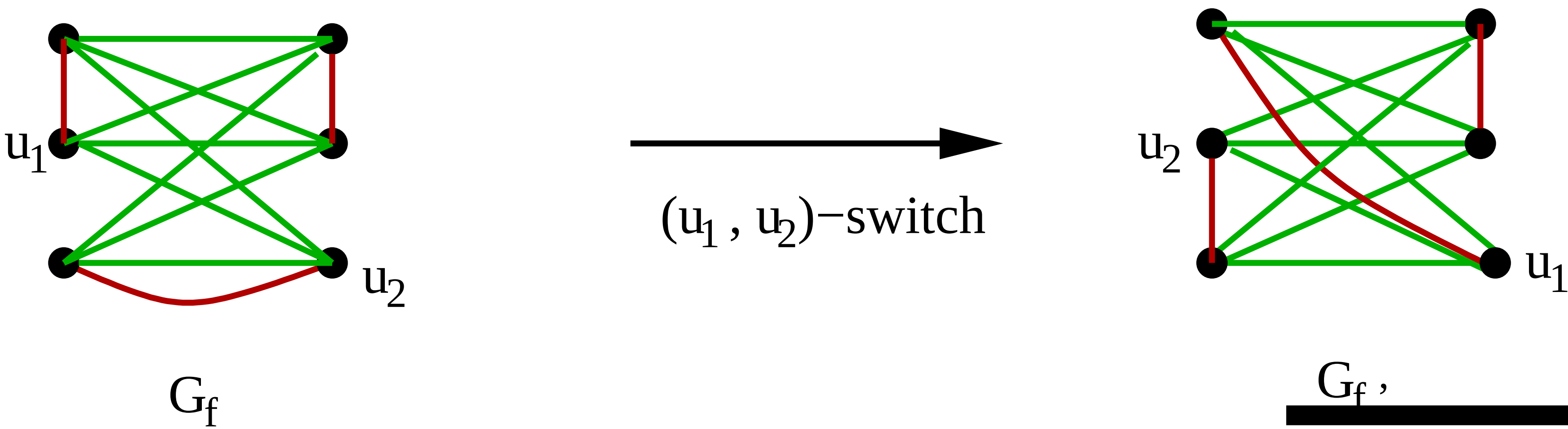

Using the Key Lemma

Consider a critical pair (G_1, G_2).

There is a bijection between $V(G_1)$ and $V(G_2)$ with exactly one conflicting edge.

Why is the Key Lemma useful?

