Error estimation for Quasi Monte Carlo Methods

Introduction of an algorithm for integration relying on the fast transform coefficients decay

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The problem

$$\left| \int_{[0,1]^d} f(x) \, dx - \frac{1}{N} \sum_{i=0}^{N-1} f(x_i) \right|$$
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- Getting the coefficients: Approximating Fourier coefficients.
- Mapping the wavenumbers.
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We define \( \tilde{\mathcal{X}} \subseteq [0, 1)^d \) and \( \mathbb{K} \subseteq \mathbb{Z}^d \) such that,

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We define \( \tilde{\mathcal{X}} \subseteq [0, 1)^d \) and \( \mathbb{K} \subseteq \mathbb{Z}^d \) such that,

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- \( (\mathbb{K}, \oplus_2) \) is a commutative group.
- \( \otimes : \mathbb{K} \times \mathcal{X} \to [0, 1) \) which returns zero if either argument is zero and has both distributive properties.
### Examples: Integration lattices and digital nets

<table>
<thead>
<tr>
<th>Integration lattices</th>
<th>Digital nets</th>
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<tbody>
<tr>
<td>( \mathcal{X} = [0,1)^d )</td>
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<tr>
<td>( K = \mathbb{Z}^d )</td>
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<td>( x \oplus_1 t = x + t \pmod{1} )</td>
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<td>( k \oplus_2 l = k + l )</td>
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<tr>
<td>( k \otimes x = k^T x \pmod{1} )</td>
<td>( k \otimes x = \left( \left[ \frac{1}{b} \sum_{\ell=0}^{\infty} k_{j\ell} x_{j,\ell+1} \right] \mod 1 \right)^d_{j=1} )</td>
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</table>
Node set $\mathcal{P}$

$\mathcal{P}$ is any finite subgroup of $\widetilde{X}$.

**Dual set**

$\mathcal{P}^\perp = \{ k \in \mathbb{K} : k \otimes x = 0 \ \forall x \in \mathcal{P} \}$.

**Dual cosets (only $|\mathcal{P}|$)**

$\mathcal{P}^\perp_k = \{ l \in \mathbb{K} : l \oplus_2 k \in \mathcal{P}^\perp \} \ \forall k \in \mathbb{K}$.

**Shifted set**

$\mathcal{P}_{\Delta} = \{ x + \Delta : x \in \mathcal{P} \}$. 
Fourier series

Let \( \{ \varphi(\cdot, k) \in L_2(\mathcal{X}) : k \in \mathbb{K} \} \) be some complete orthonormal basis for \( L_2(\mathcal{X}) \). In particular, let

\[
\varphi(x, k) = e^{2\pi \sqrt{-1} k \otimes x}, \quad k \in \mathbb{K}, x \in \mathcal{X}.
\]

Then any function in \( L_2 \) may be written in series form as

\[
f(x) = \sum_{k \in \mathbb{K}} \hat{f}(k) \varphi(x, k), \quad \text{where} \quad \hat{f}(k) = \langle f, \varphi(\cdot, k) \rangle_2, \quad (1)
\]
We can define the approximation of a Fourier coefficient as,

\[ \hat{f}(k) := \frac{1}{|P_\Delta|} \sum_{x \in P_\Delta} e^{-2\pi \sqrt{-1} k \otimes x} f(x) \]

\[ = \ldots \]

\[ = \hat{f}(k) + \sum_{l \in P_k^\perp \setminus \{k\}} \hat{f}(l)e^{2\pi \sqrt{-1}(l \ominus k) \otimes \Delta} \]
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Consider the situation where there is a sequence of nested sets,

$$\mathcal{P}_0 = \{0\} \subset \mathcal{P}_1 \subset \mathcal{P}_2 \subset \cdots, \quad |\mathcal{P}_s| = b^s$$

Then the dual sets are nested in the opposite direction,

$$\mathcal{P}_0^\perp = \mathbb{K} \supset \mathcal{P}_2^\perp \supset \mathcal{P}_2^\perp \supset \cdots.$$ 

Furthermore, the equivalence classes also obey this nesting:

$$\mathcal{P}_{s,k}^\perp = \{l \in \mathbb{K} : l \ominus k \in \mathcal{P}_s^\perp\},$$

$$\mathcal{P}_{0,k}^\perp = \mathbb{K} \supset \mathcal{P}_{1,k}^\perp \supset \mathcal{P}_{2,k}^\perp \supset \cdots \quad \forall k \in \mathbb{K}.$$
Mapping wavenumbers

For every $\kappa \in \mathbb{N}_0$, we assign a wavenumber $k(\kappa) \in \mathbb{K}$ iteratively according to the following constraints:

- Let $k(0) = 0$.
- For any $s, \lambda \in \mathbb{N}$, $\kappa = 0, \ldots, b^s - 1$, assign $k(\kappa)$ and $k(\kappa + \lambda b^s)$ such that $\mathcal{P}_{s,k(\kappa)} = \mathcal{P}_{s,k(\kappa + \lambda b^s)}$.

