# The Gap Between the List-Chromatic and Chromatic Numbers


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Joint work with

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# List Coloring

- List coloring was introduced independently by Vizing (1976) and Erdős, Rubin, and Taylor (1979), as a generalization of usual graph coloring.
- For graph $G$ suppose each $v \in V(G)$ is assigned a list, $L(v)$, of colors. We refer to $L$ as a list assignment. An acceptable L-coloring for $G$ is a proper coloring, $f$, of $G$ such that $f(v) \in L(v)$ for all $v \in V(G)$.
- When an acceptable $L$-coloring for $G$ exists, we say that $G$ is L-colorable or L-choosable.

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# List Chromatic Number

- The list chromatic number of a graph $G$, written $\chi_\ell(G)$, is the smallest $k$ such that $G$ is $L$-colorable whenever $|L(v)| \geq k$ for each $v \in V(G)$.

- When $\chi_\ell(G) = k$ we say that $G$ has list chromatic number $k$ or that $G$ is k-choosable.

- We immediately have that if $\chi(G)$ is the typical chromatic number of a graph $G$, then

$$\chi(G) \leq \chi_\ell(G).$$

- A graph is chromatic choosable if $\chi(G) = \chi_\ell(G)$. But we know the gap between $\chi(G)$ and $\chi_\ell(G)$ can be arbitrarily large

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# A Motivating Result

## Theorem (Folklore, 1970s)

$\chi_\ell(K_{a,b}) = a + 1$ *if and only if* $b \geq a^a$

- When $b \geq a$, we know $\chi_\ell(K_{a,b}) \leq \mathrm{Col}(K_{a,b}) = a + 1$.
- So, for fixed $a$, this theorem tells us the smallest value of $b$ such that $\chi_\ell(K_{a,b})$ is as large as possible (i.e., far from being chromatic-choosable).

- We can construct a sequence of graphs with increasing list chromatic number starting from chromatic number 2: $\chi(K_{a,a^a}) = \chi(K_{1,1}) = 2 = \chi_\ell(K_{1,1}) < 3 = \chi_\ell(K_{2,4}) < 4 = \chi_\ell(K_{3,27}) < \ldots < a + 1 = \chi_\ell(K_{a,a^a})$

**Question:** Can we construct such a sequence starting from chromatic number $k > 2$?
We will give an answer motivated by the Theorem above.

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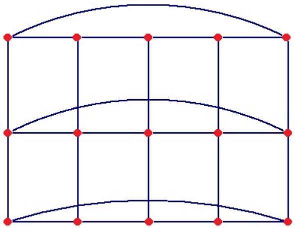
# Cartesian Product of Graphs

- The Cartesian Product $G \square H$ of graphs $G$ and $H$ is a graph with vertex set $V(G) \times V(H)$.
  Two vertices $(u, v)$ and $(u', v')$ are adjacent in $G \square H$ if either $u = u'$ and $vv' \in E(H)$ or $uu' \in E(G)$ and $v = v'$.
- Here's $C_5 \square P_3$:



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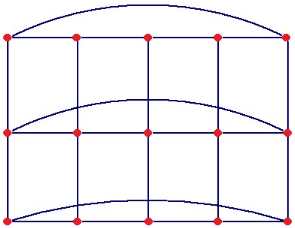

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# Coloring the Cartesian Product of Graphs

Theorem (Borowiecki, Jendrol, Kral, Miskuf (2006))
$\chi_\ell(G \square H) \leq \min\{\chi_\ell(G) + \mathrm{Col}(H), \mathrm{Col}(G) + \chi_\ell(H)\} - 1$

An easy inductive argument proves this theorem.

- For fixed $G$, $a$:
  $\chi_\ell(G \square K_{a,b}) \leq \chi_\ell(G) + \mathrm{Col}(K_{a,b}) - 1 = \chi_\ell(G) + a$

**Question:** Does there always exist a $b$ such that this upper bound is attained?

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Theorem (Borowiecki, Jendrol, Kral, Miskuf (2006))

$\chi_\ell(G \square K_{a,b}) = \chi_\ell(G) + a$, *whenever* $b \geq (\chi_\ell(G) + a - 1)^{a|V(G)|}$

**Question:** Can we improve the lower bound on $b$?

**Question:** For which graphs $G$, can we give a characterization of such $b$?

The folklore theorem from earlier gives the characterization when $G = K_1$.

- Our main tools are list color function and strongly chromatic choosable graphs.

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# The List Color Function

- For $k \in \mathbb{N}$, let $P(G, k)$ denote the number of proper colorings of $G$ with colors from $\{1, \ldots, k\}$.
- It is known that $P(G, k)$ is a polynomial in $k$ of degree $|V(G)|$. We call $P(G, k)$ the chromatic polynomial of $G$.
- The list color function of $G$, $P_\ell(G, k)$, is the minimum number of $k$-list colorings of $G$ where the minimum is taken over all $k$-list assignments for $G$.
- Recall, $P(K_{2,4}, 2) = 2$, and yet $P_\ell(K_{2,4}, 2) = 0$
- For every graph $G$ and each $k \in \mathbb{N}$, $P_\ell(G, k) \leq P(G, k)$.

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# Some Results on the List Color Function

**Theorem (Kostochka and Sidorenko (1990))**

*If G is a chordal graph (i.e. all cycles contained in G with 4 or more vertices have a chord), then $P_\ell(G, k) = P(G, k)$ for each $k \in \mathbb{N}$.*

$P_\ell(G, k)$ need not be a polynomial.

