Finding Large Subgraphs

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The maximum subgraph problem

The maximum subgraph problem for a Graph property Π asks:

Given a graph G, find a subgraph H of G satisfying property Π that has the maximum number of edges.

The maximum induced subgraph problem

The maximum induced subgraph problem for a Graph property Π asks:

Given a graph G, find an induced subgraph H of G satisfying property Π that has the maximum number of vertices.

In other words, find the minimum number of vertices to remove from G such that the remaining subgraph satisfies the property Π.

Graph Properties

The following Graph properties are commonly considered:

- Forest (cycles are forbidden)
- Bipartite subgraph (odd cycles are forbidden)
- Planar subgraph ($\{K_5, K_{3,3}\}$-minors are forbidden)
- Complete subgraph

  There is no difference between induced and non-induced versions for this.
- Independent set

  This is meaningful only for the induced version.

All these properties are hereditary, every subgraph of a graph with property Π also has property Π.

Connectedness is an example of a property that is not hereditary.

Finding Large Subgraphs

Except for the largest Forest subgraph problem,
all these largest subgraph problems are NP-hard.

In case of the largest induced subgraph problem,
Lewis and Yannakakis (1980) showed that:

The largest induced subgraph problem is NP-hard for every non-trivial hereditary property.

What about approximate solutions?

Approximation Algorithms

Algorithm $\mathcal{A}$ for a maximization problem MAX achieves an approximation factor α if

for all inputs G, we have: $\frac{\mathcal{A}(G)}{OPT(G)} \leq \alpha$,
where $\mathcal{A}(G)$ is the value of the output generated by the algorithm $\mathcal{A}$,
and $OPT(G)$ is the optimal value.

A α-approximation algorithm for MAX is a polynomial time algorithm that achieves the approximation factor α.

To show $\mathcal{A}$ achieves approximation factor α, we typically show that: $\mathcal{A}(G) \geq L$ and $OPT(G) \leq U$, so $\alpha \geq L/U$.

Approximation Algorithms for Large Subgraphs

For example
for the largest bipartite subgraph problem:

Goemans and Williamson (1995): 0.878-approximation algorithm

Hastad (1997): If $P \neq NP$ then there is no α-approximation algorithm for any $\alpha > 0.941$.

for the largest clique subgraph problem:
Feige (2005): $O(n(loglogn)^2/(logn)^3)$-approximation algorithm

Feige et al. (1996): It is hard to approximate MAX-Clique for any constant factor.

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Approximation Algorithms for Large Induced Subgraphs

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Comparatively, very little research has been done on approximation algorithms for these problems.

For example,

For the maximum induced bipartite subgraph problem: Some results for very special classes of graphs -

Zhu (2009): $5/7$ approximation factor algorithm over triangle-free subcubic graphs.

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Approximation Algorithms for Large Induced Subgraphs

For the maximum induced Planar subgraph problem:

Calinescu et al. (1998): There exists an $\epsilon > 0$ such that there is no $1 - \epsilon$- approximation algorithm unless $P = NP$.

Edward and Farr (2007): $3/(d+1)$-approximation algorithm on graphs of average degree at most $d \geq 4$, [in fact they find an induced series-parallel subgraph (more about these later)].

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Till 1990's a number of algorithms were studied but none gave an approximation ratio better than $1/3$, which can be trivially achieved by the Spanning Tree algorithm.

$ST(G) = n - 1$ and $OPT(G) \leq 3n - 6$

Calinescu et al. (1998): $4/9$-approximation algorithm, which is still the best known.
In fact, this algorithm generates an outerplanar subgraph (which gives a $2/3$-approximation algorithm for the maximum outerplanar graph problem).

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Large Series-Parallel Subgraphs

Planar graphs are characterized as having no $\{K_5, K_{3,3}\}$ minors or subdivisions.

Outerplanar graphs are characterized as having no $\{K_4, K_{2,3}\}$ minors or subdivisions.

How about subgraphs with no K_4 minors or subdivisions? These will be planar but not outerplanar.

These are Series-Parallel graphs.

H is a minor of G if a graph isomorphic to H can be obtained from G by contracting some edges, deleting some edges, and deleting some isolated vertices.

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Large Series-Parallel Subgraphs

Series-Parallel graphs are characterized as:

- No K_4 minor or subdivision.
- Arises from a forest by adding parallel edges, subdividing edges, and at the end removing any parallel edges to keep the graph simple.
- tree width ≤ 2 (subgraph of 2-tree).

Large Series-Parallel Subgraphs

The maximum Series-Parallel subgraph problem is *NP*-hard.

Since, the number of edges of a Series-Parallel graph on n vertices is bounded above by $2n - 3$,
the spanning tree algorithm gives a $1/2$-approximation algorithm.

Can we do better?

