The Art Gallery Problem

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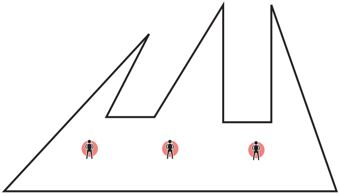
The Art Gallery Problem

The original art gallery problem (V. Klee, 1973) asked for the minimum number of guards sufficient to see every point of the interior of an n-vertex simple polygon.

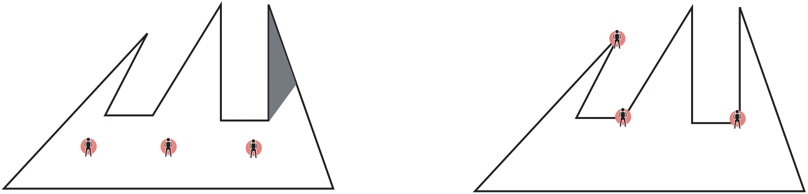
A simple polygon is a simply-connected closed region whose boundary consists of a finite set of line segments.

Two points u and v in a polygon are said to be mutually visible if the line segment joining u and v lies inside the polygon.

How many guards are needed



How many guards are needed

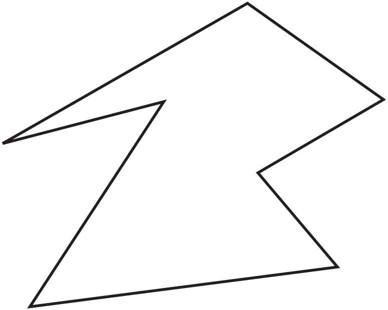


The Art Gallery Theorem

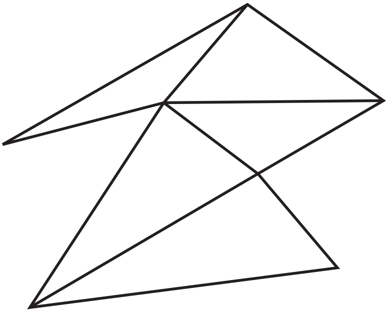
Chvátal (1975) proved that $\lfloor \frac{n}{3} \rfloor$ guards are always sufficient for any n vertex simple polygon.

Fisk gave a simple proof using graph coloring in 1978.

Proof of Art Gallery Theorem

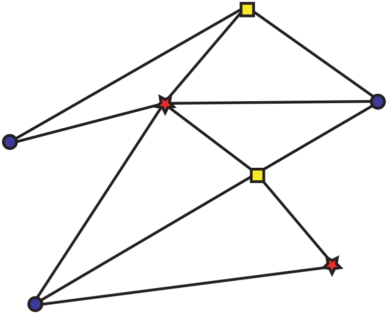


Proof of Art Gallery Theorem



Step 1: Triangulation

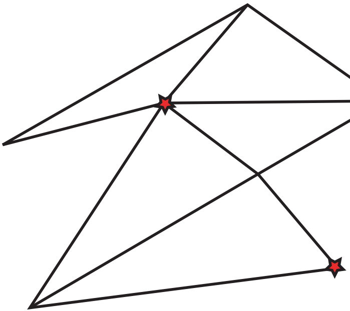
Proof of Art Gallery Theorem



Step 2: Proper coloring using 3 colors.

Proper coloring of the vertices means adjacent vertices get different colors.

Proof of Art Gallery Theorem



Step 3: Choose minimum size color class

Where to place the guards?

Note that Fisk's proof gives an explicit placement of guards on the vertices (corners) of the polygon.

Vertex guard means the guard should be positioned on a vertex.

Other variations on guards include:

Point guard means the guard can be placed anywhere in the polygon.

Edge guard means the guard can be placed anywhere along an edge of the polygon.

Mobile guard means the guard is allowed to patrol along a line segment lying in the polygon.

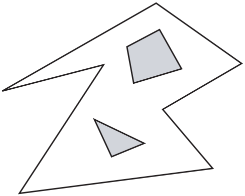
What does an Art Gallery look like?

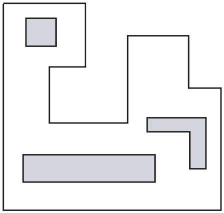
Walls tend to be perpendicular to each other!