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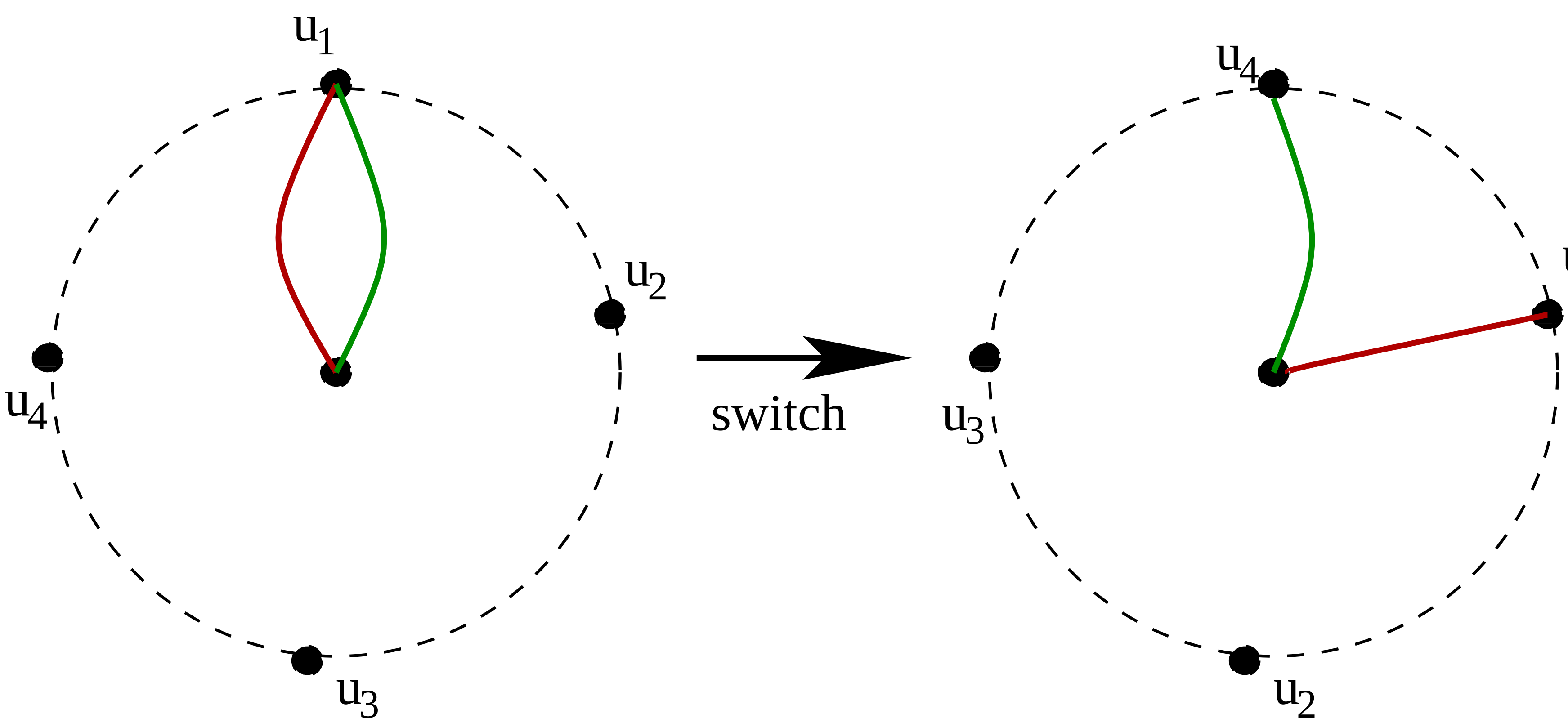
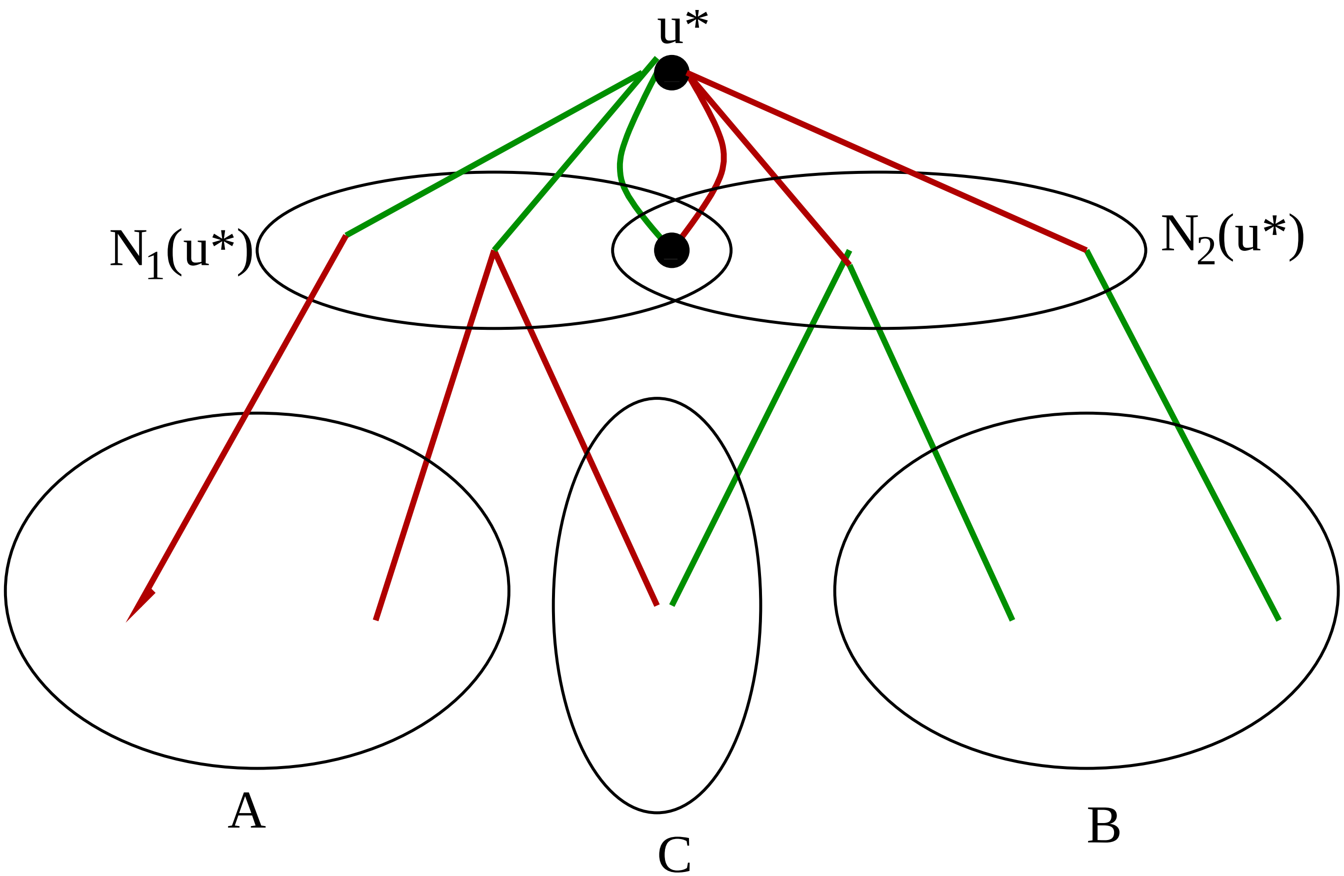


