# Polynomial Method for DP-coloring of Graphs

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# Graph Coloring

- Color vertices so that any vertices with an edge between them must get different colors.
- A proper $m$-coloring of a graph $G$ is a labeling $c : V(G) \to [m]$, such that $c(u) \neq c(v)$ whenever $u$ and $v$ are adjacent in $G$.
- Minimum number of colors needed for such a coloring is called the chromatic number $\chi(G)$ of the graph $G$.
- Each vertex has the same list of colors $[m]$ available to it.

# List Coloring

- List coloring was introduced independently by Vizing (1976) and Erdős, Rubin, and Taylor (1979), as a generalization of usual graph coloring.

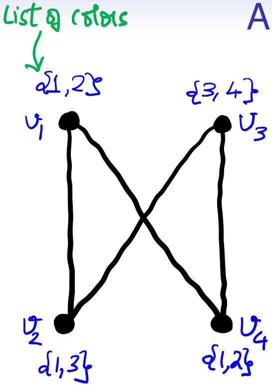
# List Coloring

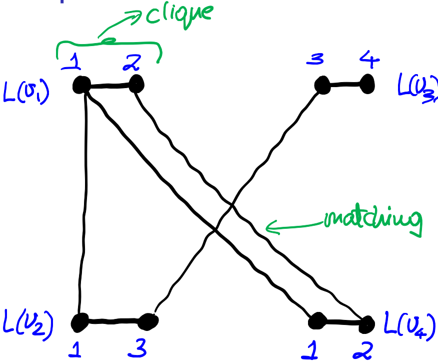
- For graph $G$ suppose each $v \in V(G)$ is assigned a list, $L(v)$, of colors. We refer to $L$ as a list assignment. If all the lists associated with the list assignment $L$ have size $m$, we say that $L$ is an $m$-assignment.
- An L-coloring for $G$ is a proper coloring, $f$, of $G$ such that $f(v) \in L(v)$ for all $v \in V(G)$.

# List Coloring

- The list chromatic number of a graph $G$, written $\chi_\ell(G)$, is the smallest $m$ such that $G$ is $L$-colorable whenever $|L(v)| \geq m$ for each $v \in V(G)$.
- Since usual coloring corresponds to a constant list assignment,
$$\chi(G) \leq \chi_\ell(G).$$

# A Different Perspective




**Proper L-coloring**

$v_1 \to 2$
$v_2 \to 1$
$v_3 \to 3$
$v_4 \to 1$

corresponds to

**Independent set of size 4 here.**

$2 \in L(v_1)$, $1 \in L(v_2)$,
$3 \in L(v_3)$, $1 \in L(v_4)$

# DP-Coloring

- In 2015, Dvořák and Postle introduced DP-coloring (they called it correspondence coloring) of graphs.
- Intuitively, DP-coloring considers the worst-case scenario of how many colors we need in the lists if we no longer can identify the names of the colors. Each vertex still gets a list of colors but identification of which colors are different can vary from edge to edge.

- A (DP-)cover of $G$ is a pair $\mathcal{H} = (L, H)$ consisting of a graph $H$ and a function $L : V(G) \to \mathcal{P}(V(H))$ satisfying:

  (1) the set $\{L(u) : u \in V(G)\}$ is a partition of $V(H)$;
  (2) for every $u \in V(G)$, the graph $H[L(u)]$ is complete;
  (3) if $E_H(L(u), L(v))$ is nonempty, then $u = v$ or $uv \in E(G)$;
  (4) if $uv \in E(G)$, then $E_H(L(u), L(v))$ is a matching (the matching may be empty).

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- See also "covering graphs", "Lifts". Studied since 1990s.

- Intuition:
  Blow up each vertex $u$ in $G$ into a clique of size $|L(u)|$;
  Add a matching (possibly empty) between any two such cliques for vertices $u$ and $v$ if $uv$ is an edge in $G$.