An orthogonal art gallery is a polygon whose edges are all aligned with a pair of orthogonal coordinate axes, which we take to be horizontal and vertical. Internal angles are either $\frac{\pi}{2}$ or $\frac{3\pi}{2}$.

What does an Art Gallery look like?

There are usually obstructions to visibility in the interior.





Polygons with holes

Guards can not see through holes.

A Short Survey I General Polygons w. holes

O'Rourke 1987 : $\lfloor \frac{n+2h}{3} \rfloor$ vertex guards are sufficient for a polygon with n vertices and h holes.

Shermer's conjecture I, 1982

Any polygon with n vertices and h holes can be guarded by $\lfloor \frac{n+h}{3} \rfloor$ vertex guards.

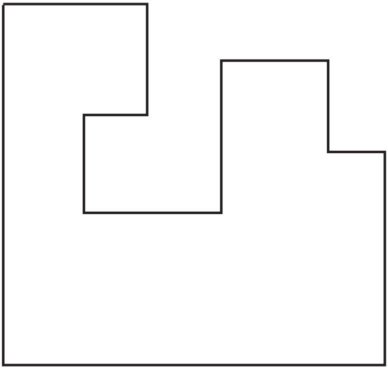
Hoffmann, Kaufmann, Kriegel 1991 : $\lfloor \frac{n+h}{3} \rfloor$ point guards are sufficient for a polygon with n vertices and h holes.

This is sharp.

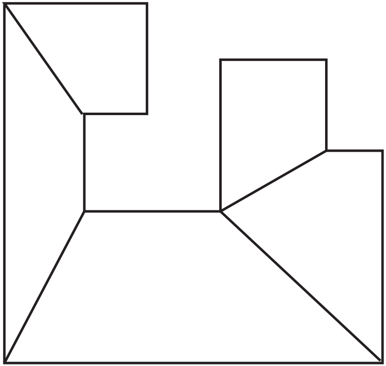
Orthogonal Art Gallery Theorem

Kahn, Klawe and Kleitman (1983) proved that $\lfloor \frac{n}{4} \rfloor$ vertex guards are sufficient for a n-vertex orthogonal art gallery.

Proof of Orthogonal Art Gallery Theorem

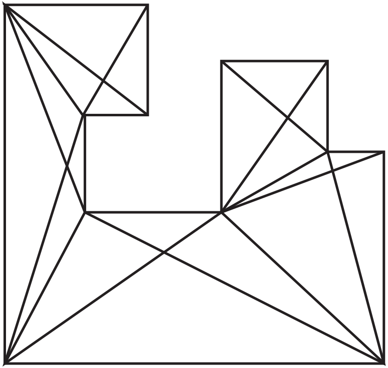


Proof of Orthogonal Art Gallery Theorem



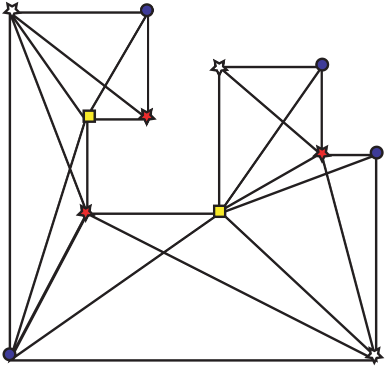
Step 1: Convex Quadrilateralization

Proof of Orthogonal Art Gallery Theorem



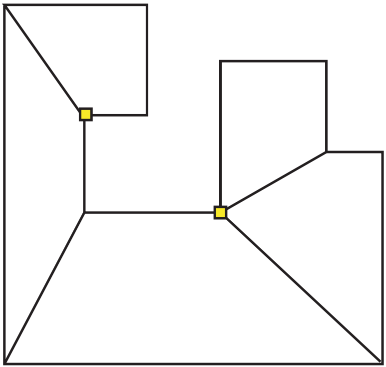
Step 2a: Add diagonals

Proof of Orthogonal Art Gallery Theorem



Step 2b: Proper Coloring using 4 colors.

