# New Results on Graph Packing

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# Introduction

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be graphs of order at most $n$, with maximum degree $\Delta_1$ and $\Delta_2$, respectively.

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We may assume $|V_1| = |V_2| = n$ by adding isolated vertices.

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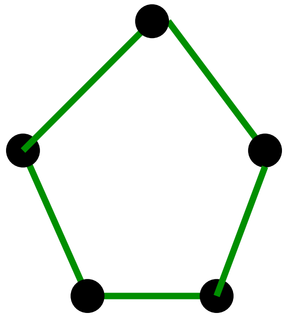
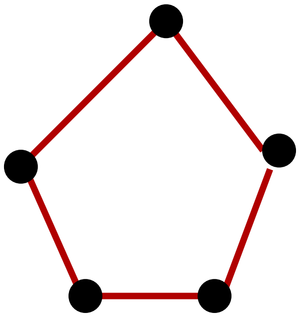
such that the images of the edge sets do not intersect.

- there exists a bijection $V_1 \leftrightarrow V_2$ such that $e \in E_1 \Rightarrow e \notin E_2$.
- $G_1$ is a subgraph of $\overline{G_2}$.

# Examples and Non-Examples



$C_5$



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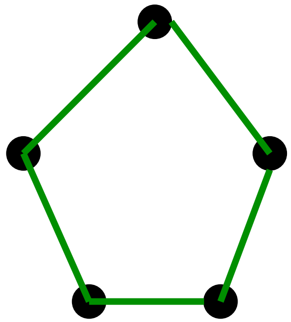

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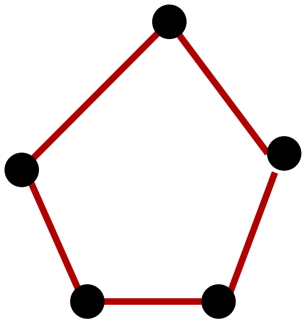

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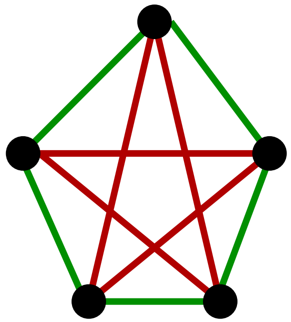

