Walking through colored rooms to a fixed point

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IIT SIAM Lecture Series: My Favorite Theorem

Brouwer's Fixed Point Theorem

How can we guarantee a solution to the equation $f(x) = x$? Such a solution, if it exists, is called a fixed point of the function f.

f should be continuous ("no breaks") at the bare minimum, else no hope.

It turns out beyond continuity of f, what truly matters is the topology ("shape") of the domain and co-domain of f.

Theorem (Brouwer's Fixed Point Theorem 1912)

Let S be a compact and convex set in $\mathbb{R}^d$.
Every continuous function $f : S \to S$ has a fixed point.

I will focus on 2-dimensions, but ALL ideas generalize in straightforward fashion to d-dimensions.

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Compact and Convex Sets

- Convex set has the property a line segment between any two points in the set lies completely inside the set. Think of polygonal or circular regions in $\mathbb{R}^2$.

- Compact set is bounded ("does not go to infinity") and closed ("includes the boundary of the region"). Again think of polygonal or circular regions in $\mathbb{R}^2$.

Theorem (Brouwer's Fixed Point Theorem 1912)

Let Δ be a triangular region in $\mathbb{R}^2$.
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In d-dimensions, we use a Simplex ("polytope that generalizes triangle and tetrahedron (pyramid)"), $\{\lambda_1 \vec{e_1} + \ldots \lambda_{k+1} \vec{e_{k+1}} \mid \sum \lambda_i = 1 \text{ and } 0 \leq \lambda_i \leq 1\}$.

Since every compact and convex set in $\mathbb{R}^d$ is homeomorphic ("can be changed in a continuous manner from one region to the other, and vice-versa") to a simplex (or to a ball) in $\mathbb{R}^d$, its enough to just consider any one such nicer region.

- NOTE: with such minimal requirements, we are guaranteeing a fixed point for any reasonable function!

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Applications of Brouwer's FPT

- "You are here!" Take an ordinary map of a country and place it flat on ground inside that country. Then there exists a point on the map that is placed exactly on top of the same point in the country.

- "Stirred not shaken!" Stir a cocktail. When the liquid comes to rest, there will be a molecule that ends up at its original position.

- Many applications, including existence of solutions of initial value problems (Differential Equations), Perron-Frobenius theorem in Linear Algebra, a simple proof of notoriously hard-to-prove Jordan Curve Theorem ("every simple closed continuous curve divides the plane into two connected regions - inside and outside"), and much more.

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- John Nash won the Nobel prize for Economics for his proof of existence of (Nash) Equillibrium in non-cooperative n-person games. The proof fits in a single page as a direct consequence of Brouwer's Fixed point theorem. It was the main part of Nash's 26 page(!) PhD thesis. He is the only Mathematician to win both Nobel and Abel prizes.

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  A non-cooperative game studies the outcome of interaction between many decision makers (player), each acting according to her/his own strategy and benefit.

  John Nash's insight was that there exists a strategy for each player such that no player benefits from unilaterally changing her/his own strategy, the players are in "equilibrium".

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  This idea pervades modern economic thought and is applied in situations ranging from war and arms race (see mutually assured destruction, prisoner's dilemma), traffic flow (see Wardrop's principle), auctions (see auction theory), environmental regulation (see tragedy of commons), and much more.

WHY and HOW of Brouwer's Fixed Point Theorem

Why is Brouwer's Fixed Point Theorem true?

How can we find a fixed point?

Most proofs of Brouwer's FPT are existential/ non-constructive, including Brouwer's original proof. Which is ironic since Brouwer was the founder of Intuitionist School of constructivist foundations of mathematics.

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Sperner's Lemma on the sphere

- Draw a bunch of points on the surface of a ball.
- Connect the dots into triangles in any manner that you like. This forms a triangulated sphere.
- Color the dots using three colors, Red, Blue, Green, in any manner that you like.
- If there exists one RGB triangle (a triangle with all three colors), then Sperner's Lemma says that there must be another RGB triangle.