Proof of Orthogonal Art Gallery Theorem



Step 3: Choose minimum size color class

A Short Survey II Orthogonal Polygons w. holes

O'Rourke 1987 : $\lfloor \frac{n+2h}{4} \rfloor$ vertex guards are sufficient for an orthogonal polygon with n vertices and h holes.

Shermer's conjecture II, 1982

Any orthogonal polygon with n vertices and h holes can be guarded by $\lfloor \frac{n+h}{4} \rfloor$ vertex guards.

Hoffmann, Kriegel 1996 : $\lfloor \frac{n}{3} \rfloor$ vertex guards are sufficient for an orthogonal polygon with n vertices and h holes.

Hoffmann 1990 : $\lfloor \frac{n}{4} \rfloor$ point guards are sufficient for a polygon with n vertices and h holes.

These are not sharp.

A Short Survey III Edge and Mobile Guards

In 1981, Godfried Toussaint formulated the problem for edge and mobile guards.

O'Rourke 1983 : $\lfloor \frac{n}{4} \rfloor$ mobile guards are sufficient for a simple polygon with n vertices.

This is sharp.

Toussaint's conjecture, 1981

Except for a few polygons, every simple polygon with n vertices can be guarded by $\lfloor \frac{n}{4} \rfloor$ edge guards.

A Short Survey III contd. Edge and Mobile Guards

Aggarwal 1984 : $\lfloor \frac{3n+4}{16} \rfloor$ mobile or edge guards are sufficient for a simple orthogonal polygon with n vertices.

Improved to just edge guards by Bjorling-Sachs, 1998

Gyori et al. 1996 : $\lfloor \frac{3n+4h+4}{16} \rfloor$ mobile guards are sufficient for an orthogonal polygon with n vertices and h holes.

These results are sharp.

A Short Survey IV

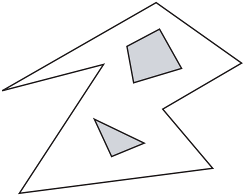
Other variations:

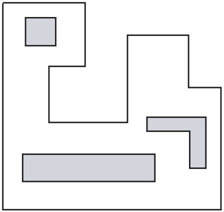
- Art galleries with interior walls (diagonal joining two corners with small doorway placed arbitrarily).
- Half-guard (fixed 180° field of vision): $\frac{n}{3}$ point half-guards.
- Guarded guards (each guard is also visible to at least one other guard): $\frac{3n-1}{7}$ vertex guarded-guards; $\frac{n}{3}$ vertex guarded-guards for orthogonal polygons.
- 3-D polytopes.
- Fortress Problem: Guards on edges so that exterior is visible.
- Prison yard Problem: Guards on edges so that both interior and exterior are visible.

Orthogonal Art Gallery Problem

Kahn, Klawe and Kleitman 1983: $\lfloor \frac{n}{4} \rfloor$ vertex guards are sufficient for a n-vertex orthogonal art gallery.

What about Orthogonal Art galleries with holes?





Shermer's conjecture

How many guards are needed for guarding an orthogonal art gallery with n vertices and h holes?

Shermer's conjecture, 1982

Any orthogonal polygon with n vertices and h holes can be guarded by $\lfloor \frac{n+h}{4} \rfloor$ vertex guards.

Known results

Shermer's conjecture for $h = 0$

Kahn, Klawe and Kleitman 1983: $\lfloor \frac{n}{4} \rfloor$ vertex guards are sufficient for a n-vertex orthogonal art gallery.

Shermer's conjecture for $h = 1$

Agarwal 1984: $\lfloor \frac{n+1}{4} \rfloor$ vertex guards are sufficient for an orthogonal polygon with n-vertices and 1-hole.

O'Rourke 1987: $\lfloor \frac{n+2h}{4} \rfloor$ vertex guards are sufficient for an orthogonal polygon with n vertices and h holes.

P. Zylinski 2006: $\lfloor \frac{n+h}{4} \rfloor$ vertex guards are sufficient if the dual graph of orthogonal polygon is a cactus.

A cactus is a graph whose any two cycles share at most one vertex and any such vertex is a cut-vertex.

Our Result

Y. Jo, K. 2009+: $\left\lfloor \frac{n+\frac{5}{3}h}{4} \right\rfloor$ vertex guards suffice for an orthogonal gallery with n vertices with h holes.

