Proportional Choosability: A New List Analogue of Equitable Coloring

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Joint work with

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Equitable Coloring

- The study of equitable vertex coloring began with a 1964 conjecture of Erdős and was formally introduced by Meyer in 1973. It asks for color classes to be of roughly equal size.

- An equitable k-coloring of a graph G is a proper k-coloring of G such that the sizes of the color classes differ by at most 1.

- If f is an equitable k-coloring of G then each of the k color classes associated with f are of size

$$\left\lfloor \frac{|V(G)|}{k} \right\rfloor \text{ or } \left\lceil \frac{|V(G)|}{k} \right\rceil$$

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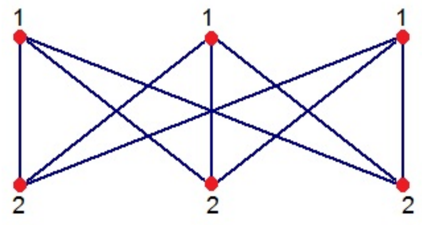
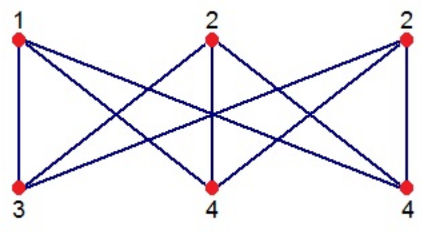
A Simple Example

- An equitable 2-coloring of $K_{3,3}$:



- An equitable 4-coloring of $K_{3,3}$:



- $K_{3,3}$ is not equitably 3-colorable.

Monotonicity?

- The existence of an equitable k-coloring does not imply the existence of an equitable $(k+1)$-coloring. (e.g. $K_{3,3}$ is equitably 2-colorable but not equitably 3-colorable.)
- We get monotonicity in k when k is large enough.

Theorem (Hajnál and Szemerédi (1970))
Every graph G is equitably k-colorable for all $k \geq \Delta(G) + 1$.

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List Coloring

- For graph G a list assignment for G, L, assigns each $v \in V(G)$ a list, $L(v)$, of available colors.
- A proper L-coloring for G is a proper coloring, f, of G such that $f(v) \in L(v)$ for all $v \in V(G)$.
- If all the lists associated with the list assignment L have size k, we say that L is a k-assignment.
- A graph G is said to be k-choosable if a proper L-coloring for G exists whenever L is a k-assignment for G.

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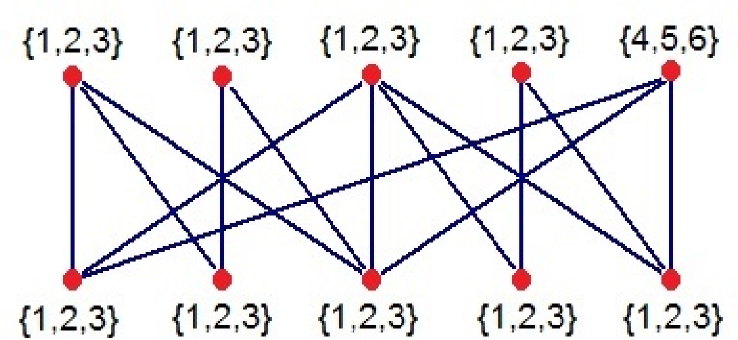
How to Obtain a List Analogue of Equitable Coloring?



Equitable Choosability

- In 2003 Kostochka, Pelsmajer, and West introduced a list analogue of equitable coloring called equitable choosability. They use equitable to capture the notion that no color is used excessively often.

- Suppose L is a k-assignment for graph G. A proper L-coloring for G is equitable if it uses each color at most $\lceil |V(G)|/k \rceil$ times. Such a coloring is called an equitable L-coloring.

- A graph is equitably k-choosable if an equitable L-coloring for G exists whenever L is a k-assignment for G.

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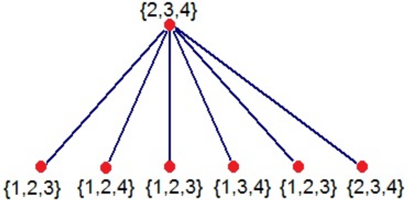
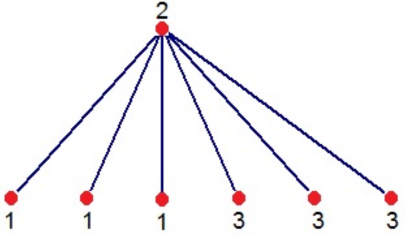
- Consider a copy of $K_{1,6}$ and the following 3-assignment.