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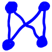
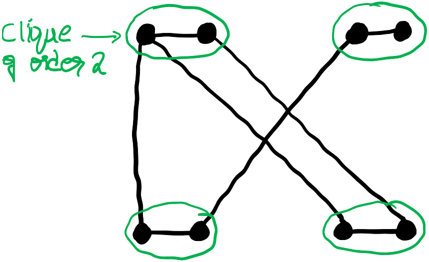
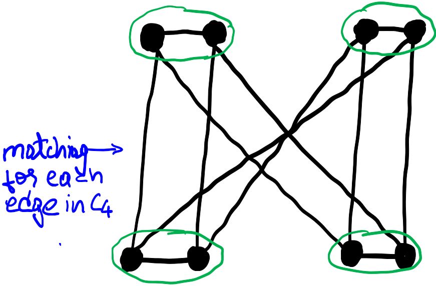
- A cover $\mathcal{H} = (L, H)$ is called $m$-fold if $|L(u)| = m$ for all $u$.
- Two 2-fold covers of $C_4$: 



# DP-Chromatic Number of a Graph

- Given $\mathcal{H} = (L, H)$, a cover of $G$, an $\mathcal{H}$-coloring of $G$ is an independent set in $H$ of size $|V(G)|$. Equivalently, an independent transversal in $\mathcal{H}$.
- The DP-chromatic number of a graph $G$, $\chi_{DP}(G)$, is the smallest $m$ such that $G$ admits an $\mathcal{H}$-coloring for every $m$-fold cover $\mathcal{H}$ of $G$.

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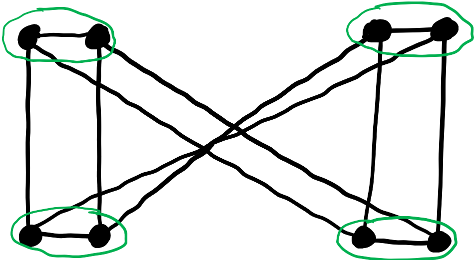
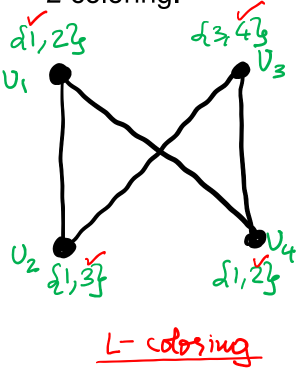
- $\chi_{DP}(C_4) > 2 = \chi_\ell(C_4)$:

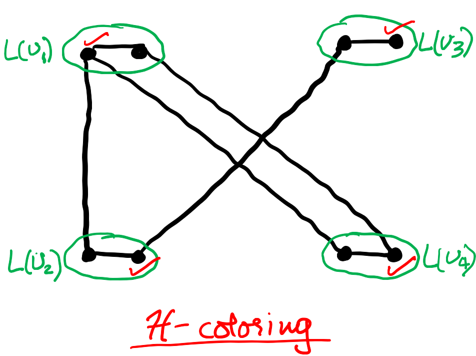


No independent set of size 4 in this 2-fold cover of $C_4$

# DP-Coloring and List Coloring

- Given an $m$-assignment, $L$, for a graph $G$, it is easy to construct an $m$-fold cover $\mathcal{H}$ of $G$ such that: $G$ has an $\mathcal{H}$-coloring if and only if $G$ has a proper $L$-coloring.




- $\chi(G) \leq \chi_\ell(G) \leq \chi_{DP}(G)$.

# Counting Colorings

- Birkhoff 1912: For $m \in \mathbb{N}$, let $P(G, m)$ denote the number of proper colorings of $G$ where the colors used come from $\{1, \ldots, m\}$. $P(G, m)$ is the chromatic polynomial of $G$.

- $P(G, L)$ be the number of proper $L$-colorings of $G$.
- Kostochka and Sidorenko 1990: The list color function $P_\ell(G, m)$ is the minimum value of $P(G, L)$ over all possible $m$-assignments $L$ for $G$.