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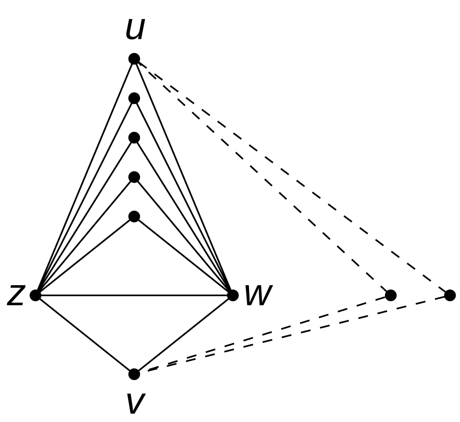
Can we do better?

New Results

Calinescu, Fernandes, K. (2009): $7/12$ approximation algorithm for the maximum Series-Parallel subgraph problem.

The output is a spruce structure: a graph each of whose blocks is either a spruce or an edge.
A spruce consists of two *base* vertices and at least one *tip* vertex, in which each tip vertex is adjacent to exactly the two base vertices.

(a)



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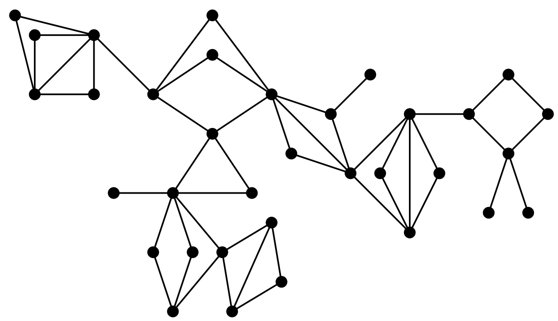

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New Ideas

Comparison with previous algorithms for Planar subgraphs:

- Unlike earlier algorithms, the subgraph we generate is not a tree or an outerplanar graph.
- Unlike earlier algorithms, we have to allow blocks of unbounded size in our subgraph.
- Unlike earlier algorithms, we sometimes have to shrink or throw away previously selected blocks.

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Unlike earlier algorithms, we have to allow blocks of unbounded size in our subgraph.

If the input graph is a *complete spruce* (spruce with an edge between the base vertices) with $n-2$ tips, then any algorithm that only generates blocks of size at most k would result in an output with a total $n+k-3$ edges.

With large n and fixed k, this is only a $1/2$-approximation.

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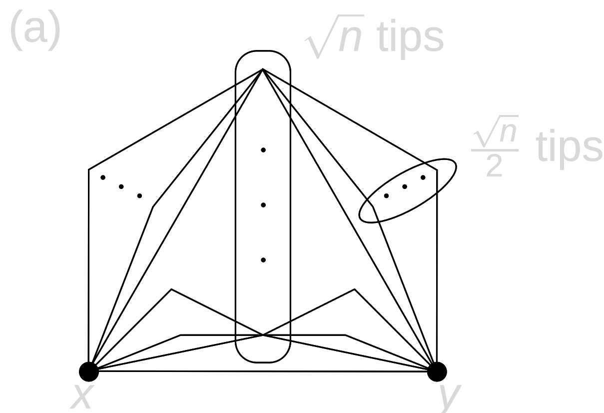

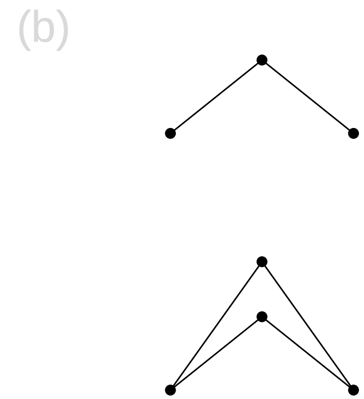

The optimum has n vertices and $2n-3$ edges.

A spruce with base vertices x and y and $\sqrt{n}$ tips. For each of its tips v, there are two complete spruces, one with base vertices x and v, and the other with base vertices v and y, each with $\sqrt{n}/2$ tips.

If an algorithm mistakenly (or greedily) selects the spruce with base vertices x and y, then it cannot add any more spruces and it ends up with about $n+\sqrt{n}$ edges — asymptotically not better than a $1/2$-approximation.