- We seek to use no color more than $\lceil 7/3 \rceil = 3$ times.



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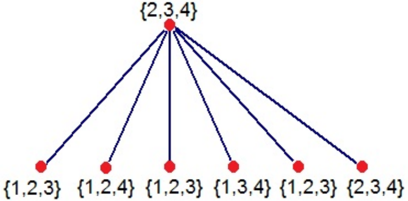

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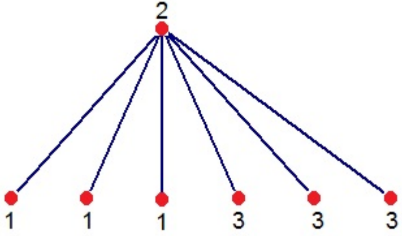

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Choosability versus Equitable Choosability

- If a graph is *k*-choosable, then the graph must be *k*-colorable.

- However, it is possible for a graph to be equitably *k*-choosable, but not equitably *k*-colorable.
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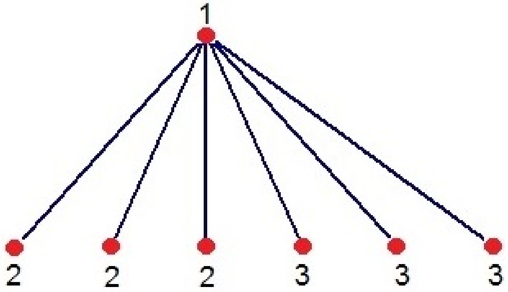

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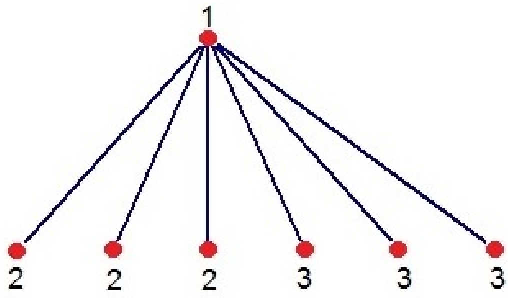

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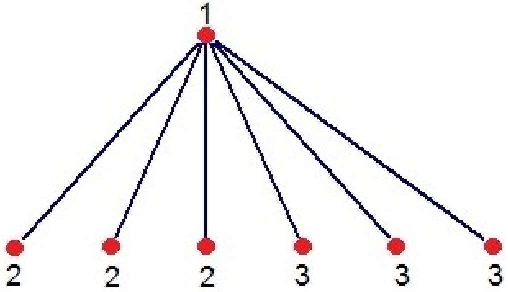

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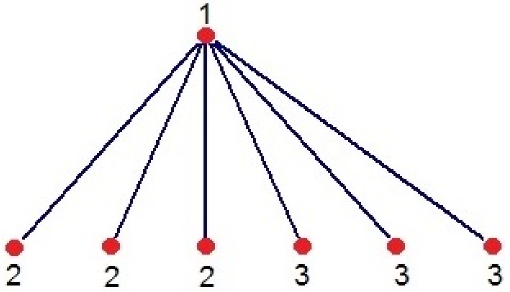

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Proportional Choosability

- Suppose L is a k-assignment for graph G, then the palette of colors associated with L is

$$\mathcal{L} = \bigcup_{v \in V(G)} L(v)$$

- For each $c \in \mathcal{L}$, the multiplicity of c in L, denoted $\eta_L(c)$ or simply $\eta(c)$ when the list assignment is clear, is $\eta(c) = |\{v : v \in V(G), c \in L(v)\}|$.