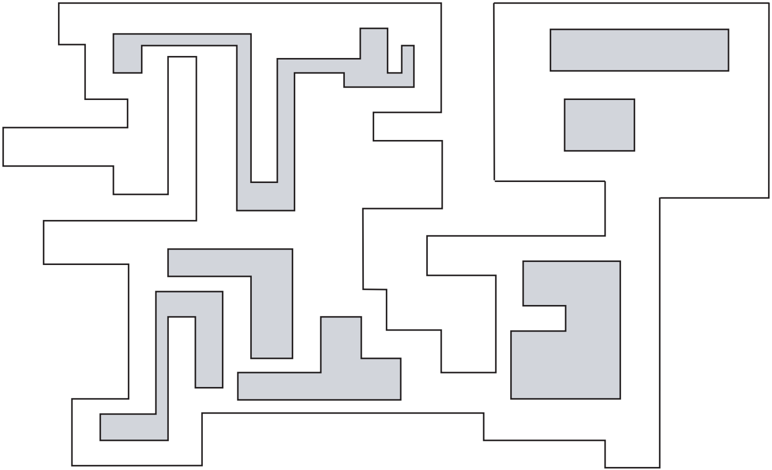
This result improves the O'Rourke bound of $\lfloor \frac{n+2h}{4} \rfloor$.

O'Rourke uses 6 extra guards per 12 holes.

We use 5 extra guards per 12 holes.

While Shermer's conjecture wants to use only 3 extra guards per 12 holes.

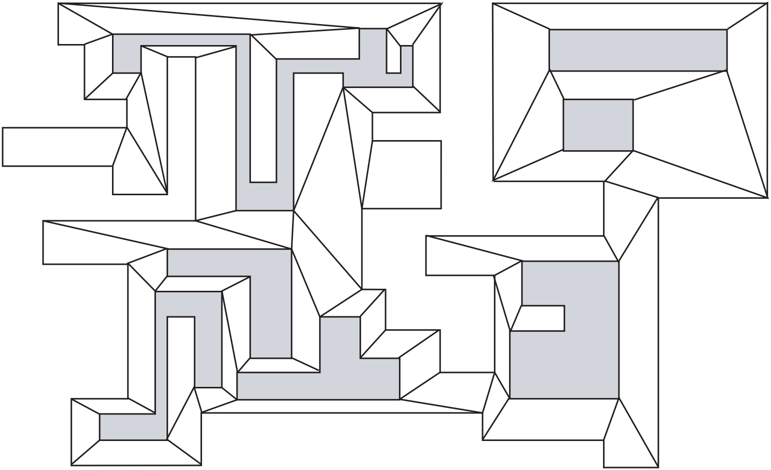
Outline of our proof



Outline of our proof

- Step 1 : Convex Quadrilateralization

Outline of our proof



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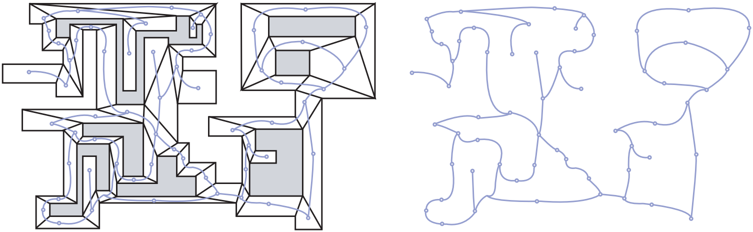


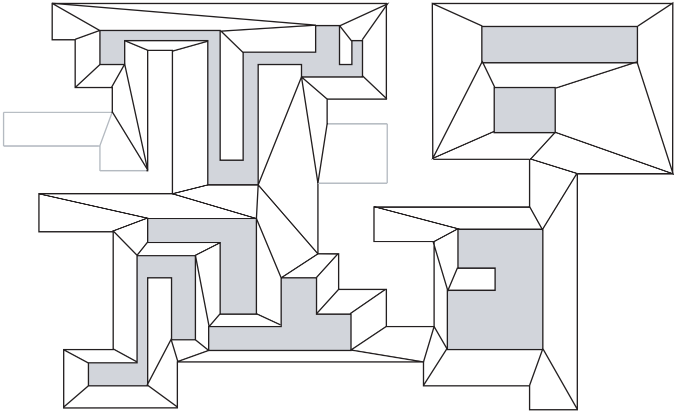
Dual Graph

Outline of our proof

- Step 2 : Removal of all the exterior leaves in the dual graph.