Structure of Counterexamples I



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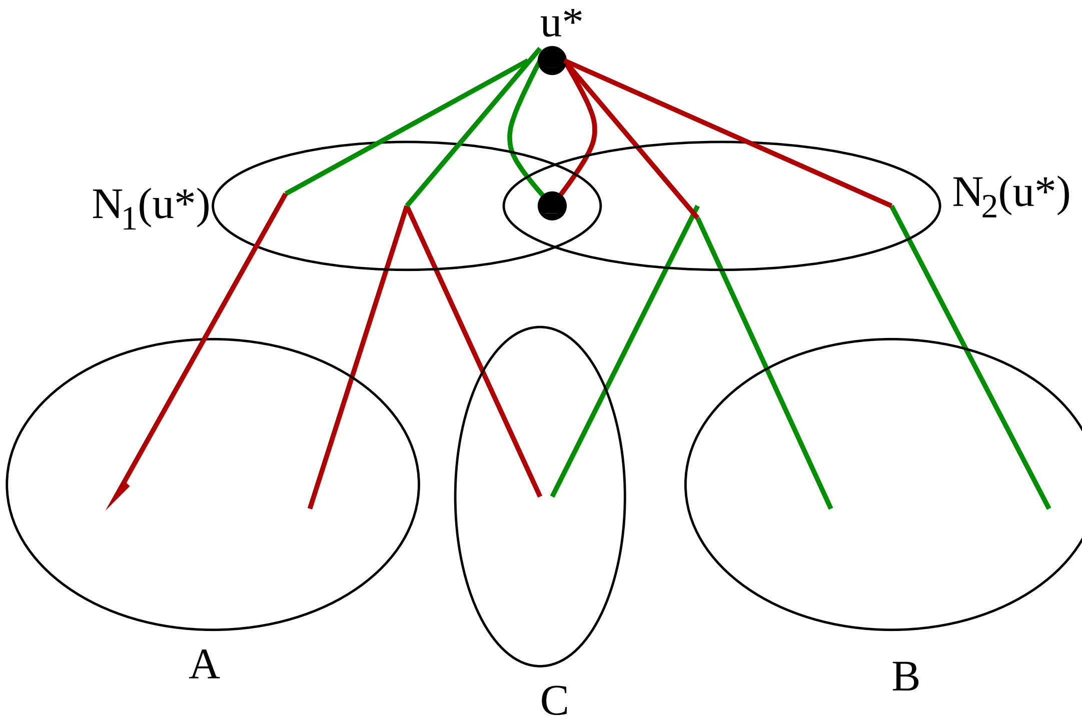