- A proportional L-coloring for G is a proper L-coloring, f, of G such that for each $c \in \mathcal{L}$,

$$|f^{-1}(\{c\})| = \left\lfloor \frac{\eta(c)}{k} \right\rfloor \text{ or } \left\lceil \frac{\eta(c)}{k} \right\rceil.$$

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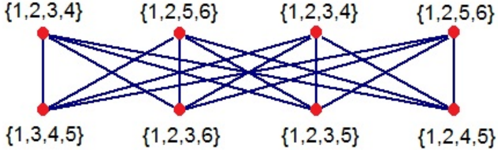


$\eta(1)=8 \Rightarrow$ must be used 2 times

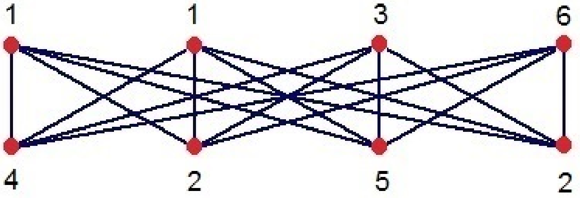
$\eta(2)=7 \Rightarrow$ must be used 1 or 2 times

$\eta(3)=5 \Rightarrow$ must be used 1 or 2 times

$\eta(4)=4 \Rightarrow$ must be used 1 time

$\eta(5)=5 \Rightarrow$ must be used 1 or 2 times

$\eta(6)=3 \Rightarrow$ must be used 0 or 1 time



Question: Can you think of a 4-assignment, L, for $K_{4,4}$ such that there is no proportional L-coloring?

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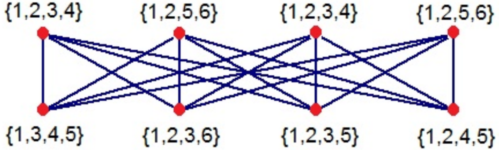

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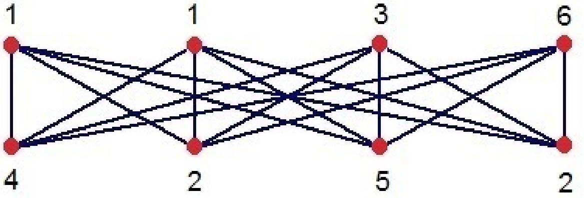

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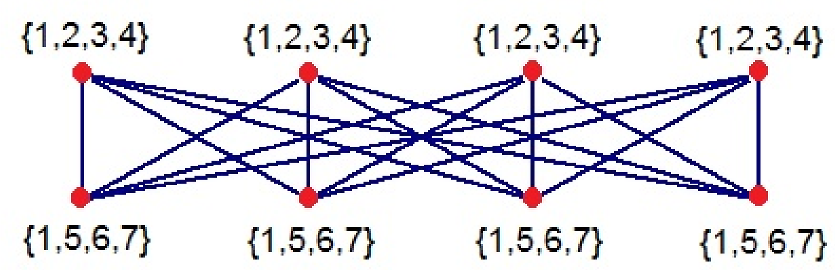

Note: 1 has to be used twice, while all the remaining six colors have to be used exactly once each.

Proportional Choosability contd.

- G is proportionally k-choosable if for any k-assignment, L, for G, there is a proportional L-coloring for G.

Proposition (K., Mudrock, Pelsmajer, Reiniger)

If G is proportionally k-choosable, then G is equitably k-choosable and G is equitably k-colorable.

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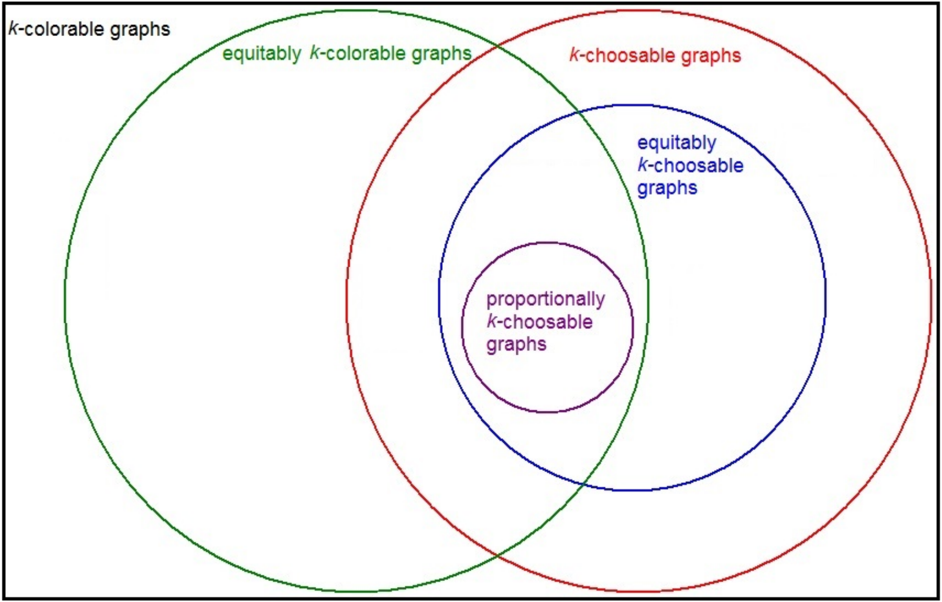