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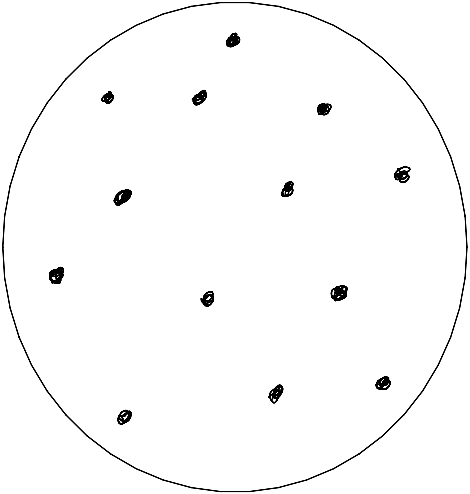
Colored Triangulation on a sphere



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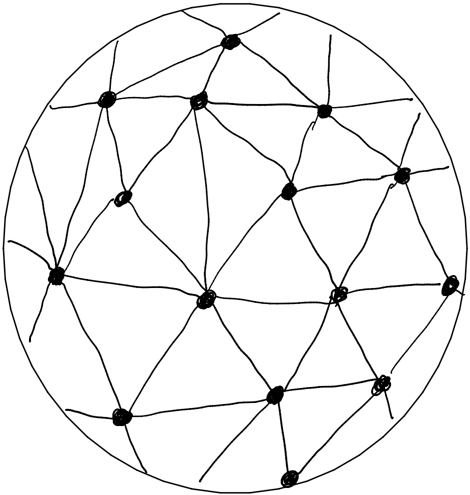

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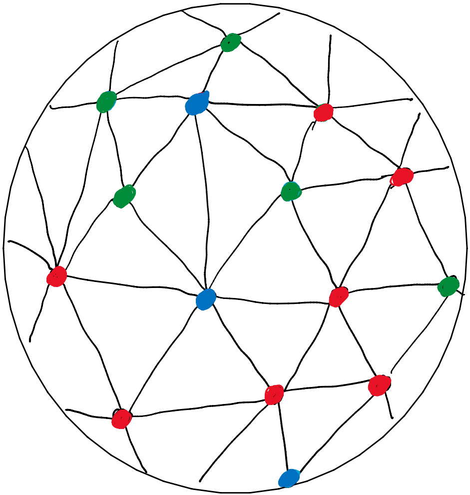

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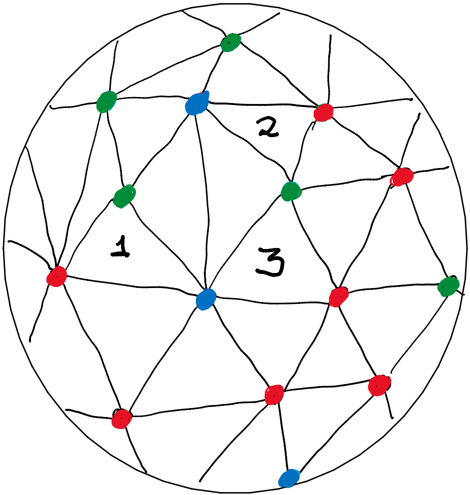
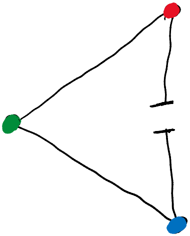


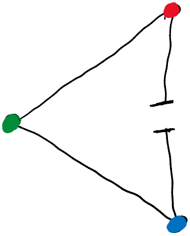
Walking through RB doors

- Think of each triangle as a room with 3 walls and a door in a wall with RB corners.



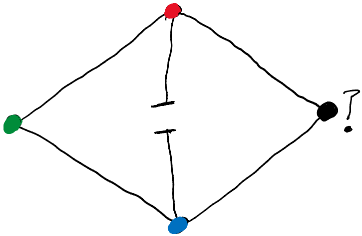
Walking through RB doors

- Think of each triangle as a room with 3 walls and a door in a wall with RB corners.
- In the given RGB room, there is exactly one RB door.



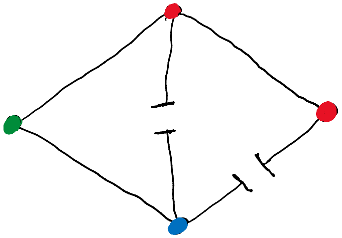
Walking through RB doors

- Starting from the given RGB room, start walking through the doors from room to room.