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- For $\mathcal{H} = (L, H)$, a cover of graph $G$, $P_{DP}(G, \mathcal{H})$ be the number of $\mathcal{H}$-colorings of $G$.
- K. and Mudrock 2021: The DP color function, $P_{DP}(G, m)$, is the minimum value of $P_{DP}(G, \mathcal{H})$ where the minimum is taken over all possible $m$-fold covers $\mathcal{H}$ of $G$.

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- $P_{DP}(G, m) \leq P_\ell(G, m) \leq P(G, m)$.

# Polynomial Method

In a survey article, Terrence Tao describes the polynomial method as:

"strategy is to capture the arbitrary set of objects in the zero set of a polynomial whose degree is in control; for instance the degree may be bounded by a function of the number of the objects."

Then we use algebraic tools to understand this zero set.

This paradigm has been used for breakthrough results in arithmetic combinatorics, additive combinatorics, number theory, graph theory, discrete geometry, and more.

# Combinatorial Nullstellentsatz

- How many zeros can a *n*-variable polynomial on a field $\mathbb{F}$ have?

## Lemma

*Let $f \in \mathbb{F}[x_1, \ldots, x_n]$. For each $i$, let the degree of $f$ in $x_i$ be at most $t_i$, and suppose $S_i$ is a set of more than $t_i$ distinct values from $\mathbb{F}$. If $f(x_1, \ldots, x_n) = 0$ for $(x_1, \ldots, x_n) \in \prod_{i=1}^{n} S_i$, then $f$ is the zero polynomial.*

Can we do better? Instead of controlling the individual degree of each variable, work with the total degree of the polynomial.

# Combinatorial Nullstellentsatz

## Theorem (Combinatorial Nullstellensatz; Alon (1999))

*Suppose that $f \in \mathbb{F}[x_1, \ldots, x_n]$, and the degree of $f$ is at most $\sum_{i=1}^{n} t_i$. For each $i \in \{1, \ldots, n\}$, suppose that $S_i$ is a set of elements in $\mathbb{F}$ with $|S_i| > t_i$.*

*If $[\prod_{i=1}^{n} x_i^{t_i}]_f \neq 0$, then $f(s_1, \ldots, s_n) \neq 0$ for some $(s_1, \ldots, s_n) \in \prod_{i=1}^{n} S_i$.*

- $[\prod_{i=1}^{n} x_i^{t_i}]_p$ denotes the element of $\mathbb{F}$ that is the coefficient of the monomial $\prod_{i=1}^{n} x_i^{t_i}$ in the expanded form of $p \in \mathbb{F}[x_1, \ldots, x_n]$.

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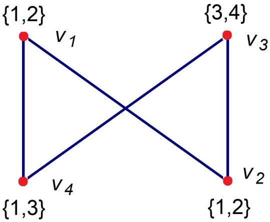
- Combinatorial Nullstellensatz has been applied to numerous problems in additive combinatorics, number theory, discrete geometry, graph theory since 1980s.

# Graph Polynomial

- The graph polynomial of $G$ with $V(G) = \{v_1, \ldots, v_n\}$ is $f_G(x_1, x_2, \ldots, x_n) = \prod_{v_i v_j \in E(G),\, j>i}(x_i - x_j)$.




$f_G(x_1, x_2, x_3, x_4) = (x_1 - x_2)(x_2 - x_3)(x_3 - x_4)(x_1 - x_4)$

- $f_G$ is homogenous of degree $|E(G)|$.
- If $L$ is a list assignment for $G$ with $L(v) \subset \mathbb{R}$ for $v \in V(G)$, then a proper $L$-coloring of $G$ exists if and only if there is a $(c_1, \ldots, c_n) \in \prod_{i=1}^{n} L(v_i)$ such that $f_G(c_1, \ldots, c_n) \neq 0$.
- $f_G(1, 2, 4, 3) = (-1)(-2)(1)(-2) = -4$ (In fact, $\chi_\ell(C_4) \leq 2$)