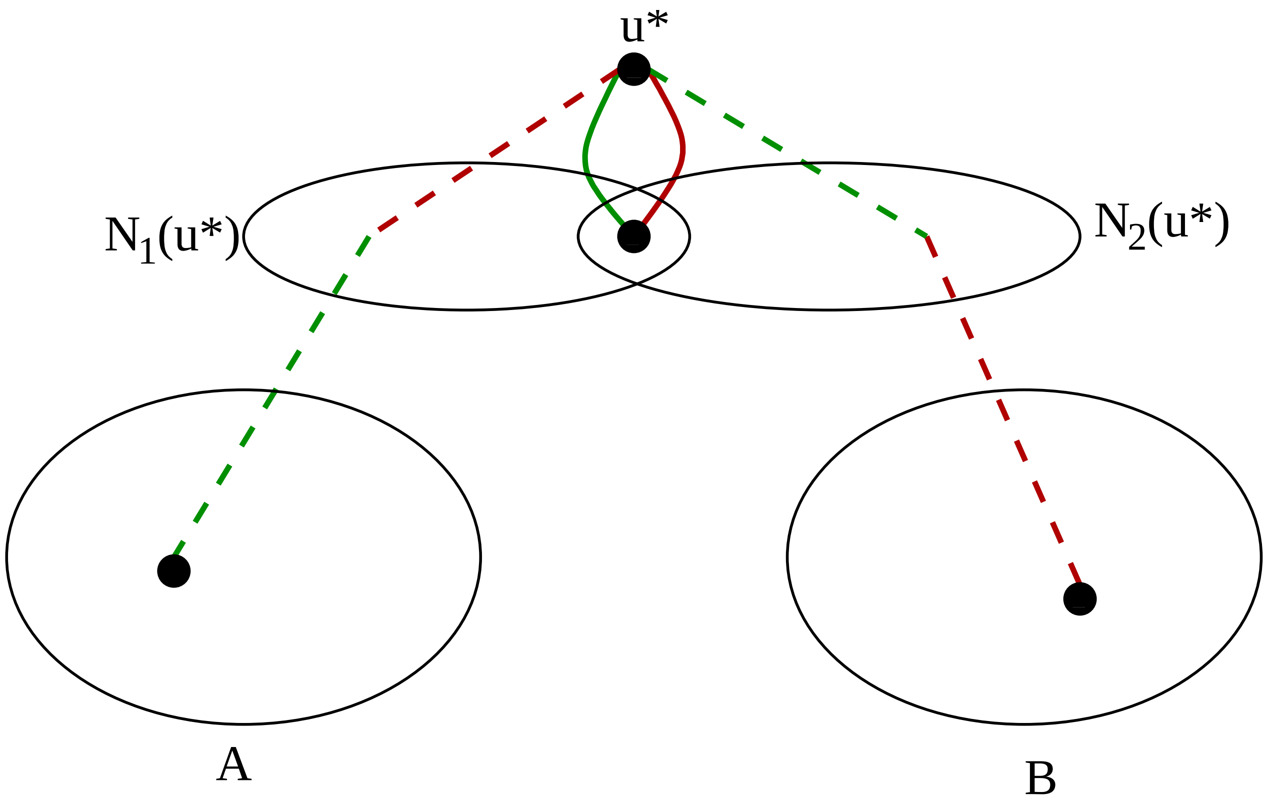
A = vertices with only green-red paths from u^*

B = vertices with only red-green paths from u^*

C = vertices with both types of paths from u^*

$|A| \geq n(1-\epsilon)$, $|B| \geq n(1-\epsilon)$, $|C| \leq n\epsilon$.

Structure of Counterexamples II



No red-green paths from u^* to A.