A leaf is a vertex with only one neighbor.

Outline of our proof



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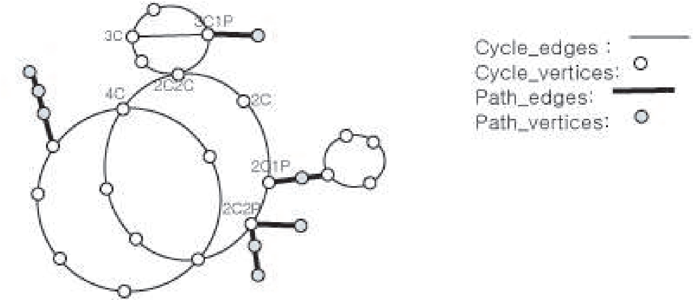
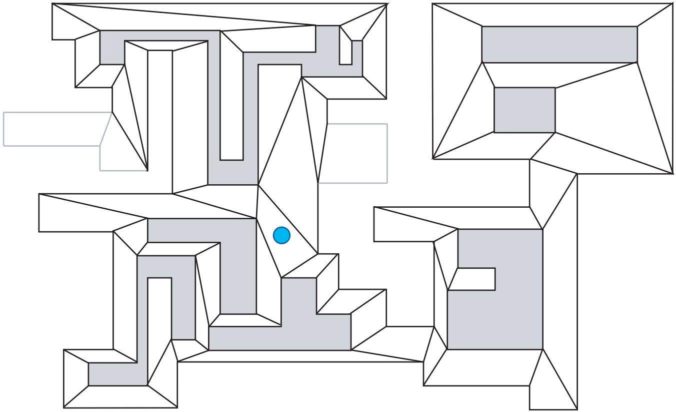


Outline of our proof

- Step 3 : Cut around a boundary vertex of type $3C$ (or $3C1P$ or $4C$) which has a neighbor of the type $2C$ or $2C1P$ or $2C2P$ or $2C2C$.
  By a lemma, such a vertex exists on the boundary of every block with more than one cycle in the dual graph.

Cutting means duplicating the vertices of the cut quadrilateral.

Outline of our proof



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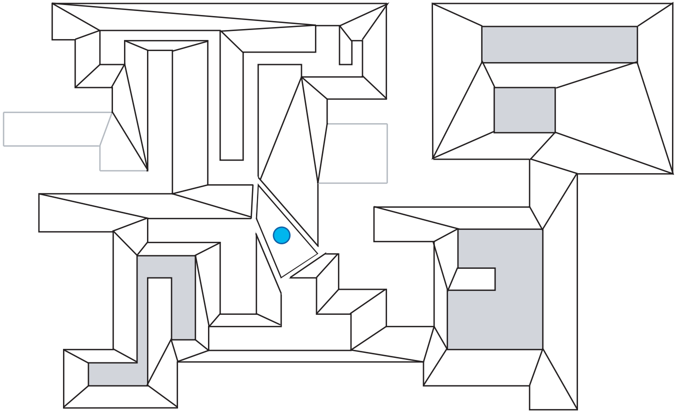

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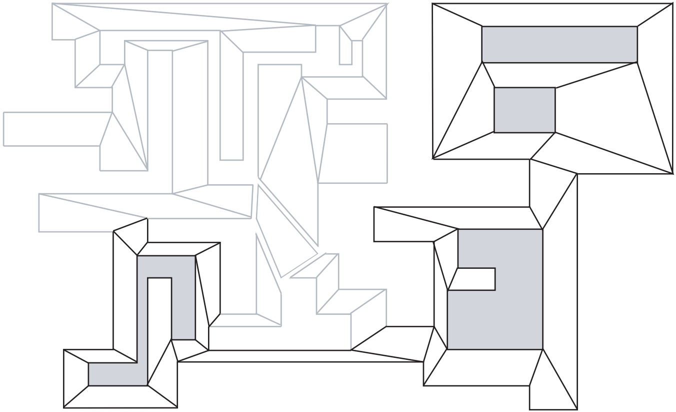