Packing

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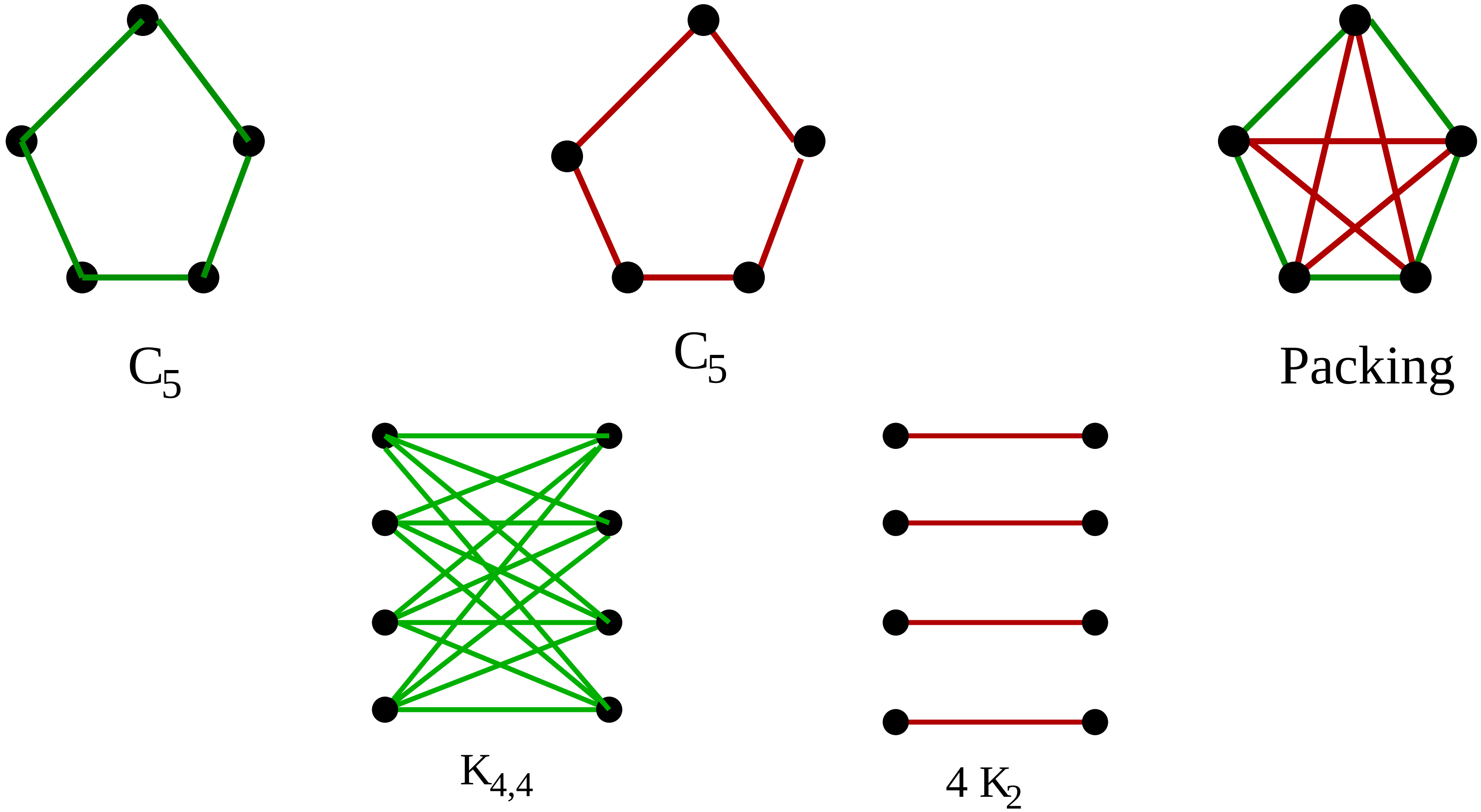

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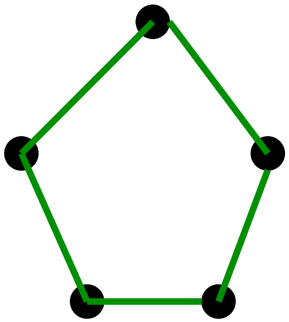

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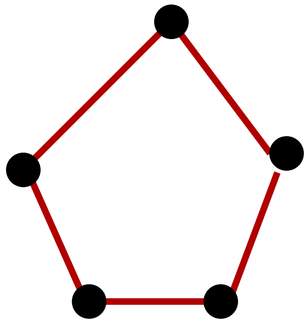

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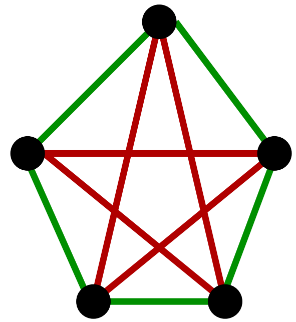