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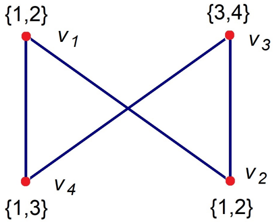


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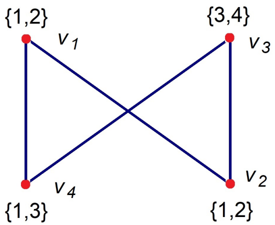


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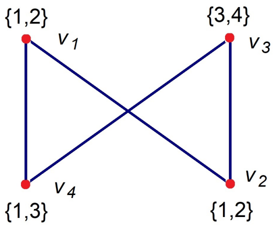


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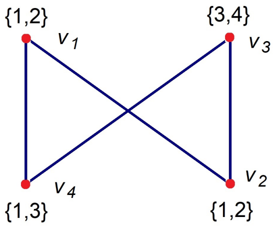


- Suppose $f \in \mathbb{R}[x_1, x_2, x_3, x_4]$ is given by $f(x_1, x_2, x_3, x_4) = (x_1 - x_2)(x_2 - x_3)(x_3 - x_4)(x_1 - x_4)$. Note $f$ has degree at most 4.
- Suppose $S_1 = S_2 = \{1, 2\}$, $S_3 = \{3, 4\}$, and $S_4 = \{1, 3\}$.
- Since $[x_1 x_2 x_3 x_4]_f = -2 \neq 0$, the CN tells us there is an element in $\prod_{i=1}^{4} S_i$ for which $f$ is nonzero.
- Alon-Tarsi (1990) famously gave a combinatorial interpretation of this non-zero coefficient of the graph polynomial. *A fundamental method for bounding the list chromatic number: $\chi_\ell(G) \leq AT(G)$.*

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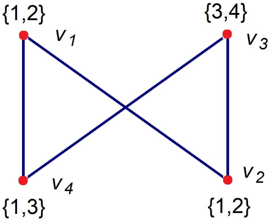


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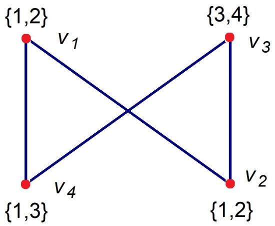


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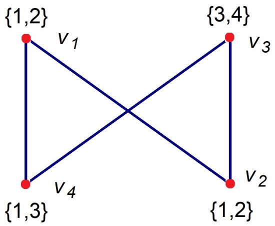


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# Combinatorial Nullstellensatz and DP Coloring

- $\chi_\ell(C_4) \leq AT(C_4) \leq 2$, but we know $\chi_{DP}(C_4) > 2$.
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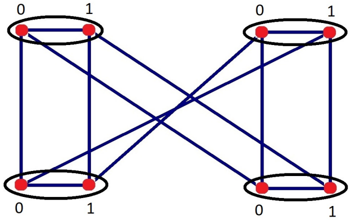


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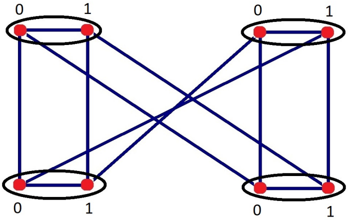


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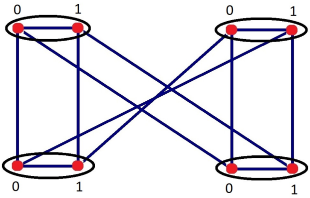
# Prime Covers