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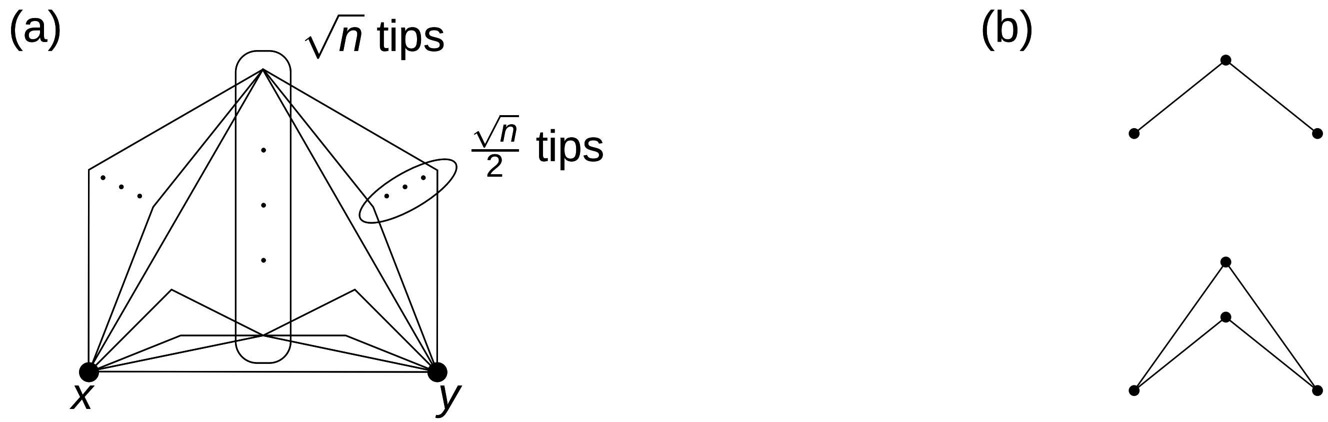

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The Algorithm - Preliminaries

$gain(S)$:= cyclomatic number

For complete spruces, *gain* is the number of tips; adjusted gain $\widehat{gain} := gain$.

For incomplete spruces, *gain* is one less than the number of tips; adjusted gain $\widehat{gain} := gain - 1$.

The Algorithm - Underlying Idea

We maintain Q, a collection of spruces.

What we add: Spruces with tips that are isolated vertices.
Let $v_1, v_2, \ldots, v_k$ be all vertices isolated in Q that are adjacent in G to both x and y.

If $k \geq 1$, let $S_Q(x,y)$ be the spruce with base vertices x and y, tips $v_1, v_2, \ldots, v_k$, and the edge xy if it exists in G. *Add* $S_Q(x,y)$ to Q.

What do we remove: For each component C of Q, the algorithm keeps a *weighted tree* T_C whose vertex set is $V(C)$ and edge set is as follows: For each spruce S in C with base vertices x and y, and tips $v_1, v_2, \ldots, v_k$, there is an edge xy with weight $\widehat{gain}$ in T_C and edges xv_i with weight 1 for $i = 1, \ldots, k$.

$index_Q(x,y)$ is an edge in T_C of minimum weight in the path in T_C from x to y. Let x' and y' be the endpoints of $index_Q(x,y)$, and C be the component of Q containing x, x', y, and y'. Let S' be the spruce in Q containing x' and y'. *If* x' and y' are the base vertices of S', then remove S' from Q.

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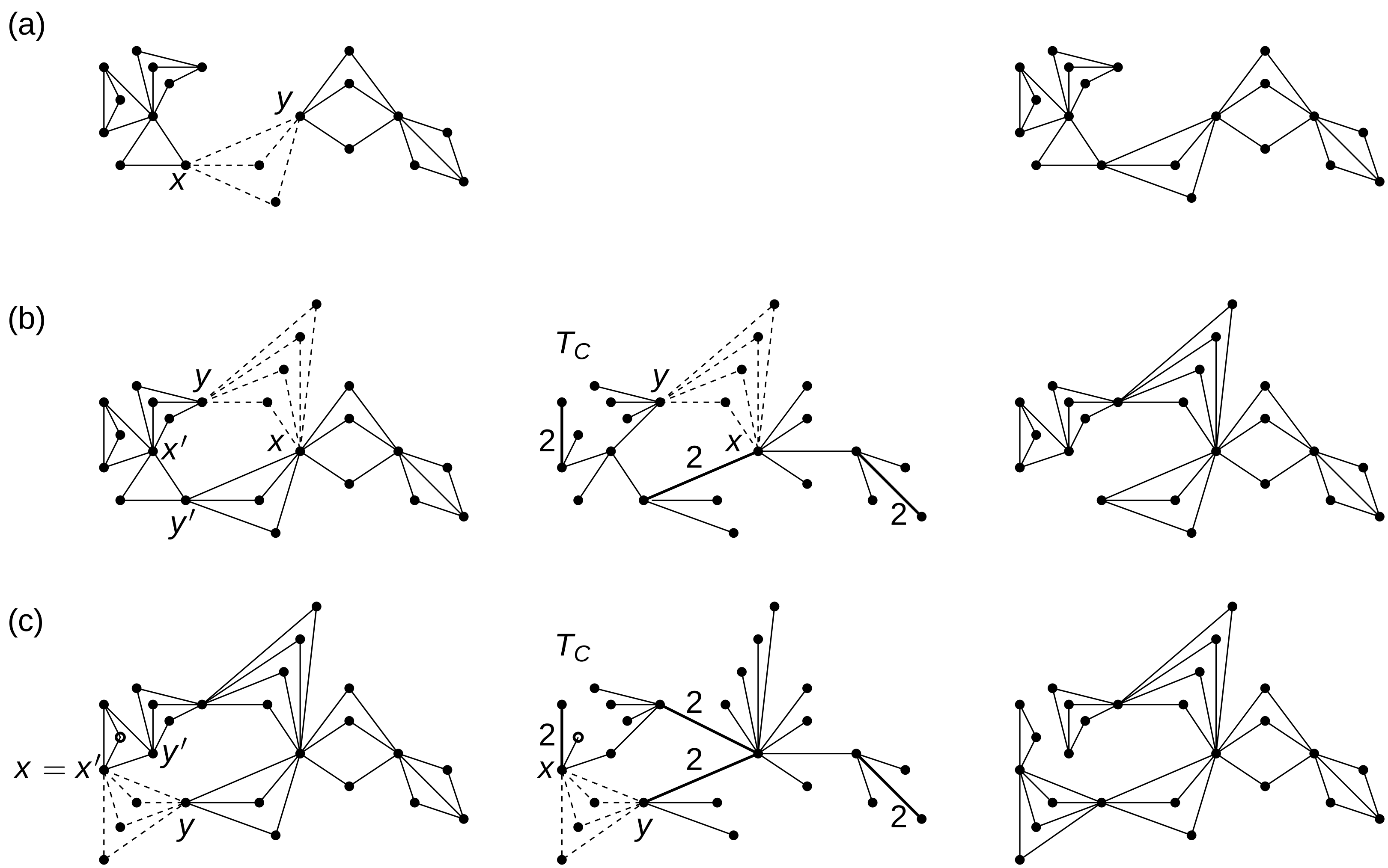
The Algorithm