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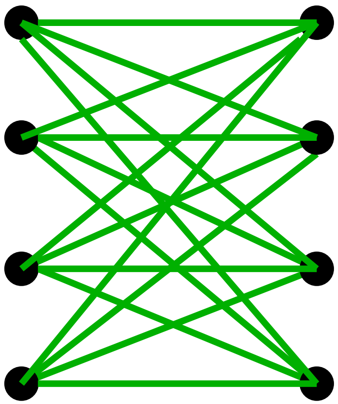
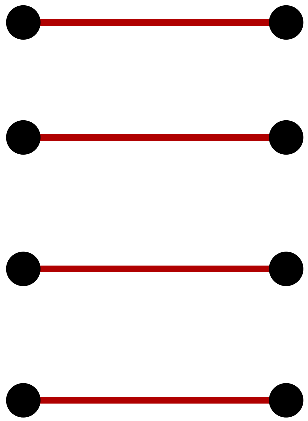
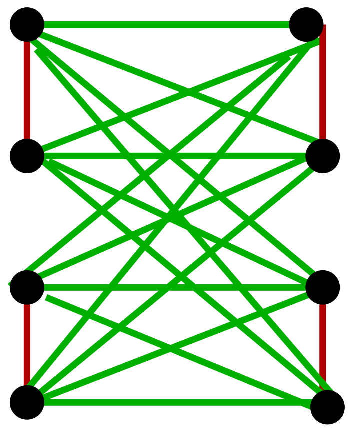


$K_{4,4}$



$4\ K_2$



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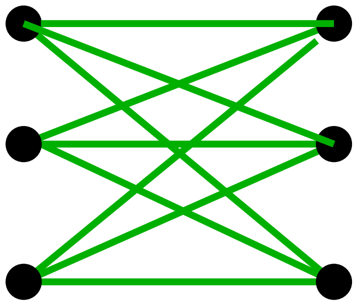
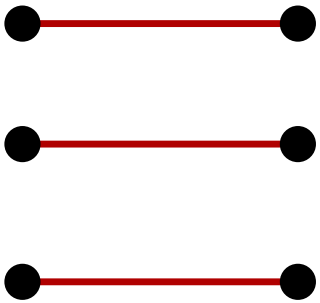


$K_{3,3}$



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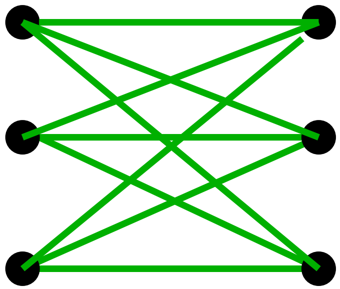

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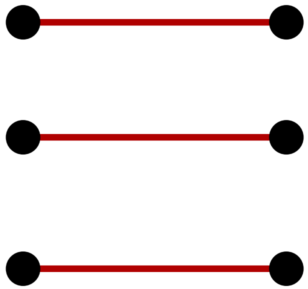

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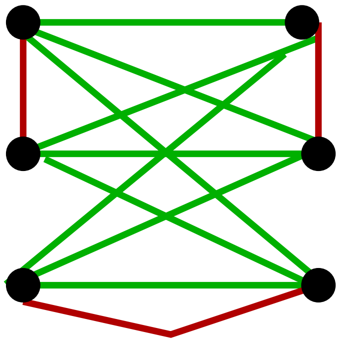

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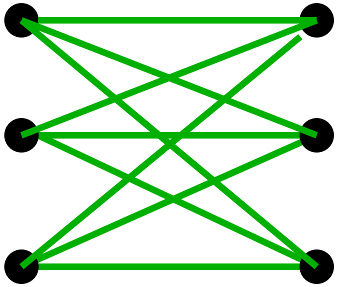

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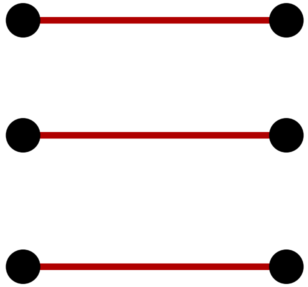

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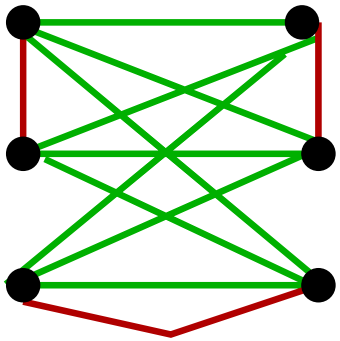