**Theorem (Thomassen (2009))**

*For any graph G, $P_\ell(G, k) = P(G, k)$ provided $k > |V(G)|^{10}$.*

**Theorem (Wang, Qian, Yan (2017))**

*For any connected graph G with m edges, $P_\ell(G, k) = P(G, k)$ provided $k > \frac{m-1}{ln(1+\sqrt{2})} \approx 1.135(m-1)$.*

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# First Result

**Theorem (K. and Mudrock)**

*$\chi_\ell(G \square K_{a,b}) = \chi_\ell(G) + a$, whenever $b \geq (P_\ell(G, \chi_\ell(G) + a - 1))^a$*

- If $G$ has at least one edge, then $P_\ell(G, \chi_\ell(G) + a - 1) < (\chi_\ell(G) + a - 1)^{|V(G)|}$; giving a (significant) improvement over the Borowiecki et al. bound.
- We can in fact prove:

**Theorem (K. and Mudrock)**

*Suppose $H$ is a bipartite graph with partite sets $A$ and $B$ where $|A| = a$ and $|B| = b$. Let $\delta = \min_{v \in B} d_H(v)$. If $b \geq (P_\ell(G, \chi_\ell(G) + \delta - 1))^a$, then $\chi_\ell(G \square H) \geq \chi_\ell(G) + \delta$.*

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**Question:** When is this bound sharp? Can we find graphs $G$ such this bound characterizes $\chi_\ell(G \square K_{a,b}) = \chi_\ell(G) + a$?

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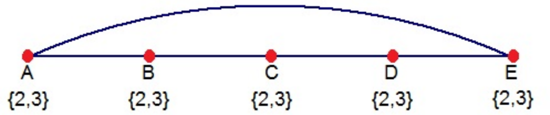
# Strong Chromatic Choosability

- List assignment, $L$, for $G$ is a bad k-assignment for $G$ if $G$ is not $L$-colorable and $|L(v)| = k$ for each $v \in V(G)$.
- List assignment, $L$, is constant if $L(v)$ is the same for each $v \in V(G)$.
- A constant (and bad) 2-assignment for a $C_5$:

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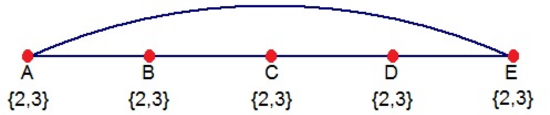


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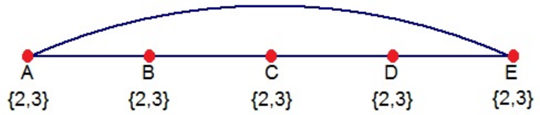


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*Let $G$ be a strong $k$-chromatic choosable graph. Then*
*(i) $\chi(G) = k = \chi_\ell(G)$ (i.e. $G$ is chromatic choosable),*
*(ii) $\chi(G - \{v\}) \leq \chi_\ell(G - \{v\}) < k$ for any $v \in V(G)$,*
*(iii) $k = 2$ if and only if $G$ is $K_2$,*
*(iv) $k = 3$ if and only if $G$ is an odd cycle,*
*(v) $G \vee K_p$ is strong $(k+p)$-chromatic choosable for any $p \in \mathbb{N}$.*

- We essentially have a notion of vertex-criticality for chromatic-choosability.
- There are many infinite families of graphs that satisfy this notion.

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*(ii) $\chi(G - \{v\}) \leq \chi_\ell(G - \{v\}) < k$ for any $v \in V(G)$,*
*(iii) $k = 2$ if and only if $G$ is $K_2$,*
*(iv) $k = 3$ if and only if $G$ is an odd cycle,*
*(v) $G \vee K_p$ is strong $(k+p)$-chromatic choosable for any $p \in \mathbb{N}$.*

- We essentially have a notion of vertex-criticality for chromatic-choosability.
- There are many infinite families of graphs that satisfy this notion.

# Second Result

## Theorem (K. and Mudrock)

*$\chi_\ell(G \square K_{a,b}) = \chi_\ell(G) + a$, whenever $b \geq (P_\ell(G, \chi_\ell(G) + a - 1))^a$*

## Theorem (K. and Mudrock)

*If $G$ is a strong $k$-chromatic choosable graph and $k \geq a + 1$, then $\chi_\ell(G \square K_{a,b}) = \chi_\ell(G) + a$ if and only if $b \geq (P_\ell(G, \chi_\ell(G) + a - 1))^a$.*

The proof idea is:

If $L$ is a $(\chi_\ell(G) + a - 1)$-assignment for $G \square K_{a,b}$, there is at most one proper $L$-coloring of the copies of G corresponding to the partite set of size $a$ that leads to a bad assignment for a given "bottom" copy of $G$.

We show if two such colorings existed, we could obtain a proper $a$-coloring of $G$.

A simple counting argument completes the proof that there exists a proper $L$-coloring of $G \square K_{a,b}$ when $b < (P_\ell(G, \chi_\ell(G) + a - 1))^a$.

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$\chi_\ell(C_{2t+1} \square K_{2,b}) = 5$ *if and only if* $b \geq (P_\ell(C_{2t+1}, 4))^2 = (3^{2t+1} - 3)^2 = 9(9^t - 1)^2$.

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*For $n \geq a+1$, $\chi_\ell(K_n \square K_{a,b}) = n + a$ if and only if* $b \geq (P_\ell(K_n, n+a-1))^a = \left(\frac{(n+a-1)!}{(a-1)!}\right)^a$

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This corollary shows the bound in the Theorem is sharp for all $a$.

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# Thank You!

## Questions?

- Define $f_a(G)$ as the smallest $b$ s.t. $\chi_\ell(G \square K_{a,b}) = \chi_\ell(G) + a$.
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- Is it always the case that $P_\ell(G, k) = P(G, k)$ when $G$ is strong chromatic choosable?
- (Thomassen 2009) Does there exist a graph $G$ and a natural number $k > 2$ such that $P_\ell(G, k) = 1$?
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