On a Local-Search Algorithm for Large Bipartite Subgraphs

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Large Bipartite Subgraphs

MAX-CUT: In a given graph $G = (V, E)$, find a bipartite subgraph with maximum number of edges.

Find a bipartition (cut) (X, Y), with $X \subseteq V(G)$ and $Y = V(G) \setminus X$, that maximizes the number of edges between X and Y.

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- NP-complete
- Approximation results: Randomized (0.87)-approximation algorithm [Goemans-Williamson, 1995].
- Hard to approximate: No (0.942)-approximation algorithm exists unless P=NP [Hastad, 1997].

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- Extremal results like Edwards-Erdős Inequalities :
  1) $b(G) \geq \frac{1}{2}m + \frac{1}{8}(\sqrt{8m+1} - 1)$, $m = e(G)$
  2) $b(G) \geq \frac{1}{2}m + \frac{1}{4}(n - 1)$, $n = n(G)$

A local search algorithm

Idea : Starting with an arbitrary vertex partition, switch a vertex from one partite set to the other if doing so increases the number of edges in the cut (the bipartite subgraph induced by the vertex partition).

Given a partition $V(G) = X \cup Y$ of the vertex set of a graph G, a local switch moves a vertex v from X to Y that has more neighbors in X than in Y.

A list of local switches performed successively is a switching sequence.

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Size of the bipartite subgraph : How big a bipartite subgraph is guaranteed at the end of a switching sequence?

Length of a switching sequence : How long can a switching sequence be?

Size of the bipartite subgraph

Theorem: [Bylka + Idzik + Tuza, 1999]
A bipartite subgraph of size $\frac{1}{2}m + \frac{1}{4}o(G)$ is guaranteed, where $o(G)$ is the number of odd degree vertices in G.

A slight modification of the local switching rules improves the guarantee to the first Edwards-Erdős Inequality :
$b(G) \geq \frac{1}{2}m + \frac{1}{8}(\sqrt{8m+1} - 1)$.

Minimum length of a switching length

Let $s(G)$ denote the minimum length of a maximal flip sequence starting from the trivial vertex partition.

Theorem [Kaul + West, 2007]:
If G is an n-vertex loopless multigraph, then $s(G) \leq n/2$.
In fact, there exists a sequence of at most $n/2$ flips that produces a globally optimal partition.

Maximum length of a switching sequence

Each switch increases the edges in the cut by at least one.

Observation: The maximum length of a switching sequence, $l(G)$, is at most $b(G) \leq e(G)$.

This is best possible, as the star $K_{1,n-1}$ achieves equality for both $b(G)$ and $e(G)$.

Bounding the length with n : A bipartite graph on n vertices has at most $\frac{n^2}{4}$ edges, so any switching sequence has length at most $\frac{n^2}{4}$.

$l(G)$: Upper Bound

To get a better upper bound, we look at the tradeoff between $\delta(G)$, the minimum degree of G, and $b(G)$, as a switching sequence progresses.

Proposition [Kaul + West, 2007]:
The length of any switching sequence is at most
$b(G) - (\frac{3}{8}\delta^2(G) + \delta(G))$.

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Given an arbitrary switching sequence. The first move gains at least $\delta(G)$ edges, the second move gains at least $\delta(G) - 1$ edges, and so on. Furthermore, we cannot move anything back until more than $\delta(G)/2$ vertices are moved.

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For example, Let G be a αn-regular graph, $\alpha \in (0, 1)$, then
$l(G) \leq (1 - \frac{3}{4}\alpha)\, e(G)$,
which is much better than the trivial upper bound.

Let G be triangle-free, then the upper bound above improves to
$b(G) - \frac{7}{16}\delta^2(G)$.

$l(G)$: Lower Bound

Can we do faster than $\frac{1}{4}n^2$ switches to reach a local optima?

Theorem: [Cowen + West, 2002]
When n is a perfect square, there exists a graph G with n vertices that has a switching sequence of length $e(G) = \frac{1}{2}n^{\frac{3}{2}}$.

A delicate construction in which each switch gained exactly one edge.

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For every n, there exists a graph G with n vertices that has a switching sequence of length at least $\frac{2}{25}(n^2 + n - 31)$.