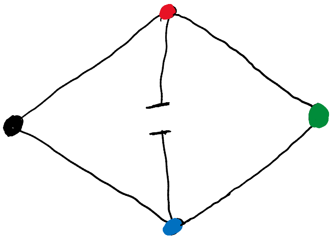
Walking through RB doors

- After entering a room from a RB door, there are only two options:
  Either there is another RB door and you use that to leave,

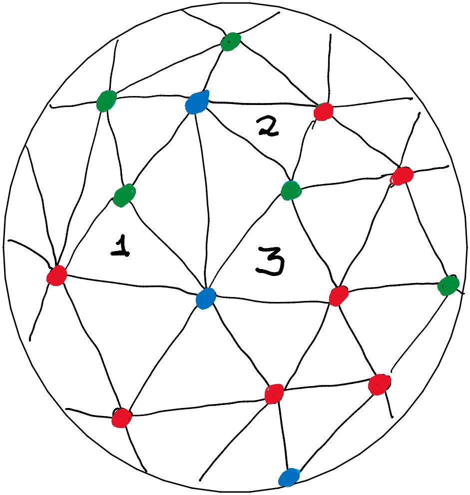


Walking through RB doors

- After entering a room from a RB door, there are only two options:
  Either there is another RB door and you use that to leave,
  Or there is no other RB door which means you have found another RGB room.



Walking through colored rooms



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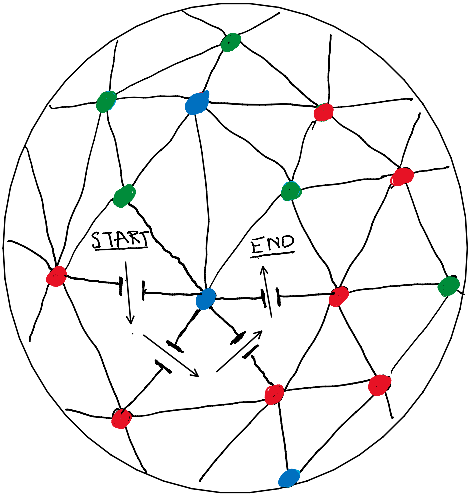

Correctness of Walking

- There are only finitely many rooms, so the walk must end somewhere, and that somewhere is another RGB room.

  Careful!

  Could we just keep walking cyclically?

- If it exists, consider the very first room we **revisited**.

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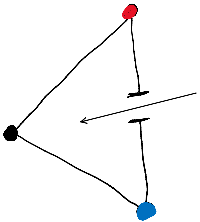
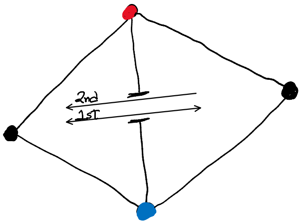


Correctness of Walking

- If it exists, consider the very **first** room we **revisited**.
- How did we enter/leave this room the first time?



If we used the same door the first time, then this was not the first room we revisited.

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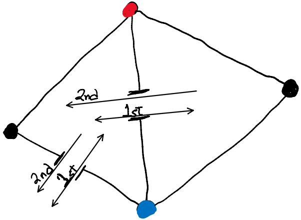

Even if this room has another door, this was not the first room we revisited.

Sperner's Lemma in the plane

- Consider a triangulation of any polygon in the plane.
- Color the vertices of the triangulation with Red, Blue, Green, with one condition:
- there are an odd number of RB edges (walls) on the boundary of the polygon.
  No conditions on the coloring in the interior of the polygon!
- Starting from outside the polygon and walking through RB doors as before will again guarantee we end up in a RGB room.
- Sperner's Lemma guarantees the existence of a RGB triangle in a colored triangulation of any polygon where the coloring satisfies the above condition.

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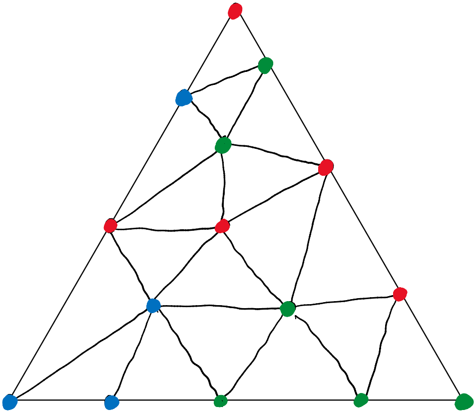

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Theorem (Sperner's Lemma 1928)

Every Sperner coloring of a triangulation of k-dimensional Simplex contains a cell colored with all $k + 1$ colors.

Think of $k = 2$ dimensions and a triangle.

Formal Proof of Sperner's Lemma

- MATH 100/454/553 students fill in the details!