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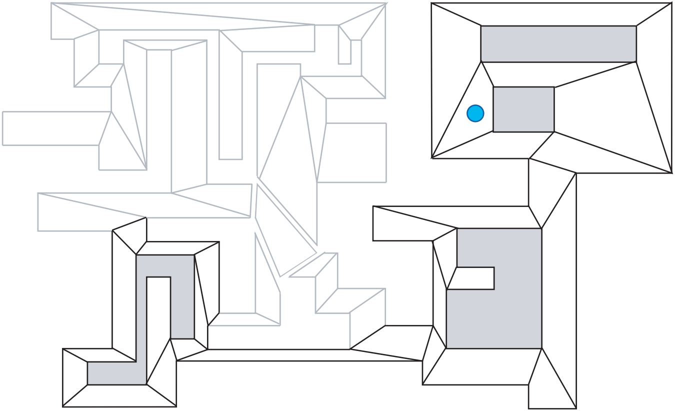

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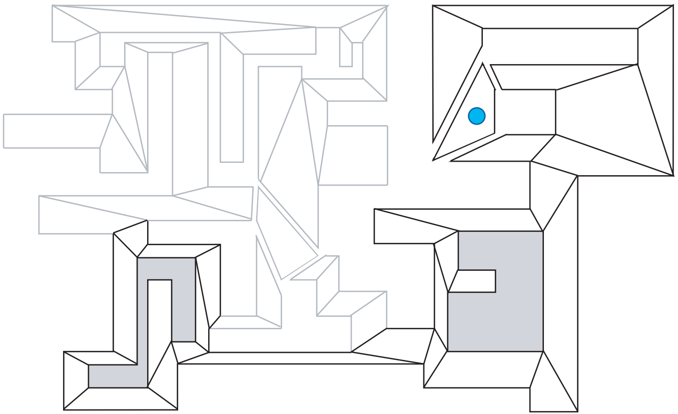

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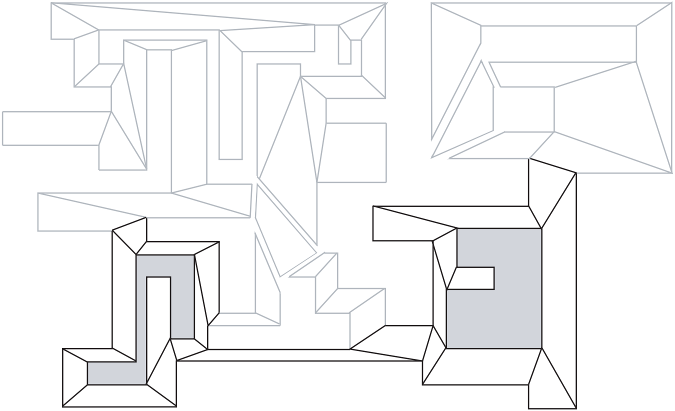
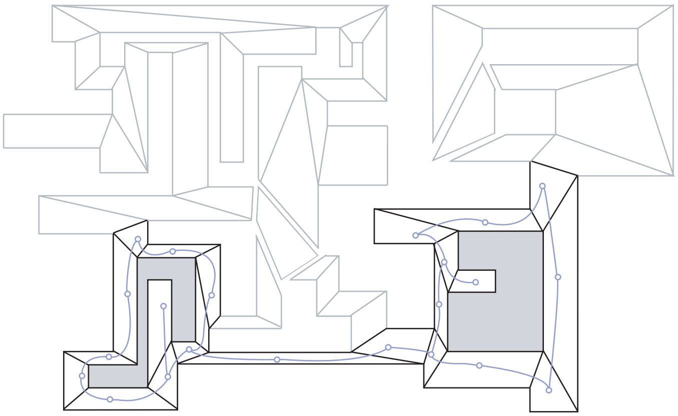


Outline of our proof

We are left with an art gallery whose dual graph is a cactus.