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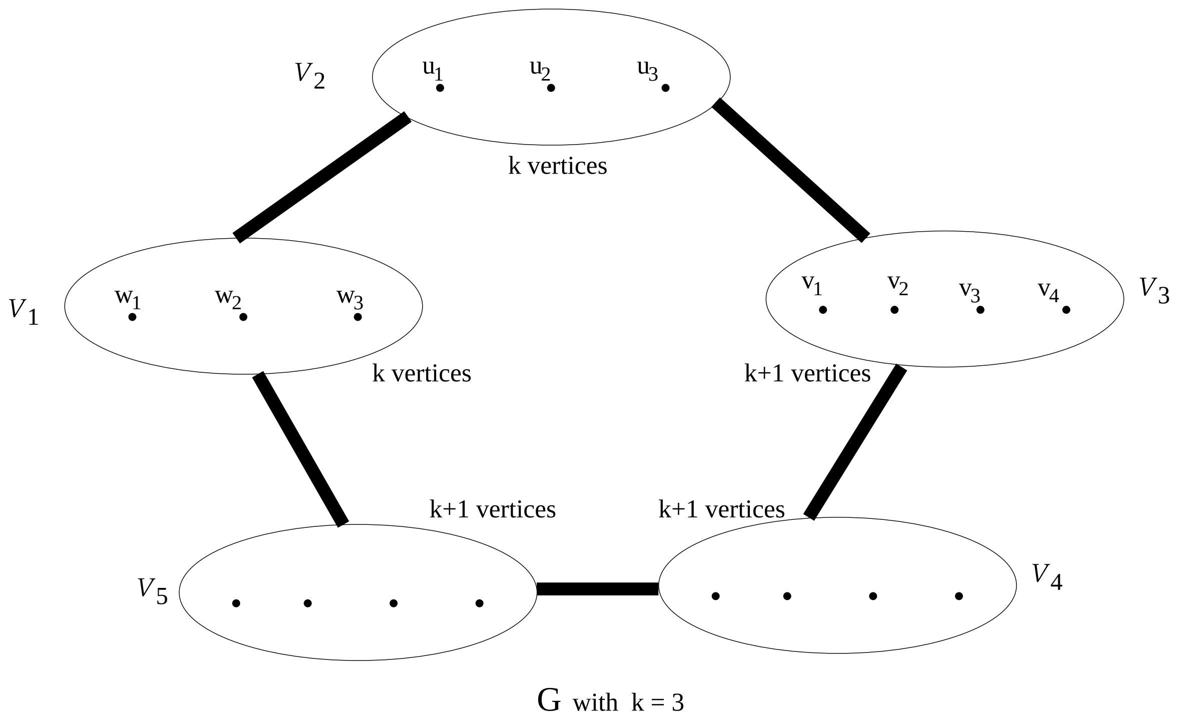
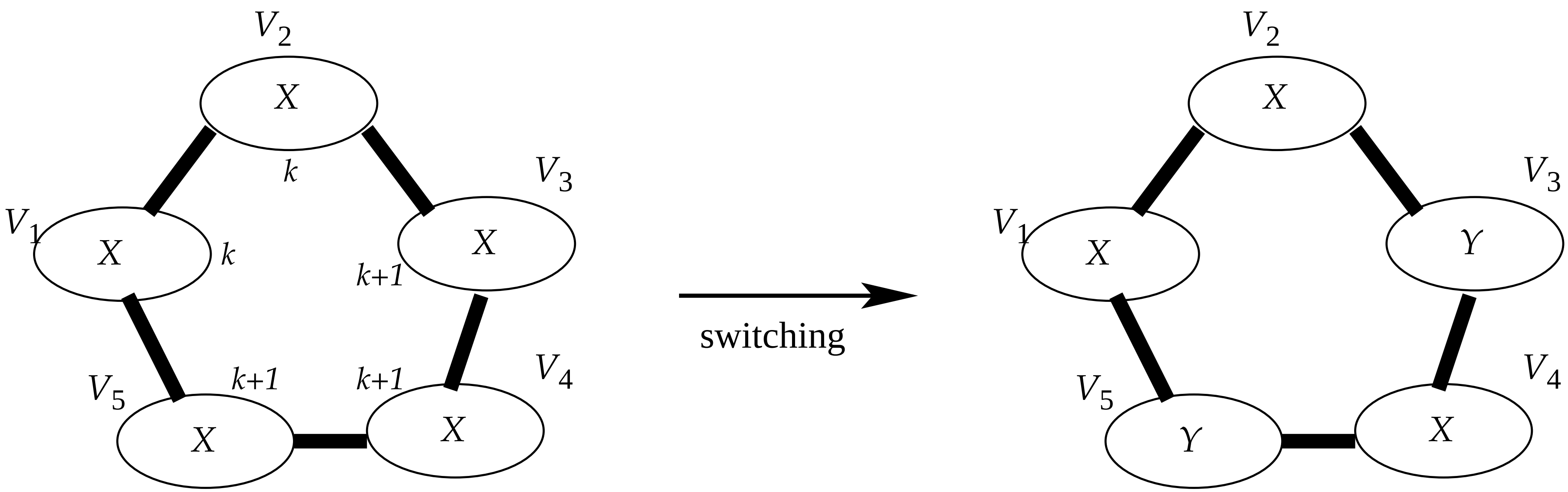