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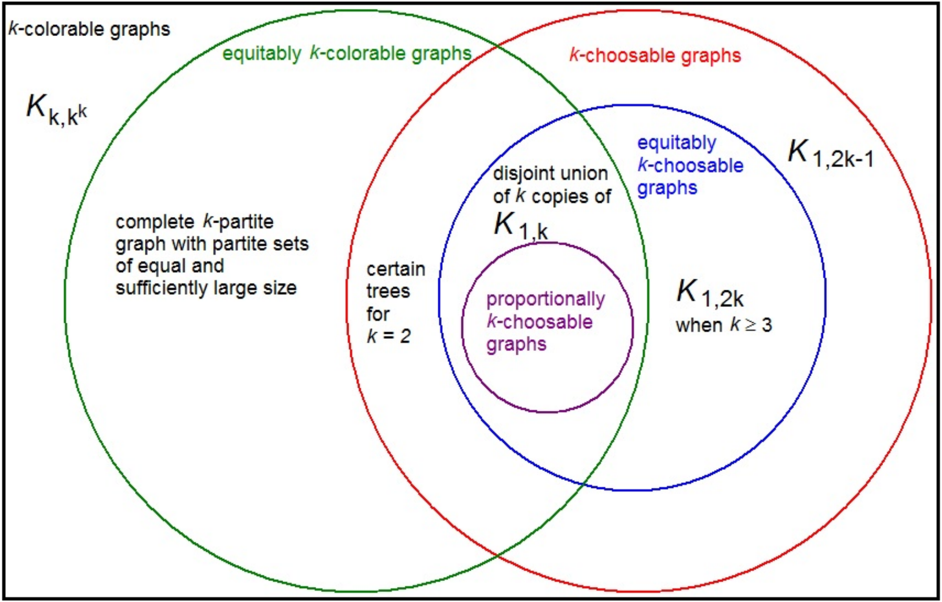

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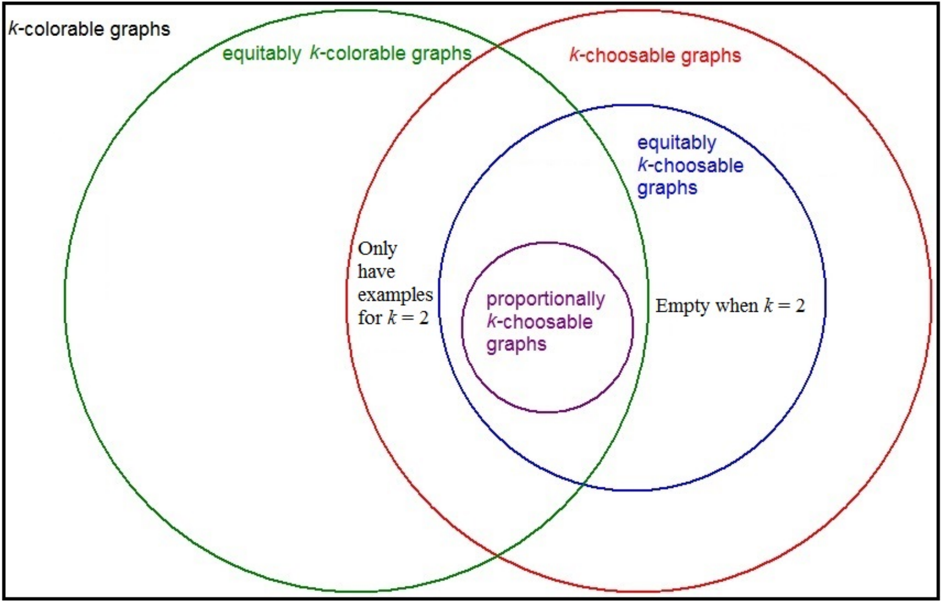

Monotone Property

Lemma (K., Mudrock, Pelsmajer, Reiniger)

Suppose H is a subgraph of G. If G is proportionally k-choosable, then H is proportionally k-choosable.