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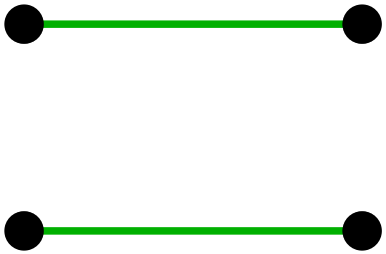
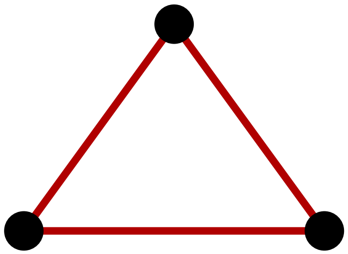


$2\ K_2$



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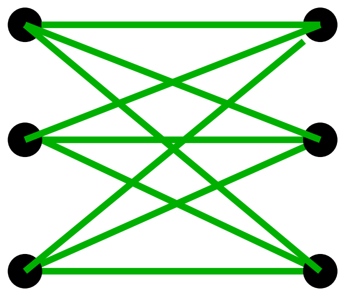

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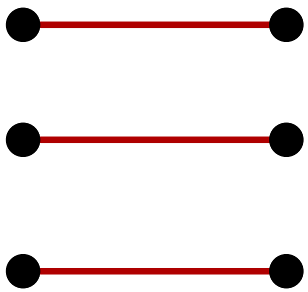

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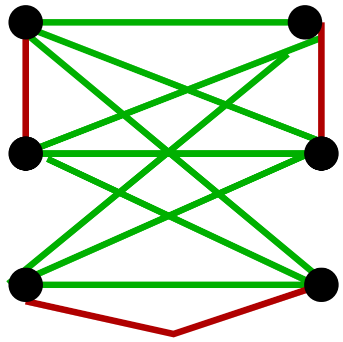

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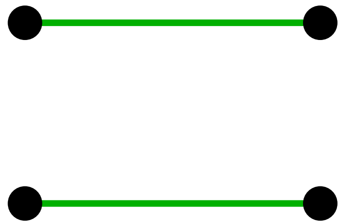

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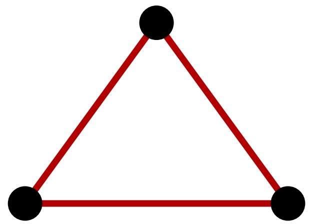

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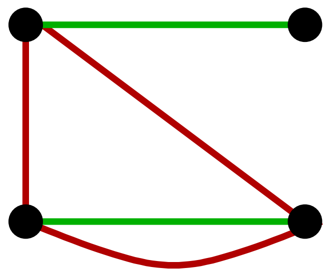

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- Turan-type problems (forbidden subgraphs).
- Ramsey-type problems.
- “most” problems in Extremal Graph Theory.

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Erdős-Sos Conjecture : Let $G$ be a graph of order $n$ and $T$ be a tree of size $k$.
If $e(G) < \frac{1}{2}n(n-k)$ then $T$ and $G$ pack.

# Sauer and Spencer's Packing Theorem

Theorem [Sauer + Spencer, 1978] :
If $2\Delta_1\Delta_2 < n$ , then $G_1$ and $G_2$ pack.

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This is sharp.

For $n$ even.

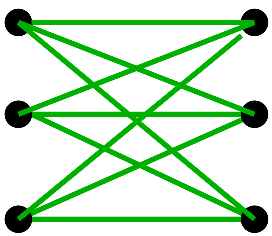
$G_1 = \frac{n}{2}K_2$ , a perfect matching on $n$ vertices.

$G_2 \supseteq K_{\frac{n}{2}+1}$, or

$G_2 = K_{\frac{n}{2},\frac{n}{2}}$ with $\frac{n}{2}$ odd.