Outline of our proof



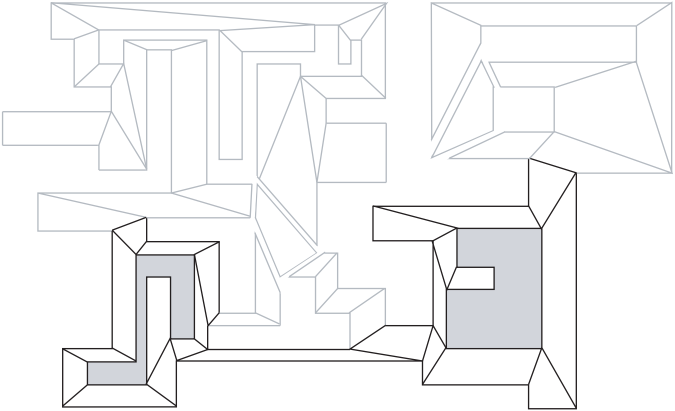
Outline of our proof

- Step 4 : Apply the P. Zylinski result to color the remaining polygon.

  By a lemma, a polygon with w holes whose corresponding dual graph is a cactus is 4-colorable using at most w duplicate vertices.

  Duplicating a vertex allows us to place two guards in the same quadrilateral if needed.

Outline of our proof



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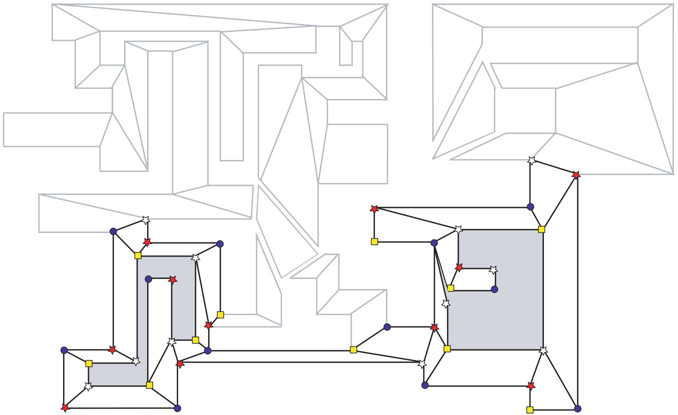
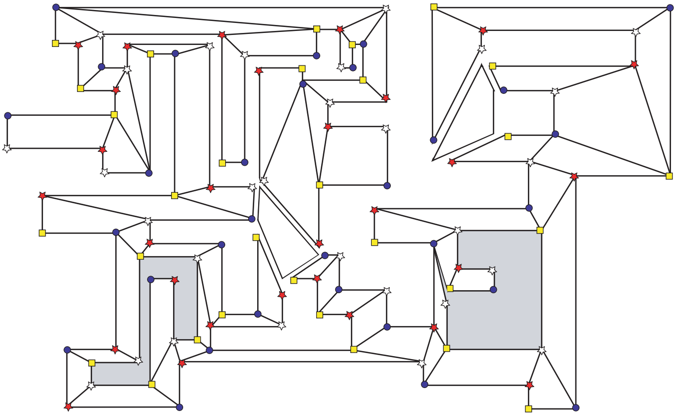


Outline of our proof

- Step 5a : Extend the coloring to all of the polygon except quadrilaterals that were cut in step 3 by reattaching the leaves of the dual graph removed in Step 2.

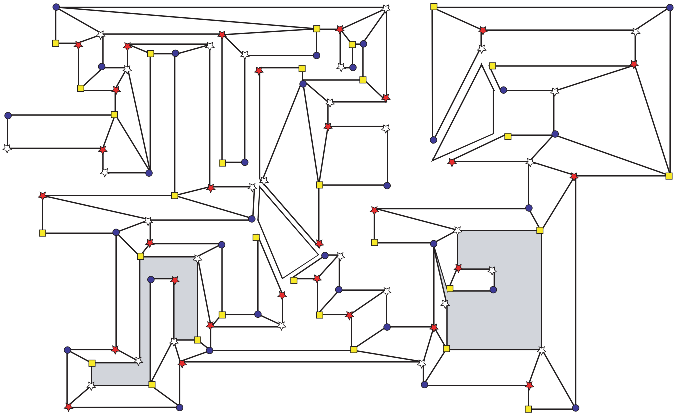
Outline of our proof



Outline of our proof

- Step 5b :
  - Attach the cut quadrilaterals to the main colored polygon in reverse order of their being cut (by application of Step 3)
  - At least 3 duplicated vertices of a cut quadrilateral are attached. (Other vertices remain duplicated.)
  - Recoloring of some vertices may be needed to maintain a proper coloring of the extended polygon.