- Given a graph $G$ and a function $f : V(G) \to \mathbb{N}$, we say $\mathcal{H} = (L, H)$ is an $f$-cover of $G$ if $|L(u)| = f(u)$ for each $u \in V(G)$. We say that $G$ is $f$-DP-colorable if $G$ is $\mathcal{H}$-colorable whenever $\mathcal{H}$ is an $f$-cover of $G$.
- An $f$-cover $\mathcal{H} = (L, H)$ of $G$ is a prime cover of $G$ of order t whenever $t$ is a power of a prime and $\max_{v \in V(G)} f(v) \leq t$.
- When the choice of $t$ is implicitly known, we simply say prime cover or prime $f$-cover.
- If $\mathcal{H} = (L, H)$ is a prime cover of $G$ of order $t$, we assume that $L(v) \subseteq \{(v, j) : j \in \mathbb{F}_t\}$ for each $v \in V(G)$.

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# Prime Covers

- An $f$-cover $\mathcal{H} = (L, H)$ of $G$ is a prime cover of $G$ of order t whenever $t$ is a power of a prime and $\max_{v \in V(G)} f(v) \leq t$.
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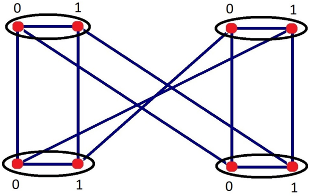


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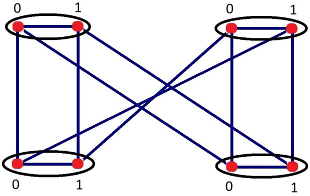


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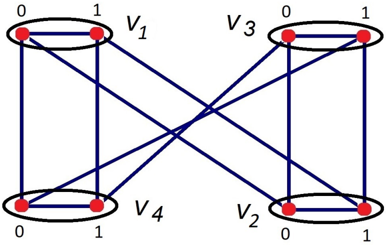
# Saturation Functions

- $V(G) = \{v_1, \ldots, v_n\}$.
- $\mathcal{H} = (L, H)$ be a prime cover of $G$ of order $t$.
- For each $v_i v_j \in E(G)$, the saturation function associated with $E_H(L(v_i), L(v_j))$ is denoted $\sigma^{\mathcal{H}}_{v_i v_j}$.




For example, $\sigma^{\mathcal{H}}_{v_3 v_4}(0) = 1$ and $\sigma^{\mathcal{H}}_{v_3 v_4}(1) = 0$.

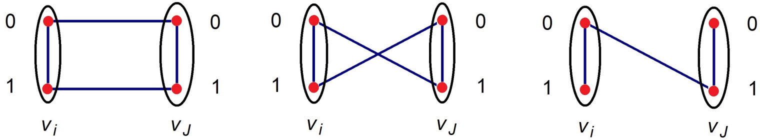
# Good Saturation Functions

- We say that $\sigma^{\mathcal{H}}_{v_i v_j}$ is good if there is a $\beta \in \mathbb{F}_t$ such that for each $a$ in the domain of $\sigma^{\mathcal{H}}_{v_i v_j}$

$$a - \sigma^{\mathcal{H}}_{v_i v_j}(a) = \beta$$

  where subtraction is performed in $\mathbb{F}_t$.

- For a 2-fold cover, each saturation function associated with a matching, must be good!



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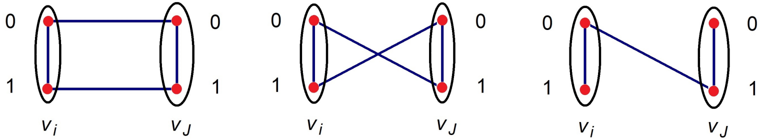
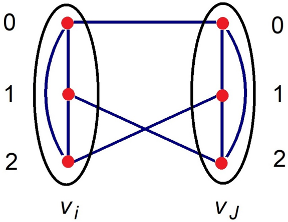


# Good Covers

- Suppose $\mathcal{H} = (L, H)$ is a prime cover of $G$ of order $t$.
- We say that $\mathcal{H}$ is a good prime cover of order $t$ if for each $v_i v_j \in E(G)$ with $j > i$, the associated saturation function $\sigma^{\mathcal{H}}_{v_i v_j}$ is good.
- For example, we know every 2-fold cover is a good prime cover of order 2.
- It **cannot** be said that every 3-fold cover is a good prime cover of order 3.