Then, $2\Delta_1\Delta_2 = n$, and $G_1$ and $G_2$ do not pack.

# Sauer and Spencer's Packing Theorem



$K_{3,3}$

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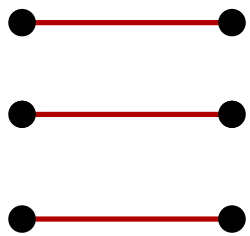
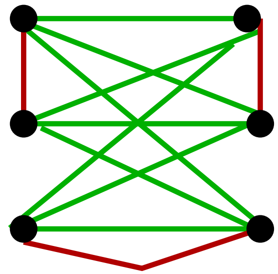


$3\ K_2$

$G_2 = \frac{n}{2} K_2$



No Packing

$2\ K_2$

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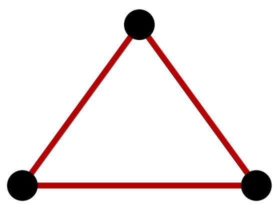

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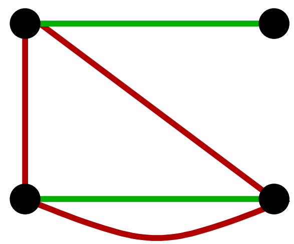

No Packing

# Extending the Sauer-Spencer Theorem

Theorem 1 [Kaul + Kostochka, 2005]:
If $2\Delta_1\Delta_2 \leq n$, then
$G_1$ and $G_2$ do not pack if and only if
one of $G_1$ and $G_2$ is a perfect matching and the other
either is $K_{\frac{n}{2},\frac{n}{2}}$ with $\frac{n}{2}$ odd or contains $K_{\frac{n}{2}+1}$.

This result characterizes the extremal graphs for the Sauer-Spencer Theorem.

To appear in *Combinatorics, Probability and Computing*.

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This result can also be thought of as a small step towards the well-known Bollobás-Eldridge conjecture.

Bollobás-Eldridge Graph Packing Conjecture :
If $(\Delta_1 + 1)(\Delta_2 + 1) \leq n + 1$ then $G_1$ and $G_2$ pack.

# Bollobás-Eldridge Conjecture

Bollobás-Eldridge Graph Packing Conjecture [1978] :
If $(\Delta_1 + 1)(\Delta_2 + 1) \leq n + 1$ then $G_1$ and $G_2$ pack.

If true, this conjecture would be sharp, and would be a considerable extension of the Hajnal-Szemerédi theorem on equitable colorings.

The conjecture has only been proved when

$\Delta_1 \leq 2$ [Aigner + Brandt (1993), and Alon + Fischer (1996)], or

$\Delta_1 = 3$ and n is huge [Csaba + Shokoufandeh + Szemerédi (2003)].

# Reformulating the Conjecture

Let us consider a refinement of the Bollobás-Eldridge Conjecture.

Conjecture : For a fixed $0 \le \epsilon \le 1$.
If $(\Delta_1 + 1)(\Delta_2 + 1) \le \frac{n}{2}(1+\epsilon) + 1$, then $G_1$ and $G_2$ pack.

For $\epsilon = 0$, this is essentially the Sauer-Spencer Theorem, while $\epsilon = 1$ gives the Bollobás-Eldridge conjecture.

For any $\epsilon > 0$ this would improve the Sauer-Spencer result (in a different way than Theorem 1).

# Towards the Bollobás-Eldridge Conjecture

Theorem 2 [Kaul + Kostochka + Yu, 2005+]:
For $\epsilon = 0.2$, and $\Delta_1, \Delta_2 \geq 400$,
If $(\Delta_1 + 1)(\Delta_2 + 1) \leq \frac{n}{2}(1 + \epsilon) + 1$, then $G_1$ and $G_2$ pack.

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In other words,

Theorem 2 [Kaul + Kostochka + Yu, 2005+]:
For $\Delta_1, \Delta_2 \geq 400$,
If $(\Delta_1 + 1)(\Delta_2 + 1) \leq (0.6)n + 1$, then $G_1$ and $G_2$ pack.