CONSTRUCT-SPRUCE-STRUCTURE (G)

$Q \leftarrow \emptyset$
while there are x and y such that $S_Q(x,y)$ is defined
  and $\widehat{gain}(S_Q(x,y)) > w(index_Q(x,y))$ **do**
  if $index_Q(x,y)$ is undefined
    then $Q \leftarrow Q \cup \{S_Q(x,y)\}$
    else let x' and y' be the endpoints of $index_Q(x,y)$
      let S' be the spruce in Q containing x' and y'
      $Q \leftarrow Q \setminus \{S'\} \cup \{S_Q(x,y)\}$
      if x' or y' is a tip of S'
        then let z be between x', y', a tip of S'
          let $\{e,f\}$ be the edges of S' touching z
          $S \leftarrow S' - \{e,f\}$
          if S is not degenerate nor single edge
            then $Q \leftarrow Q \cup \{S\}$
add bridges to Q to obtain a connected spanning subgraph of G
return Q

Local improvement examples



Running Time Analysis

If $gain(Q)$ increased in every iteration, then it would have been easy to conclude that the algorithm runs a polynomial number of iterations.
The gain of Q never decreases and, in the iterations in which the gain of Q is same, the number of components increases.

Define $\Phi(Q) = 3\,gain(Q) + c(Q)$, where $c(Q)$ is the number of components of Q when Q is seen as a spanning subgraph of G.

We prove: Every iteration of the algorithm increases the parameter Φ.

$gain(Q) \le (2n-3) - (n-1) = n-2$,
so $\Phi(Q)$ is bounded by $3(n-2) + n = 4n-6$.

Each iteration can be easily implemented in polynomial time:
$O(n^2)$ pairs x, y for which $S_Q(x, y)$ must be computed and, if possible, used in updating Q.

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Approximation ratio ideas - to beat $1/2$

- If significantly many vertices in our structure, we win.
- If OPT has significantly less than $2n$ edges, we win.
- If none of the above, the spruces of OPT have significant $\widehat{gain}$.

Approximation ratio ideas - to beat $1/2$

From OPT, construct *w*eighted Series Parallel graph with $\widehat{gain}$ on edges.

Compare to our weighted forest.

We have a maximum spanning forest in the union of the two graphs!

Thus our $\widehat{gain}$ is $1/2$ of what that of OPT.

Therefore we have significant $\widehat{gain}$.

New Questions

- Weighted maximum Series-Parallel subgraph problem.
- Maximum induced Series-parallel subgraph problem.
- For fixed r, maximum K_r-minor-free subgraph problem.
- In particular, maximum K_5-minor-free subgraph problem.
  Number of edges in such a graph are $\leq 3n - 6$.
  Also, the structural characterization is known - constructed from copies of planar graphs and Wagner's graph by gluing over k-cliques for $k \leq 3$.
- For fixed r, maximum subgraph of tree width $\leq r$.
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