- This property also holds for k-colorability and k-choosability.
- This property does not hold in the context of equitable coloring. For example, $K_{3,3}$ is equitably 2-colorable, but $K_{1,3}$ is not equitably 2-colorable.
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Proportional Choice Number

- The fact that we have monotonicity in k when it comes to proportional choosability leads us to introduce a graph invariant.

- For graph G, the proportional choice number of G, denoted $\chi_{pc}(G)$, is the smallest k such that G is proportionally k-choosable.

Proposition (K., Mudrock, Pelsmajer, Reiniger)

If G is not a complete graph, then $\chi_{pc}(G) \leq |V(G)| - 1$.

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Proportional Choosability of Small Graphs

Theorem (K., Mudrock, Pelsmajer, Reiniger)

For any graph G,

$$\chi_{pc}(G) \leq \Delta(G) + \frac{|V(G)|}{2}.$$

- We know $\chi_{pc}(G) \geq (\Delta(G)+1)/2$
- To prove this Theorem, we need two Lemmas.

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Rough Proof Idea

- We can find an appropriate L-coloring for G that doesn't use any color excessively.

Lemma (K., Mudrock, Pelsmajer, Reiniger)

Let L be a k-assignment for a graph G with $k \geq \Delta(G) + |V(G)|/2$. There is a proper L-coloring of G that uses no color $c \in \mathcal{L}$ more than $\lceil \eta(c)/k \rceil$ times.

- We give an algorithmic argument to convert an equitable L-coloring into a proportional L-coloring for a k-assignment L of G with every color having multiplicity less than $2k$.

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Suppose L is a k-assignment for G with $\max_{c \in \mathcal{L}} \eta(c) < 2k$. If there is a proper L-coloring, f, of G with $|f^{-1}(\{c\})| \leq \lceil \eta(c)/k \rceil$ for each $c \in \mathcal{L}$, then G is proportionally L-colorable.

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Suppose L is a k-assignment for G with $\max_{c \in \mathcal{L}} \eta(c) < 2k$. If there is a proper L-coloring, f, of G with $|f^{-1}(\{c\})| \leq \lceil \eta(c)/k \rceil$ for each $c \in \mathcal{L}$, then G is proportionally L-colorable.

Proportional Choosability of a Star

Proposition (Kaul, M., Pelsmajer, Reiniger)

$K_{1,m}$ is proportionally k-choosable if and only if $k \geq 1 + m/2$.

- Note that the " $\Longrightarrow$ " direction is easy. If $k < 1 + m/2$, then $k \leq (1 + m)/2$ and $\lfloor (m + 1)/k \rfloor \geq 2$.
- So, $K_{1,m}$ is not even equitably k-colorable when $k \leq (m + 1)/2$.

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- Let $G = K_{1,m}$, and L be a k-assignment for G with $k = 1 + \lceil m/2 \rceil$.
- Let $\{v_0\}$ be the partite set of size 1.
- Suppose $L(v_0)$ contains only colors with multiplicity greater than k. In this case we apply:

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- Suppose $L(v_0)$ contains a color with multiplicity at most k. In this case we use some classic matching theory.
- Recall the following classic Corollaries of Hall's Theorem.

Corollary

For $k > 0$, every k-regular bipartite multigraph has a perfect matching.