This is work in progress.

# Some Proof Ideas for Theorem 1

Theorem 1 [Kaul + Kostochka, 2005]:
If $2\Delta_1\Delta_2 \leq n$, then
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one of $G_1$ and $G_2$ is a perfect matching and the other either is $K_{\frac{n}{2},\frac{n}{2}}$ with $\frac{n}{2}$ odd or contains $K_{\frac{n}{2}+1}$.

We have to analyze the 'minimal' graphs that do not pack (under the condition $2\Delta_1\Delta_2 \leq n$).

$(G_1, G_2)$ is a *critical pair* if $G_1$ and $G_2$ do not pack, but for each $e_1 \in E(G_1)$, $G_1 - e_1$ and $G_2$ pack, and for each $e_2 \in E(G_2)$, $G_1$ and $G_2 - e_2$ pack.

# Some Proof Ideas for Theorem 1

Each bijection $f : V_1 \to V_2$ generates a (multi)graph $G_f$, with

$$\mathbf{V(G_f) = \{(u, f(u)) \ : \ u \in V_1\}}$$

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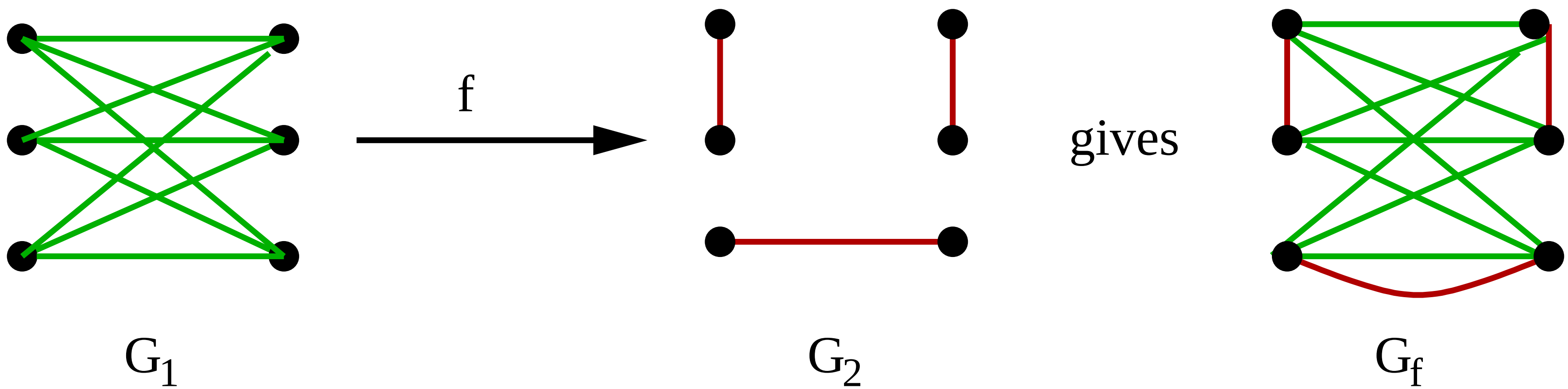

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$(u_1, u_2)$-*switch* means replace $f$ by $f'$, with

$$f'(u) = \begin{cases} f(u) & , \quad u \neq u_1, u_2 \\ f(u_2) & , \quad u = u_1 \\ f(u_1) & , \quad u = u_2 \end{cases}$$

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$2$-neighbors of $u_1 \longleftrightarrow 2$-neighbors of $u_2$