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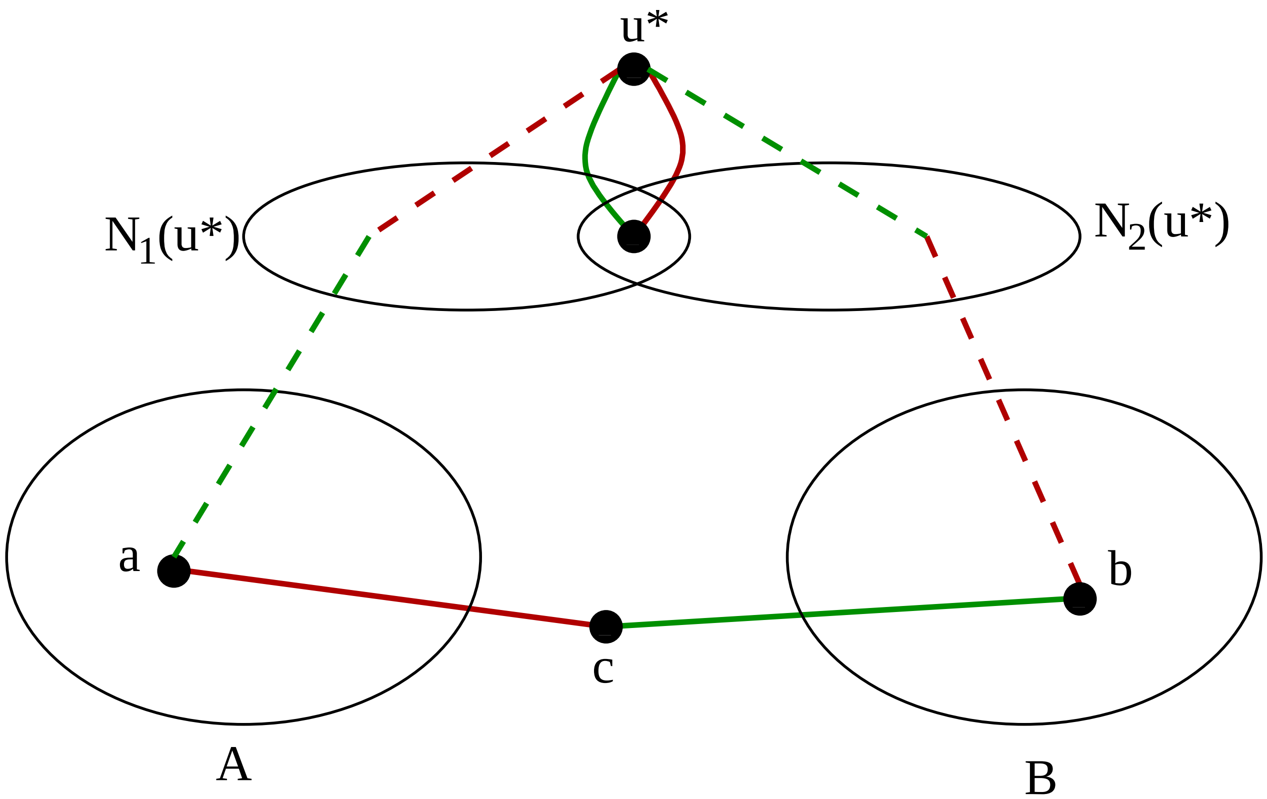
Structure of Counterexamples II



Unique red-green paths from A to B.

$$(u^*, a, c, b) - switch$$

The Primary Inequality

Let N be the number of pairs of vertices in $A \times B$ with *exactly one* red-green path between them.

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Compare the lower bound and the upper bound of N.
$|A|\,|B| - |A|(\Delta_1\Delta_2 - |B|) \leq M\Delta_1\Delta_2$
Get an inequality for ϵ, leading to a contradiction.

Need an upper bound on M !

Outline of the Proof of Theorem 1

Theorem 1 [Kaul + Kostochka, *CPC 2007*]:
If $\Delta_1\Delta_2 \leq \frac{1}{2}n$, then
G_1 and G_2 do not pack if and only if
one of G_1 and G_2 is a perfect matching and the other
either is $K_{\frac{n}{2},\frac{n}{2}}$ with $\frac{n}{2}$ odd or contains $K_{\frac{n}{2}+1}$.

Outline of the Proof of Theorem 1

The Key Lemma – this is the idea of the proof of the Sauer-Spencer Theorem.

Lemma 1 : Let (G_1, G_2) be a critical pair and $2\Delta_1\Delta_2 \leq n$. Given any $e \in E_1$, in a $e-$packing of (G_1, G_2) with $e = u_1u_1'$, the following statements are true.

