Conflict-free Allocation of Limited Resources

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Some Mysterious Formulae

With my graduate student Jeff Mudrock, we recently discovered some interesting looking formulae, as a corollary to a more general theorem.

Corollary

$$\chi_\ell(C_{2l+1} \square K_{1,s}) = \begin{cases} 3 & \text{if } s < 2^{2l+1} - 2 \\ 4 & \text{if } s \geq 2^{2l+1} - 2. \end{cases}$$

Corollary

$$\chi_\ell(K_n \square K_{1,s}) = \begin{cases} n & \text{if } s < n! \\ n+1 & \text{if } s \geq n!. \end{cases}$$

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$$\chi_\ell((K_n \vee C_{2l+1}) \square K_{1,s}) = \begin{cases} n+3 & \text{if } s < \frac{1}{3}(n+3)!(4^l - 1) \\ n+4 & \text{if } s \geq \frac{1}{3}(n+3)!(4^l - 1). \end{cases}$$

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Our story starts with ...

Allocation of classrooms (limited resource) to courses (entities) so that courses with overlapping-time (conflict) are given different rooms.

Allocation of radio channels (limited resource) to radio stations (entities) so that stations with proximity interference (conflict) are given different channels.

Allocation of colors (limited resource) to regions (entities) in a map so that regions with common boundary (conflict) are given different colors.

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Coloring a Graph

- Conflict-free partition of “entities” under study.
- Entities ↔ Vertices.
  Conflicts ↔ Edges.
- Color vertices so that any vertices with an edge between them must get different colors.

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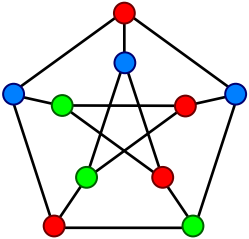

Chromatic number of a Graph

- Entities $\leftrightarrow$ Vertices.
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- Color vertices so that any vertices with an edge between them must get different colors.
- Partition the set of all vertices into independent sets (edge-free sets/ “conflict-free” sets)

- Resources $\leftrightarrow$ Colors.

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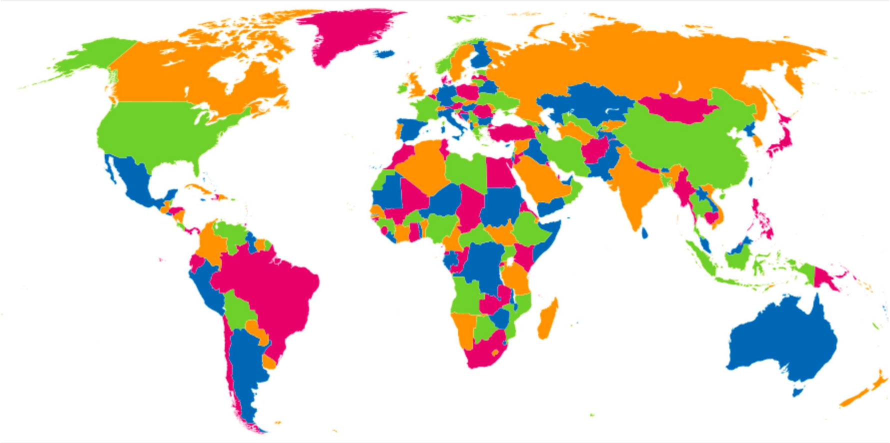
Four Colors for the World Map



(c) Wikimedia

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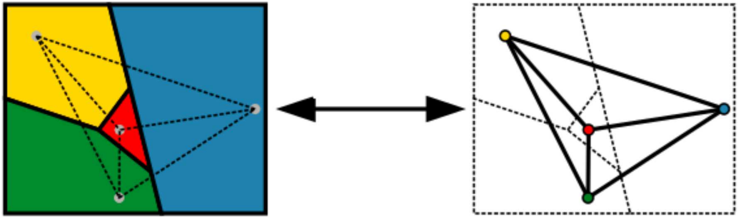

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List Coloring a Graph

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- In many applications, available resources might vary from entity to entity.
  That is, each vertex might have its own list of colors (resources) available to it.
  Think of Radio stations and Radio frequencies, or Courses and Classrooms, etc.
- Minimum number of colors needed for such a list coloring (no matter what colors those lists contain) is called the list chromatic number $\chi_\ell(G)$ of the graph G.