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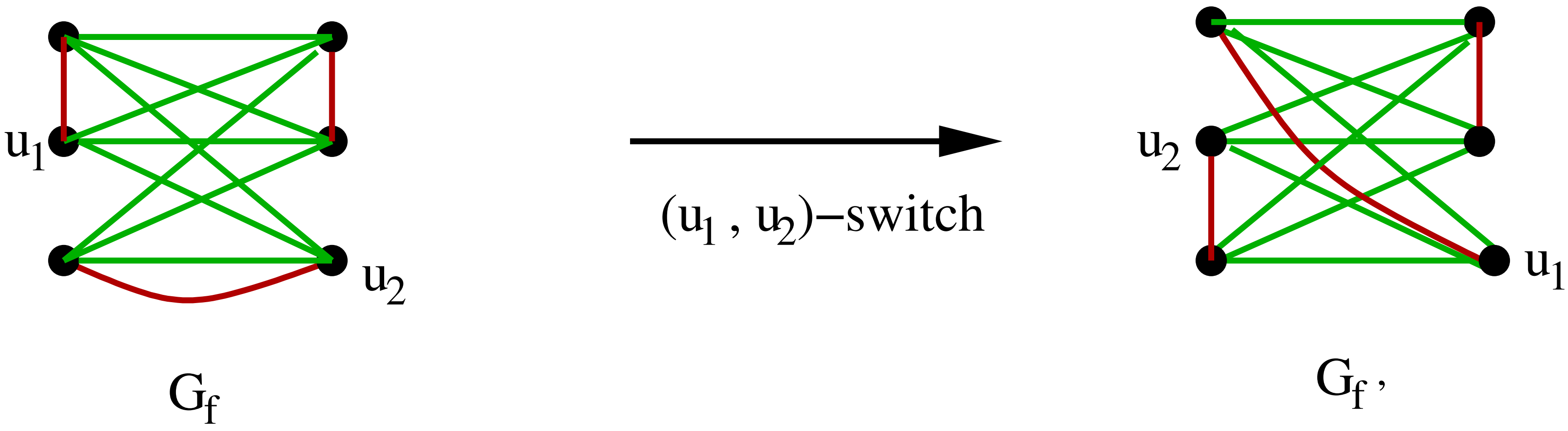


# Some Proof Ideas for Theorem 1

An important structure that we utilize in our proof is -

$(u_1, u_2; 1, 2)$-*link* is a path of length two (in $G_f$) from $u_1$ to $u_2$ whose first edge is in $E_1$ and the second edge is in $E_2$.

For $e \in E_1$, an $e$-*packing* (*quasi-packing*) of $(G_1, G_2)$ is a bijection $f$ between $V_1$ and $V_2$ such that $e$ is the only edge in $E_1$ that shares its incident vertices with an edge from $E_2$.

Such a packing exists for every edge $e$ in a critical pair.

# Outline of the Proof of Theorem 1

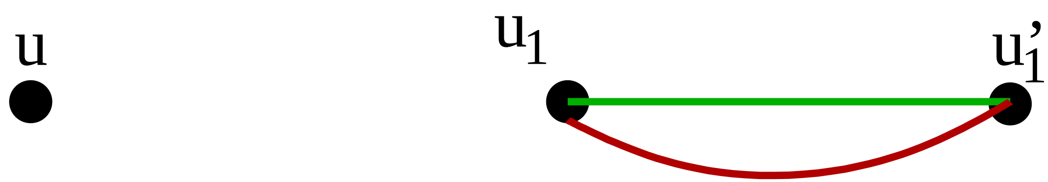
The main tool –

Lemma 1 : Let $(G_1, G_2)$ be a critical pair and $2\Delta_1\Delta_2 \leq n$. Given any $e \in E_1$, in a $e$–packing of $(G_1, G_2)$ with $e = u_1u_1'$, the following statements are true.

(i) For every $u \neq u_1'$, there exists either a unique $(u_1, u; 1, 2)$–link or a unique $(u_1, u; 2, 1)$–link,

(ii) there is no $(u_1, u_1'; 1, 2)$–link or $(u_1, u_1'; 2, 1)$–link,

(iii) $\mathbf{2deg_{G_1}(u_1)deg_{G_2}(u_1) = n}$.