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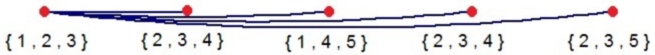

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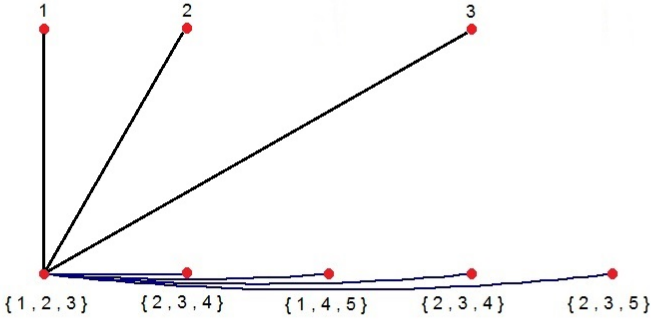

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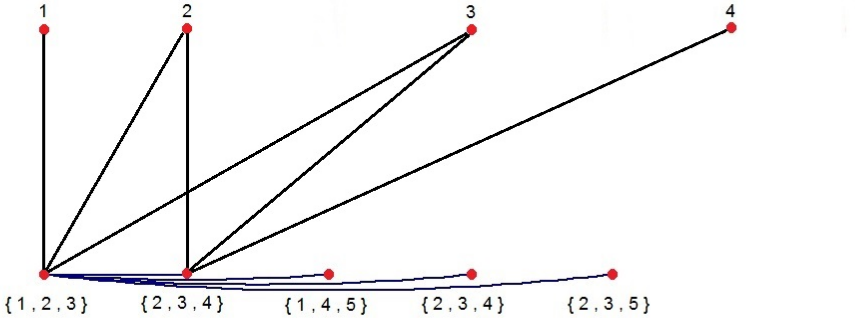

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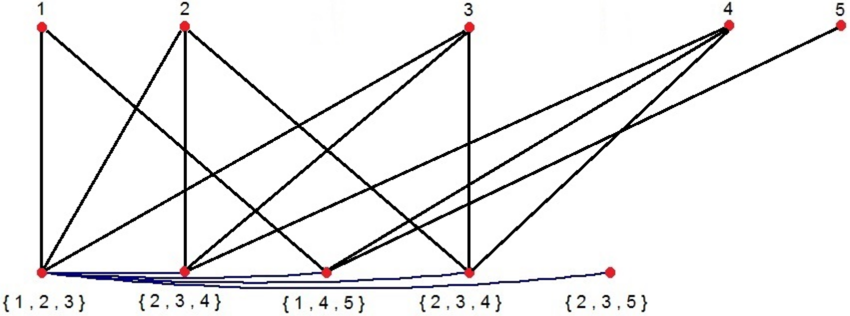

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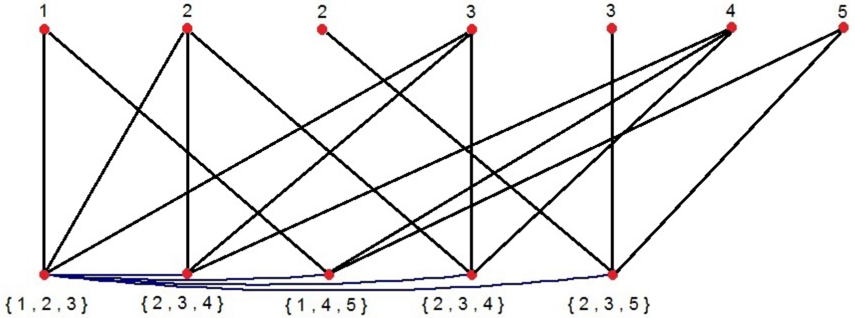

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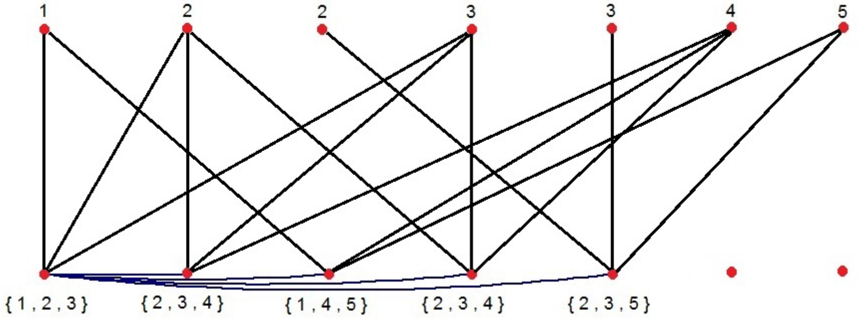