Local optima

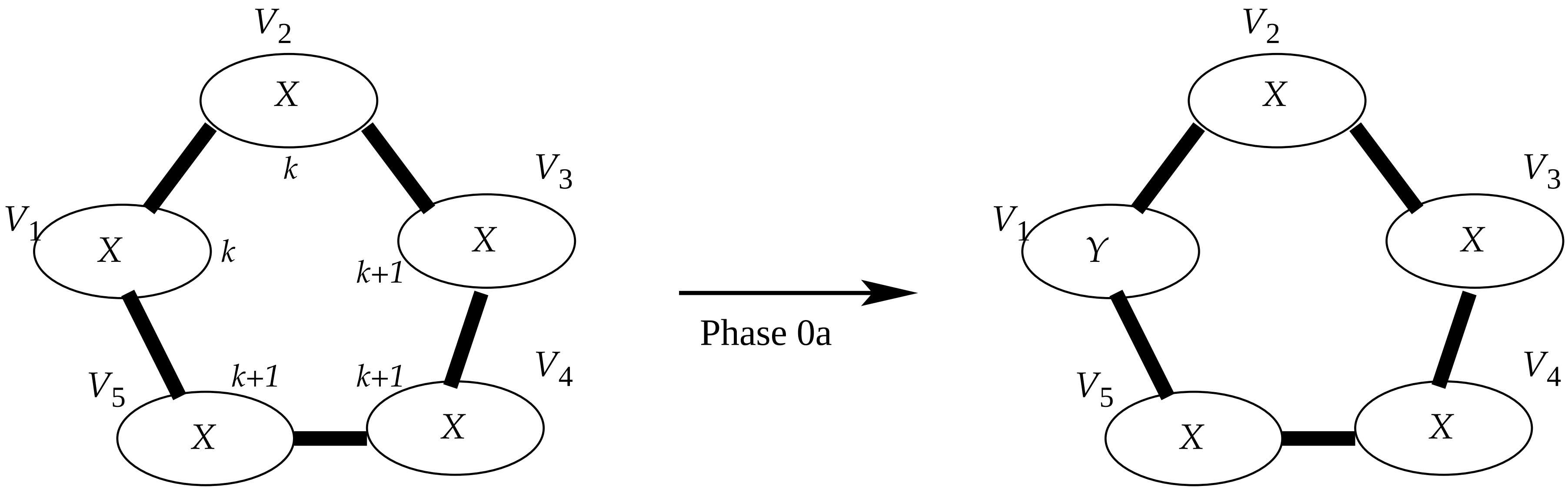
Initialize $X = V(G)$ and $Y = \varnothing$ and with each local switch dynamically update the membership of X and Y.

At the end of the switching sequence $X = V_1 \cup V_2 \cup V_4$ and $Y = V_3 \cup V_5$.