- Define a graph with a vertex corresponding to each "face" (all triangles as well as the unbounded external region).
- Two vertices are adjacent if their faces share a common boundary with colors R and B on its ends (our doors :-).
- The degree of the vertex corresponding to the external face is odd. (Check!)
- A graph must have an even number of odd degree vertices.
- So there exists another (interior) vertex of odd degree.
- This vertex corresponds to a RGB face (triangle)! (Check!)

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Applications of Sperner's Lemma

- Constructive proof of Brouwer's fixed point theorem.

Applications of Sperner's Lemma

- **Fair Division.** There exists an "envy-free" division of a "cake" among *n* people, each with their own consistent preferences. [Look up Rental Harmony, Divide your Rent Fairly in NYT.]

- **Game of HEX.** Two players Alice and Bob take turns claiming empty cells in a $n \times n$ hexagonal grid.
  Alice wins if she is able to claim a path of cells from the left end to the right end of the grid. Bob wins if he has a path from the top to the bottom of the grid.
  John Nash (1948) proved that HEX always has a winner.

- **Monsky's Theorem.** A square can not be subdivided into an odd number of triangles of equal area.

- **PPAD complexity class.** Sperner's lemma along with Brouwer's FPT, Nash Equiilibrium, Market Equillibrium, Fair Division problems, etc. are all PPAD-complete.

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Brouwer's from Sperner

Theorem (Brouwer's Fixed Point Theorem 1912)

Let Δ be a triangular region in $\mathbb{R}^2$.
Every continuous function $f : \Delta \to \Delta$ has a fixed point.

Think of the triangle Δ as being in $\mathbb{R}^3$ with corners given by $A = \vec{e_1} = (1, 0, 0)$, $B = \vec{e_2} = (0, 1, 0)$, $C = \vec{e_3} = (0, 0, 1)$.

$\Delta = \{\lambda_1 \vec{e_1} + \lambda_2 \vec{e_2} + \lambda_3 \vec{e_3} \mid \sum \lambda_i = 1 \text{ and } 0 \leq \lambda_i \leq 1\}$

So $f(x) = (f_1(x), f_2(x), f_3(x))$ with each $f_i : \Delta \to [0, 1]$ and $f_1(x) + f_2(x) + f_3(x) = 1$.

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Color each vertex p of the triangulation as follows:

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- It is easy to show that this is a Sperner coloring with corners $A = (1, 0, 0)$ colored by 1, $B = (0, 1, 0)$ colored by 2, $C = (0, 0, 1)$ colored by 3.

- Sperner's Lemma tells us that this triangle (with corners colored 1, 2 and 3) contains another triangle with corners colored 1, 2, and 3.

- We can keep repeating this argument within a smaller triangle, which will push us closer to a fixed point as the diameter of the triangulations gets smaller.
  This can be made precise using the Bolzano-Weirstrass Theorem (sequential compactness).

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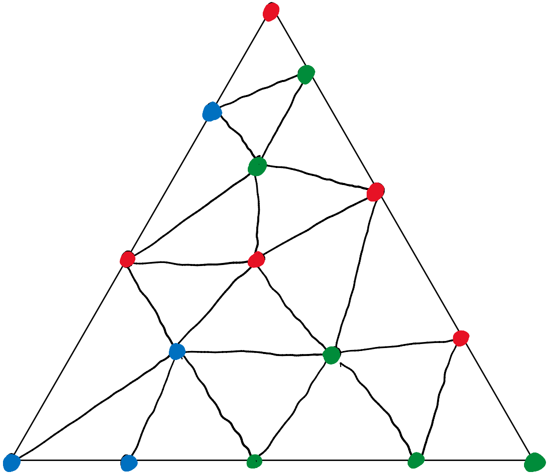

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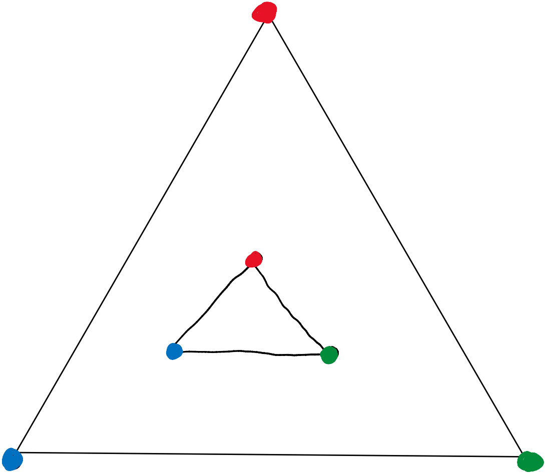

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