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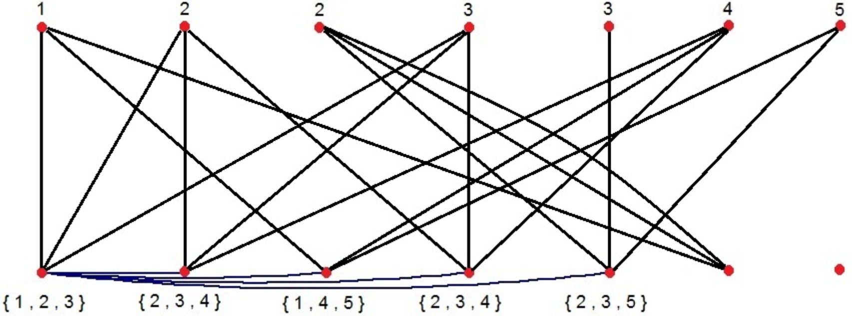

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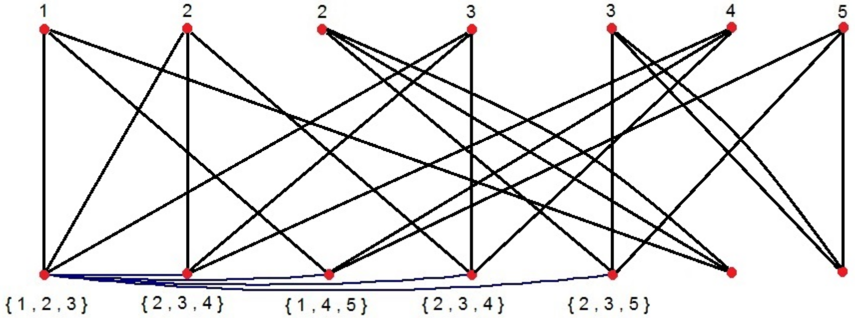

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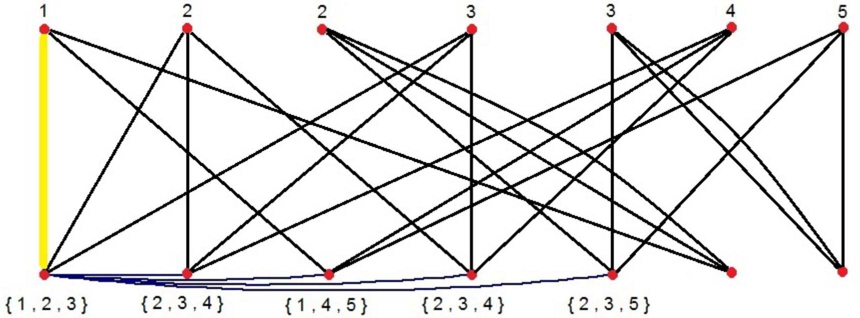

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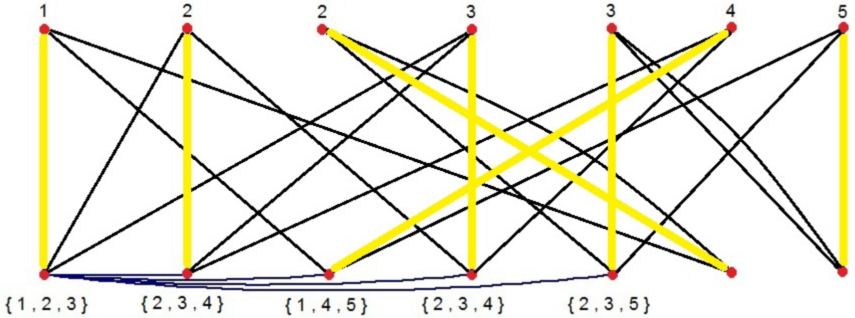

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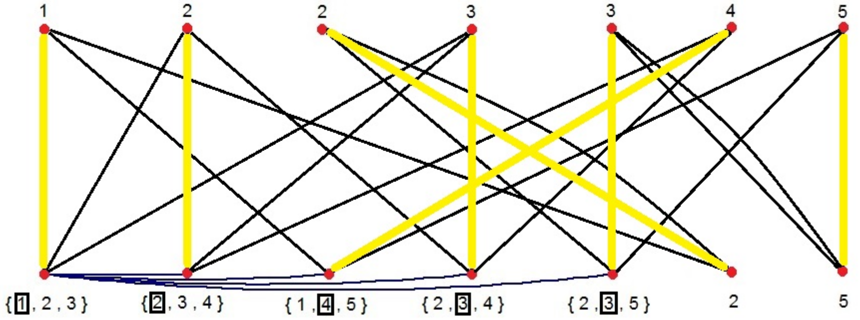