Then we can rewrite:

$$\tilde{f}_{s,\kappa} = \hat{f}_\kappa + \sum_{\lambda=1}^{\infty} \hat{f}_{\kappa + \lambda b^s} e^{2\pi \sqrt{-1}(k(\kappa + \lambda b^s) \Theta k(\kappa)) \otimes \Delta}.$$
Sums of series coefficients and assumptions

Consider the following sums of the series coefficients defined for \( r, s \in \mathbb{N}, \ r \leq s \):

\[
S(r) = \sum_{\kappa = b^r - 1}^{b^r - 1} |\hat{f}_{\kappa}|, \quad \hat{S}(r, s) = \sum_{\kappa = b^r - 1}^{b^r - 1} \sum_{\lambda = 1}^{\infty} |\hat{f}_{\kappa} + \lambda b^s|,
\]

\[
\tilde{S}(r, s) = \sum_{\kappa = b^r - 1}^{b^r - 1} |\tilde{f}_{s, \kappa}|.
\]

and assume that, for a fix \( r_* \),

\[
S(s) \leq \omega(s - r)S(r), \quad \hat{S}(r, s) \leq \hat{\omega}(s - r)S(r), \quad r, s \in \mathbb{N}, \ r_* \leq r \leq s,
\]

for some functions \( \omega \) and \( \hat{\omega} \) with \( \lim_{s \to \infty} \omega(s) = \lim_{s \to \infty} \hat{\omega}(s) = 0 \).
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Bounding $S(r)$

Note that with these definitions:

$$S(r) = \sum_{\kappa = br^{-1}}^{b^r - 1} |\hat{f}_\kappa| = \sum_{\kappa = br^{-1}}^{b^r - 1} |\tilde{f}_{s,\kappa} - \sum_{\lambda=1}^{\infty} \hat{f}_{\kappa + \lambda b^s} e^{2\pi \sqrt{-1} (l(\kappa + \lambda b^s) \Theta k(\kappa)) \otimes \Delta}|$$

$$\leq \sum_{\kappa = br^{-1}}^{b^r - 1} |\tilde{f}_{s,\kappa}| + \sum_{\kappa = br^{-1}}^{b^r - 1} \sum_{\lambda=1}^{\infty} |\hat{f}_{\kappa + \lambda b^s}| = \tilde{S}(r, s) + \hat{S}(r, s)$$

$$\leq \tilde{S}(r, s) + \hat{\omega}(s - r) S(r)$$

$$S(r) \leq \frac{\tilde{S}(r, s)}{1 - \hat{\omega}(s - r)}$$

provided that $\hat{\omega}(s - r) < 1$. 
Bounding the error

It then follows that,

\[
\left| \int f(x) \, dx - \frac{1}{b^s} \sum_{x \in \mathcal{P}_{s, \Delta}} f(x) \right| = \left| \hat{f}_0 - \tilde{f}_{s,0} \right| = \left| \sum_{\lambda=1}^{\infty} \hat{f}_{\lambda b^s} e^{2\pi \sqrt{-1} \mathbf{l}(\lambda b^s) \otimes \Delta} \right| \leq \sum_{\lambda=1}^{\infty} |\hat{f}_{\lambda b^s}|
\]

\[
\leq \sum_{\kappa=b^s}^{b^r - 1} \sum_{r'=s+1}^{r' \leq r + b^r} |\hat{f}_{\kappa}| = \sum_{r'=s+1}^{\infty} S(r')
\]

\[
\leq \sum_{r'=s+1}^{\infty} \omega(r' - r) S(r) = \sum_{r'=1}^{\infty} \omega(r' + s - r) S(r) = \Omega(s - r) S(r)
\]

\[
\leq \frac{\tilde{S}(r, s) \Omega(s - r)}{1 - \hat{\omega}(s - r)}.
\]
Any questions?