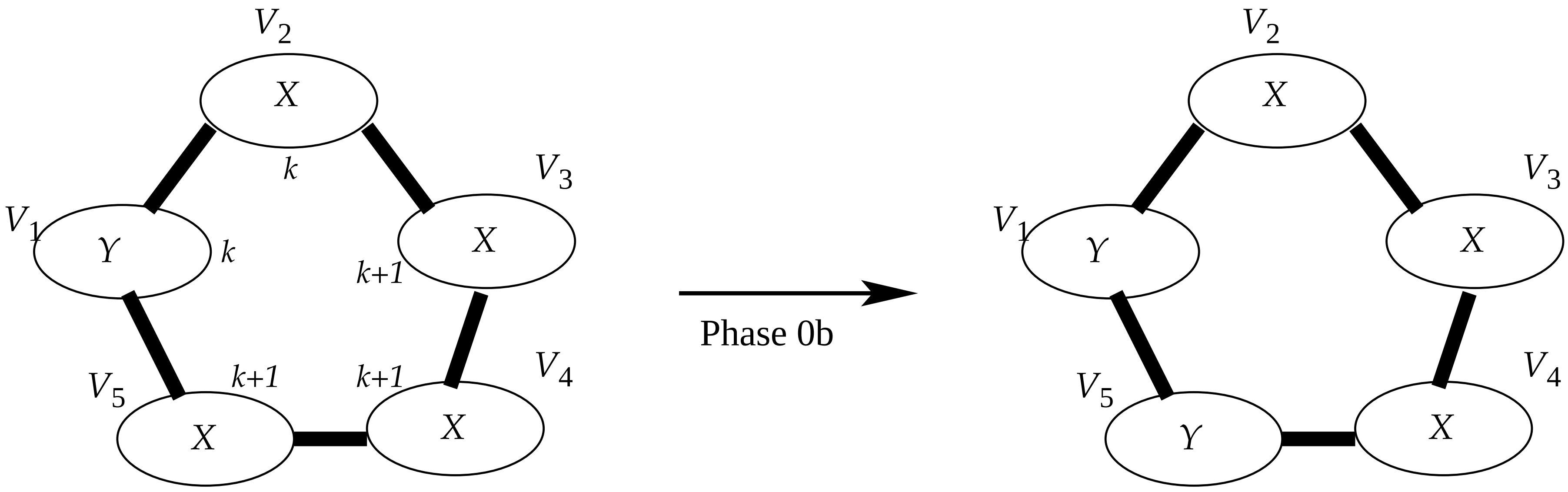
Switching sequence- Preprocessing

Phase 0a. Move each vertex in V_1 from X to Y. This is possible because all the neighbors of each vertex in V_1 are in X.



Switching sequence- Preprocessing

Phase 0b. Move each vertex in V_5 from X to Y. This is possible because each vertex in V_5 has $k+1$ neighbors (from V_4) in X and k neighbors (from V_1) in Y.



Henceforth, always $V_4 \subseteq X$ and $V_5 \subseteq Y$.

Switching sequence- Main Phase

For $i = 1, \ldots, k+1$,

Phase i. At the start of Phase i,
$X = \{w_j : j < i\} \cup V_2 \cup \{v_j : j \geq i\} \cup V_4$,
$Y = \{w_j : j \geq i\} \cup \{v_j : j < i\} \cup V_5$.