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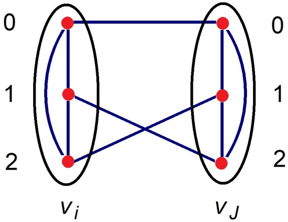

# Key Observations

- Suppose $G$ is a graph with $V(G) = \{v_1, \ldots, v_n\}$ and $\mathcal{H} = (L, H)$ is a good prime cover of $G$ of order $t$.
- For each $v_i v_j \in E(G)$ with $j > i$, there is a $\beta_{i,j} \in \mathbb{F}_t$ such that $a - \sigma^{\mathcal{H}}_{v_i v_j}(a) - \beta_{i,j} = 0$ for each $a$ in the domain of $\sigma^{\mathcal{H}}_{v_i v_j}$.
- Let $\hat{f}(x_1, \ldots, x_n) = \prod_{v_i v_j \in E(G),\, j>i} (x_i - x_j - \beta_{ij})$.
- An $\mathcal{H}$-coloring of $G$ exists if there is a $(p_1, \ldots, p_n) \in \prod_{i=1}^n P_i$ such that $\hat{f}(p_1, p_2, \ldots, p_n) \neq 0$, where $P_i = \{j \in \mathbb{F}_t : (v_i, j) \in L(v)\}$.
- Note that if $\sum_{i=1}^n t_i = |E(G)|$, then $\left[\prod_{i=1}^n x_i^{t_i}\right]_{\hat{f}} = \left[\prod_{i=1}^n x_i^{t_i}\right]_{f_G}$. So, the Combinatorial Nullstellensatz can be applied.

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# Combinatorial Nullstellensatz for DP Coloring

## Theorem (K., Mudrock (2020))

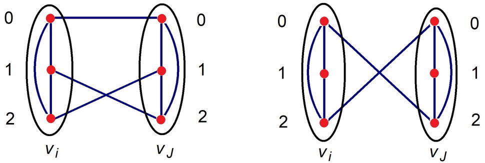
*Let $\mathcal{H} = (L, H)$ be a good prime cover of order $t$ of a graph $G$. Suppose that $f_G \in \mathbb{F}_t[x_1, \ldots, x_n]$. If $[\prod_{i=1}^{n} x_i^{t_i}]_{f_G} \neq 0$ and $|L(v_i)| > t_i$ for each $i \in [n]$, then there is an $\mathcal{H}$-coloring of $G$.*

With applications to:

- $f$-DP-coloring of a cone of a connected bipartite graph.
- DP-coloring analogue of a Theorem of Akbari et al. (2006) on a sufficient condition for f-choosability in terms of unique colorability.
- Completely determine the DP-chromatic number of squares of all cycles.
- Algebraic sufficient condition for DP-3-colorability.

# Three-fold Covers

- Suppose that $\mathcal{H}$ is a prime cover of $G$ of order 3.
- If $\sigma^{\mathcal{H}}_{v_i v_j}$ is bad, there is a $\beta_{i,j} \in \mathbb{F}_3$ so that $a + \sigma^{\mathcal{H}}_{v_i v_j}(a) = \beta_{i,j}$.




- So, for any $v_i v_j \in E(G)$ there is a $c_{i,j}, \beta_{i,j} \in \mathbb{F}_3$ so that

$$a + (-1)^{c_{i,j}} \sigma^{\mathcal{H}}_{v_i v_j}(a) = \beta_{i,j}$$

for each $a$ in the domain of $\sigma^{\mathcal{H}}_{v_i v_j}$.