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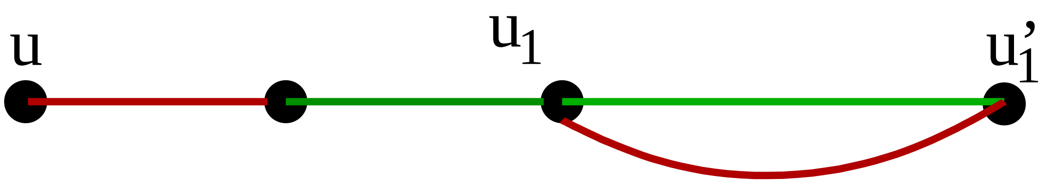
(i) For every $u \neq u_1'$, there exists either a unique $(u_1, u; 1, 2)-$link or a unique $(u_1, u; 2, 1)-$link,

(ii) there is no $(u_1, u_1'; 1, 2)-$link or $(u_1, u_1'; 2, 1)-$link,

(iii) $\mathbf{2deg_{G_1}(u_1)deg_{G_2}(u_1) = n}$.



# Outline of the Proof of Theorem 1

The main tool –

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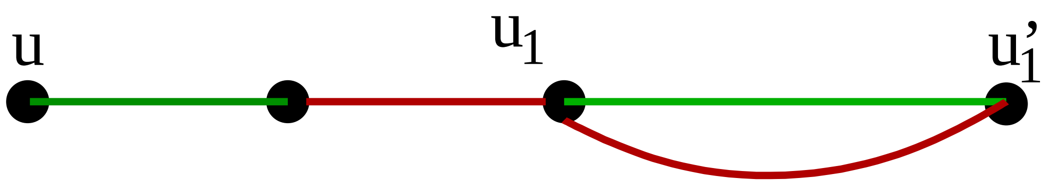

# Outline of the Proof of Theorem 1

Lemma 2 : If $2\Delta_1\Delta_2 = n$ and $(G_1, G_2)$ is a critical pair, then every component of $G_i$ is either $K_{\Delta_i,\Delta_i}$ with $\Delta_i$ odd, or an isolated vertex, or $K_{\Delta_i+1}$, $i = 1, 2$.

Lemma 2 allows us to settle the case of : $\Delta_1$ or $\Delta_2 = 1$.

Then, we have to give a packing for all remaining pairs of graphs, to eliminate their possibility.

# Outline of the Proof of Theorem 1

The following Lemma limits the possible remaining pairs of graphs.

Lemma 3 : Let $\Delta_1, \Delta_2 > 1$ and $2\Delta_1\Delta_2 = n$. If $(G_1, G_2)$ is a critical pair and the conflicted edge in a quasi-packing belongs to a component $H$ of $G_2$ isomorphic to $K_{\Delta_2,\Delta_2}$, then every component of $G_1$ sharing vertices with $H$ is $K_{\Delta_1,\Delta_1}$.

Now, we completely eliminate such graphs.

Lemma 4 : Suppose that $\Delta_1, \Delta_2 \geq 3$ and odd, and $2\Delta_1\Delta_2 = n$. If $G_1$ consists of $\Delta_2$ copies of $K_{\Delta_1,\Delta_1}$ and $G_2$ consists of $\Delta_1$ copies of $K_{\Delta_2,\Delta_2}$, then $G_1$ and $G_2$ pack.

# Outline of the Proof of Theorem 1

Now, lets eliminate the only remaining possibility.

Lemma 5 : Let $\Delta_1, \Delta_2 > 1$ and $2\Delta_1\Delta_2 = n$. If every non-trivial component of $G_i$ is $K_{\Delta_i+1}$, $i = 1, 2$, then $G_1$ and $G_2$ pack.

This would complete the proof of Theorem 1.