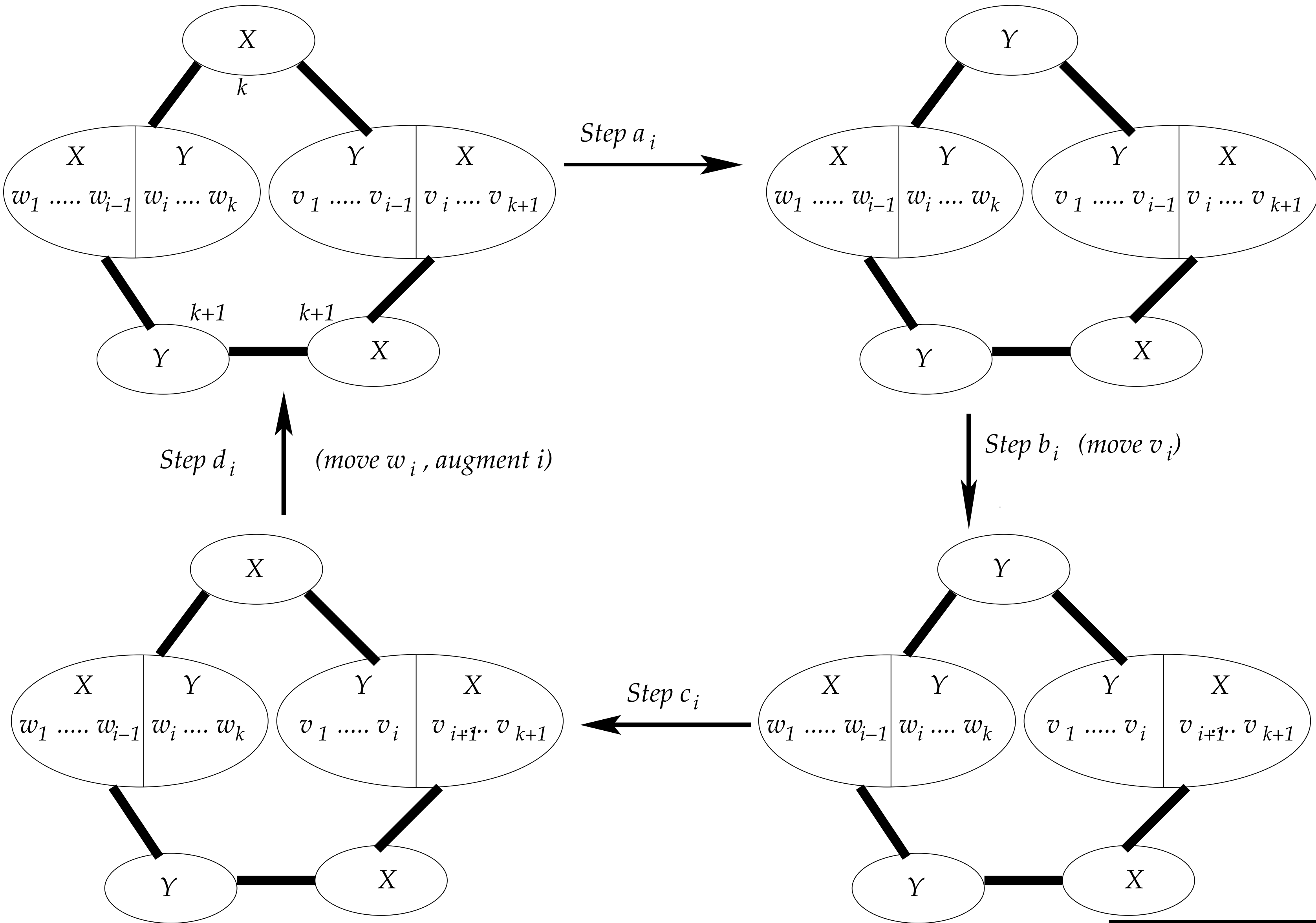
(a_i) Move each vertex in V_2 from X to Y. $k+1$ neighbors in X and k neighbors in Y.

(b_i) Move $v_i \in V_3$ from X to Y. $k+1$ neighbors (all of V_4) in X and k neighbors (all of V_2) in Y.

(c_i) Move each vertex in V_2 from Y to X. $k+1$ neighbors in Y and k neighbors in X.

(d_i) If $i < k+1$, move $w_i \in V_1$ from Y to X, otherwise stop. $k+1$ neighbors (all of V_5) in Y and k neighbors (all of V_2) in X.

Switching sequence- Main Phase



Open Questions I

Problem 1. Determine the exact constant multiple (between $\frac{8}{100}$ and $\frac{25}{100}$) of n^2 for $l(G)$.

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Problem 2. New ideas for upper bounds on $l(G)$.

Open Questions II

Modify the switching algorithm by allowing up to $k \geq 1$ vertices to be switched at a time.

How close can we get to the second Edwards-Erdős Inequality : $b(G) \geq \frac{1}{2}m + \frac{1}{4}(n-1)$?

Problem 3. [Tuza, 2001] Given k, determine the largest constant $c = c(k)$ such that the local switching algorithm guarantees a bipartite subgraph of size at least $\frac{1}{2}m + cn - o(n)$.

A construction shows that $c(k) < \frac{1}{4}$, for all k.

What is the smallest k with $c(k) > 0$? Is $c(1) > 0$?