# Three-fold Covers

### Theorem (K., Mudrock (2020))

*Suppose G is a graph with $\chi_{DP}(G) \geq 2$ and $V(G) = \{v_1, \ldots, v_n\}$. Let $\mathcal{F} \subseteq \mathbb{F}_3[x_1, \ldots, x_n]$ be the set of at most $2^{|E(G)|}$ polynomials given by:*
$\mathcal{F} = \left\{ \prod_{v_i v_j \in E(G),\, j>i} (x_i + b_{i,j} x_j) : b_{i,j} \in \{-1, 1\} \right\}.$
*If for each $f \in \mathcal{F}$ there exists $(t_1, t_2, \ldots, t_n) \in \prod_{i=1}^{n} \{0, 1, 2\}$ such that $[\prod_{i=1}^{n} x_i^{t_i}]_f \neq 0$, then $\chi_{DP}(G) \leq 3$.*

- The number of polynomials in set $\mathcal{F}$ can be reduced to $2^{|E(G)|-|V(G)|+1}$, when $G$ is a connected graph containing a cycle.

# Combinatorial Nullstellensatz for DP-color Function

**Theorem (Alon, Füredi (1993))**

*Let $\mathbb{F}$ be an arbitrary field, let $A_1$, $A_2$, ..., $A_n$ be any non-empty subsets of $\mathbb{F}$, and let $B = \prod_{i=1}^{n} A_i$. Suppose that $P \in \mathbb{F}[x_1, \ldots, x_n]$ is a polynomial of degree $d$ that does not vanish on all of $B$. Then, the number of points in $B$ for which $P$ has a non-zero value is at least $\min \prod_{i=1}^{n} q_i$ where the minimum is taken over all integers $q_i$ such that $1 \leq q_i \leq |A_i|$ and $\sum_{i=1}^{n} q_i \geq -d + \sum_{i=1}^{n} |A_i|$.*

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**Theorem (Dahlberg, K., Mudrock (2023+))**

*Let $G$ be a graph with $\chi_{DP}(G) \leq 3$. Suppose that $|V(G)| = n$, $|E(G)| = l$, and $2n \geq l$. Then, $P_{DP}(G, 3) \geq 3^{n-l/2}$.*

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### Corollary

*Let $G$ be an $n$-vertex planar graph of girth at least 5. Then, $P_{DP}(G, 3) \geq 3^{n/6}$.*

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- Such bounds have a long history going back to Birkhoff and Lewis (1946) Conjecture: Given an $n$-vertex planar graph $G$, for each real number $m \geq 4$, $P(G, m) \geq m(m-1)(m-2)(m-3)^{n-3}$.

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*There are infinitely many graphs $G$ for which $\chi_{DP}(G) = 3$, $P_{DP}(G, 3) = P(G, 3)$, and there is an $N_G \in \mathbb{N}$ such that $P_{DP}(G, m) < P(G, m)$ whenever $m \geq N_G$.*

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- Why is this interesting?

# Relationships between the counting functions

- $P_{DP}(G, m) \leq P_\ell(G, m) \leq P(G, m)$.

  Kirov and Naimi 2016: A question of stickiness - Do the list color function and the corresponding chromatic polynomial of a graph stay the same after the first point at which they are both nonzero and equal?

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**Theorem (K., Maxfield, Mudrock, Thomason (2022+))**

*If $G$ is $\Theta(2, 3, 3, 3, 2)$ or $\Theta(2, 3, 3, 3, 3, 3, 2, 2)$, then $P_{DP}(G, 3) = P(G, 3)$ and there is an $N$ such that $P_{DP}(G, m) < P(G, m)$ for all $m \geq N$.*

Only two counterexamples. But now we have infinitely many such examples!

# Combinatorial Nullstellensatz for DP-color Function

**Theorem (Dahlberg, K., Mudrock (2023+))**

*Let $G$ be a graph with $\chi_{DP}(G) \leq 3$. Suppose that $|V(G)| = n$, $|E(G)| = l$, and $2n \geq l$. Then, $P_{DP}(G, 3) \geq 3^{n-l/2}$.*

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# Thank You!

Questions?