(i) For every $u \neq u_1'$, there exists either a unique $(u_1, u; 1, 2)-$link or a unique $(u_1, u; 2, 1)-$link,

(ii) there is no $(u_1, u_1'; 1, 2)-$link or $(u_1, u_1'; 2, 1)-$link,

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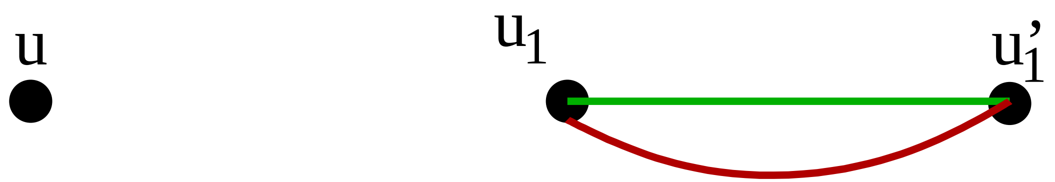

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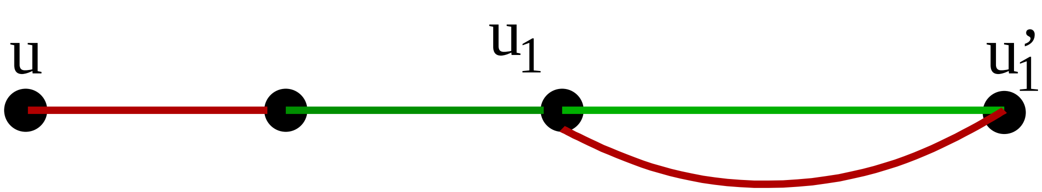

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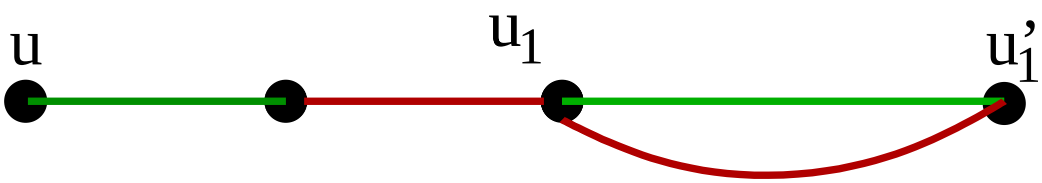

Outline of the Proof of Theorem 1

Lemma 2 : If $2\Delta_1\Delta_2 = n$ and (G_1, G_2) is a critical pair, then every component of G_i is either K_{Δ_i,Δ_i} with Δ_i odd, or an isolated vertex, or K_{Δ_i+1}, $i = 1, 2$.

Lemma 2 allows us to settle the case of : Δ_1 or $\Delta_2 = 1$.

If $\Delta_2 = 1$, i.e., G_2 is a matching. Then $\Delta_1 = \frac{n}{2}$.

If G_1 contains K_{Δ_1,Δ_1}, then simply $G_1 = K_{\frac{n}{2},\frac{n}{2}}$.
$K_{\frac{n}{2},\frac{n}{2}}$ cannot pack with a matching iff the matching is perfect and $\frac{n}{2}$ is odd.

If G_1 consists of $K_{\frac{n}{2}+1}$ and $\frac{n}{2} - 1$ isolated vertices, then it does not pack with a matching iff the matching is perfect.

Outline of the Proof of Theorem 1

Now, we have to give a packing for all remaining pairs of graphs, to eliminate their possibility.

The following Lemma says K_{Δ_1,Δ_1} exists only when K_{Δ_2,Δ_2} does, and vice-versa.

Lemma 3 : Let $\Delta_1, \Delta_2 > 1$ and $2\Delta_1\Delta_2 = n$. If (G_1, G_2) is a critical pair and the conflicted edge in a quasi-packing belongs to a component H of G_2 isomorphic to K_{Δ_2,Δ_2}, then every component of G_1 sharing vertices with H is K_{Δ_1,Δ_1}.

Now, we pack such graphs.

Outline of the Proof of Theorem 1

Lemma 4 : Suppose that $\Delta_1, \Delta_2 > 1$ and odd, and $2\Delta_1\Delta_2 = n$. If G_1 consists of Δ_2 copies of K_{Δ_1,Δ_1} and G_2 consists of Δ_1 copies of K_{Δ_2,Δ_2}, then G_1 and G_2 pack.

Outline of the Proof of Theorem 1

Lets eliminate the only remaining possibility.

Lemma 5 : Let $\Delta_1, \Delta_2 > 1$ and $2\Delta_1\Delta_2 = n$. If every non-trivial component of G_i is K_{Δ_i+1}, $i = 1, 2$, then G_1 and G_2 pack.

This would complete the proof of Theorem 1.