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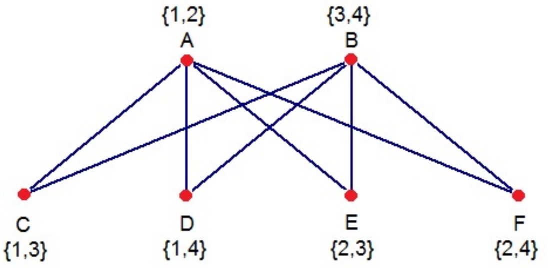
Chromatic Number and List Chromatic Number

The gap between $\chi(G)$ and $\chi_\ell(G)$ can be arbitrarily large.

$\chi(K_{m,n}) = 2$.

Suppose that we wish to find the list chromatic number of $K_{2,4}$. Consider the following list assignment:



The above list assignment shows that $\chi_l(K_{2,4}) > 2$.

In fact, $\chi_\ell(K_{m,m^m}) = m + 1$.

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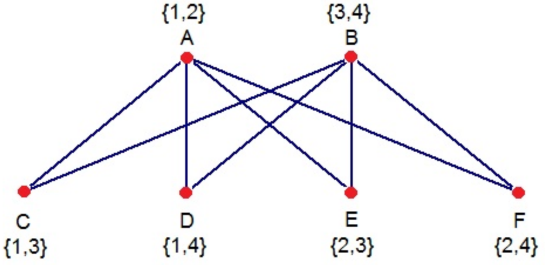

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Chromatic Choosability

- List coloring (finding $\chi_\ell(G)$) is a much harder problem than usual coloring (finding $\chi(G)$).
  $\chi(G) \leq \chi_\ell(G)$.
- A graph is chromatic choosable if $\chi(G) = \chi_\ell(G)$.
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Factorizing Graphs

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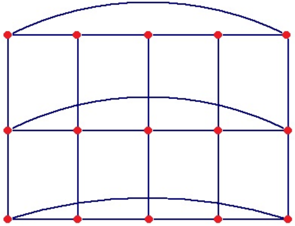
- The Cartesian Product $G \square H$ of graphs G and H is a graph with vertex set $V(G) \times V(H)$.
  Two vertices (u, v) and (u', v') are adjacent in $G \square H$ if either $u = u'$ and $vv' \in E(H)$ or $uu' \in E(G)$ and $v = v'$.
- Here's $C_5 \square P_3$:



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Factorizing Graphs

- Every connected graph has a unique factorization under the Cartesian product which can be found in polynomial time: $G \cong G_1^{p_1} \square G_2^{p_2} \square \ldots \square G_d^{p_d}$.

- It is well known that $\chi(G \square H) = \max\{\chi(G), \chi(H)\}$.

- We wish to find G and H such that $\chi_\ell(G \square H) = \chi(G \square H)$.

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Idea 1: Criticality

- Criticality for usual coloring means $\chi(G) = k$ but $\chi(G - v) < k$ for all vertices v of G.

- How to do this for list coloring, particularly chromatic choosable graphs?

- We introduce how to do this with a new notion of criticality: A graph G is said to be strong k-chromatic choosable if $\chi(G) = \chi_\ell(G) = k$ and if the usual coloring is the obstacle to list coloring of G. This gives $\chi(G - v) \leq \chi_\ell(G - v) < k$ for all vertices v of G.

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Idea 2: Counting list Colorings

- We need to be able to count how many different ways we can color a graph from its assignment of color lists.
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Theorem (Kaul and Mudrock)

Suppose that G is a strong k-chromatic choosable graph with $k \geq 2$. Then,

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