Outline of our proof



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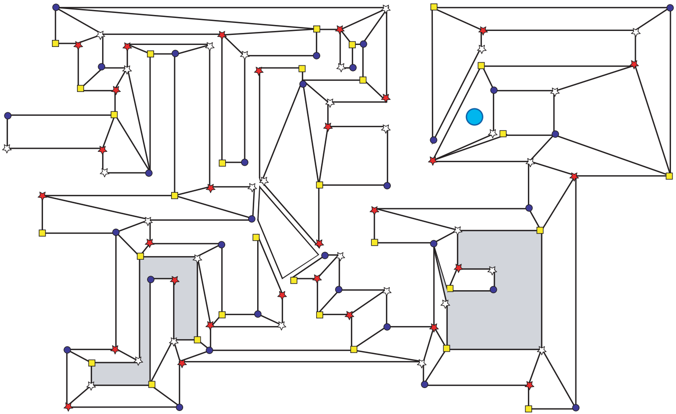

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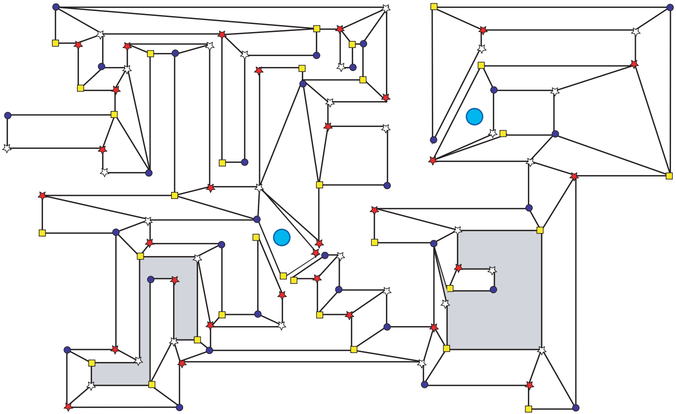

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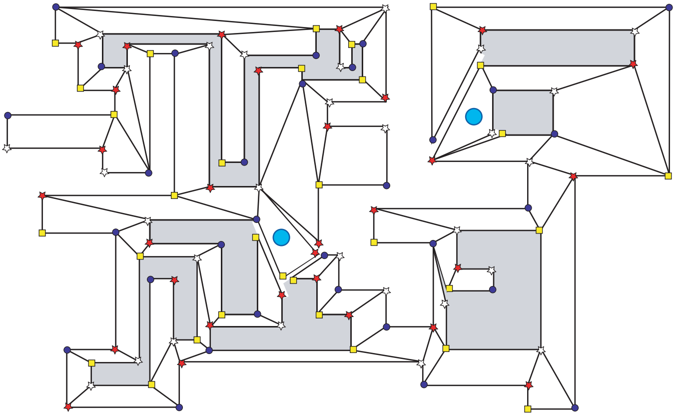

- Let k vertices of type $3C$ be cut.
  Let l vertices of type $3C1P$ be cut.
  Let m vertices of type $4C$ be cut in Step 3.
- Let w be the number of holes remaining when the dual graph becomes a cactus.
- Number of duplicate vertices used for the 4-coloring :
  $w + (6k - 3k) + (6l - 3l) + (8m - 3m) = w + 3k + 3l + 5m$
- Total number of holes in original polygon :
  h= $w + 2k + 2l + 3m$
- So, the number of duplicate vertices is
  $w + 3k + 3l + 5m \leq \frac{5}{3}h$.
  Hence, $\left\lfloor \frac{n+\frac{5}{3}h}{4} \right\rfloor$ is the maximum size of the smallest color class.