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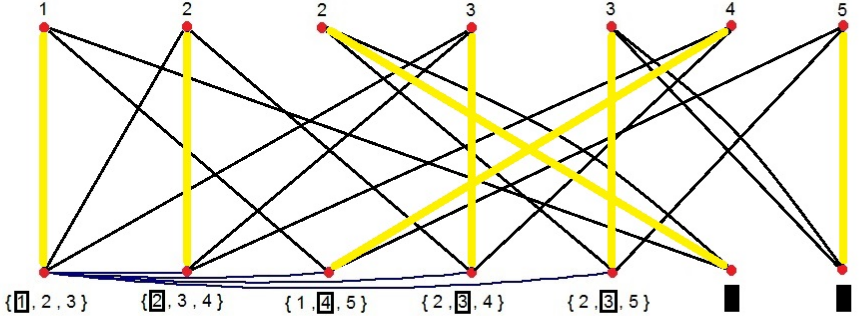

A Comment on Disconnected Graphs

- Understanding proportional choosability may be difficult on a disconnected graph even when we completely understand the proportional choosability of each component.

- For $m \geq 2$, since $m \geq 1 + m/2$, we know $K_{1,m}$ is proportionally m-choosable.

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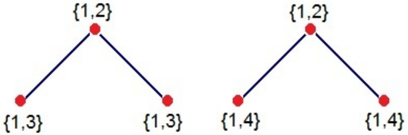
A Comment on Disconnected Graphs contd.

Proposition (K., Mudrock, Pelsmajer, Reiniger)

Let $H_1, H_2, \ldots, H_m$ be m pairwise vertex disjoint copies of $K_{1,m}$. If $G = \sum_{i=1}^{m} H_i$, then G is not proportionally m-choosable.



$\eta(1) = 6 \Rightarrow$ must be used 3 times

$\eta(2) = 2 \Rightarrow$ must be used 1 time

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A Result for Disconnected Graphs

- Another result we have obtained via the matching ideas involves the disjoint union of cliques.

Theorem (K., Mudrock, Pelsmajer, Reiniger)

If G is a graph such that each of its components have at most t vertices, then G is proportionally t-choosable.

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Suppose G is the disjoint union of cliques and the largest component of G has t vertices. Then G is proportionally t-choosable.

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Proportional 2-Choosability

- Characterizations of 2-colorable, equitably 2-colorable, and 2-choosable graphs are known.

Theorem (K., Mudrock, Pelsmajer, Reiniger)

Graph G is proportionally 2-choosable if and only if G is a disjoint union of paths where the largest component of G has at most 5 vertices and all the other components of G have 2 or fewer vertices.

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- We know $K_{1,2k-1}$ is not proportionally k-choosable for each k. So, $\chi_{pc}(H) > \frac{\Delta(H)+1}{2}$.
- Hence, Proportional 2-choosable graphs have $\Delta(G) \leq 2$.
- Since proportional 2-choosability implies 2-colorability, G consists of paths and even cycles.
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- [Proportional Analogue of Obha] If G is equitably k-colorable and $|V(G)| \leq 2k - 1$, must it be that G is proportionally k-choosable?
- [Paths!] For each $n \geq 6$, what is the value of $\chi_{pc}(P_n)$? We know its between 3 and $n/2 + 2$. Does there exist a constant C such that $\chi_{pc}(P_n) \leq C$ for all n?
- [Disjoint Unions] Suppose G is proportionally k-choosable. If H is a graph that is vertex disjoint from G with $|V(H)| \leq k$, must it be the case that the disjoint union of these graphs, $G + H$, is proportionally k-choosable?
- [Equitable Choosability] Find a characterization of equitably 2-choosable graphs. (In 2004 Wang and Lih claimed that a connected graph G is equitably 2-choosable if and only if (1) G is 2-choosable and (2) G has a bipartition X, Y such that $||X| - |Y|| \leq 1$. But we have a counterexample to this.)

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