Numerical Methods for the Solution of the Hilbert-Schmidt Integral Eigenvalue Problem

Haocheng Bian

Department of Applied Mathematics
Illinois Institute of Technology

hbian1@hawk.iit.edu

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Overview

1. Hilbert-Schmidt Integral Eigenvalue Problem
2. Collocation method
3. Different choices of basis functions
4. Quadrature method
5. Determine error without using real solutions
6. Current/Future work
Hilbert-Schmidt Integral Eigenvalue Problem

Hilbert-Schmidt Eigenvalue Problem:

\[ \int_{\Omega} K(x, z) \varphi(z) p(z) dz = \lambda \varphi(x) \]

Usually we let the weight function \( p(z) = 1 \),

Then we have equation:

\[ \int_{\Omega} K(x, z) \varphi(z) dz = \lambda \varphi(x) \]
What is Hilbert-Schmidt Integral Eigenvalue Problem

The following equation can be solved only in some special cases.

\[ \int_{\Omega} K(x, z) \varphi(z) dz = \lambda \varphi(x) \]

In general, it’s hard to find analytic solutions. Therefore, we need to introduce numerical method to help us solve this problem.
Collocation Method

Making the assumption that eigenfunctions can be represented by the finite-dimensional basis:

\[ \mathcal{H} = \{ h_1, h_2, \ldots, h_N \}, \]

Then we know eigenfunction can be written as

\[ \varphi(x) = \sum_{j=1}^{N} c_j h_j(x). \]

Replace the eigenfunction by the linear combination of basis functions, we get:

\[ \sum_{j=1}^{N} c_j \int_{\Omega} K(x, z) h_j(z) dz = \lambda \sum_{j=1}^{N} c_j h_j(x) \]
To turn this into a system of equations and solve this problem, we need to choose collocation points \( x_k \in \Omega, 1 \leq k \leq M \)

\[
\sum_{j=1}^{N} c_j \int_{\Omega} K(x_k, z) h_j(z) \, dz = \lambda \sum_{j=1}^{N} c_j h_j(x_k), \quad 1 \leq k \leq M
\]

Write this system of equations into vector form:

\[
\begin{pmatrix}
\sum_{j=1}^{N} c_j \int_{\Omega} K(x_1, z) h_j(z) \, dz \\
\vdots \\
\sum_{j=1}^{N} c_j \int_{\Omega} K(x_M, z) h_j(z) \, dz
\end{pmatrix}
= 
\begin{pmatrix}
\lambda \sum_{j=1}^{N} c_j h_j(x_1) \\
\vdots \\
\lambda \sum_{j=1}^{N} c_j h_j(x_M)
\end{pmatrix}
\]
Collocation Method

Let \( c^T = (c_1, \cdots, c_N) \), then we will have following matrices:

\[
\begin{pmatrix}
\int_{\Omega} K(x_1, z) h_1(z) dz & \cdots & \int_{\Omega} K(x_1, z) h_N(z) dz \\
\int_{\Omega} K(x_2, z) h_1(z) dz & \cdots & \int_{\Omega} K(x_2, z) h_N(z) dz \\
\int_{\Omega} K(x_M, z) h_1(z) dz & \cdots & \int_{\Omega} K(x_M, z) h_N(z) dz
\end{pmatrix}
\begin{pmatrix}
c_1 \\
\vdots \\
c_N
\end{pmatrix}
= \lambda
\begin{pmatrix}
h_1(x_1) & \cdots & h_N(x_1) \\
\vdots & \ddots & \vdots \\
h_1(x_M) & \cdots & h_N(x_M)
\end{pmatrix}
\begin{pmatrix}
c_1 \\
\vdots \\
c_N
\end{pmatrix}
\]

We can write this more compactly by defining the right hand matrix \( H \) such that \( H_{ij} = h_j(x_i) \) and the left hand side matrix \( A \) such that \( A_{ij} = \int_{\Omega} K(x_i, z) h_j(z) dz \). Then this system is just

\[
Ac = \lambda Hc,
\]

and we have standard methods to solve this.
Different choices of basis functions

1. Polynomial basis
2. Connect-the-dot basis
3. Chebyshev basis
Different choices of basis functions

Polynomial basis:
In this case, we let basis function $h_j(x) = x^{j-1}$.

Connect-the-dot basis:
In this case, we let basis function $h_j(x) = \min(x_j, z) - x_j z$.

Chebyshev basis:
Chebyshev polynomials are defined by the recurrence relationship:

$T_0(x) = 1$

$T_1(x) = x$

$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$

In this case, we let basis function $h_j(x) = T_{j-1}(2x - 1)$. Chebyshev basis is orthogonal on [-1,1] and we are interested in the interval [0,1].
Testing Chebyshev basis

Same as testing previous basis, we use the kernel that we already know the answer to test our result. The kernel that we know the solutions:

\[ K(x, z) = \min(x, z) - xz \]

eigenvalue:

\[ \lambda = \frac{1}{n^2 \pi^2} \]

eigenfunction:

\[ \varphi(x) = \sqrt{2} \sin(n\pi x) \]
Table for the number of eigenvalues with relative error less than $10^{-3}$

The following table shows the number of eigenvalues which relative error less than $10^{-3}$ for different basis functions and different number of collocation points.

<table>
<thead>
<tr>
<th># of collocation points</th>
<th>10</th>
<th>30</th>
<th>50</th>
<th>70</th>
<th>90</th>
<th>110</th>
</tr>
</thead>
<tbody>
<tr>
<td>Polynomial</td>
<td>4</td>
<td>3</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>Connect-the-dot</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>Chebyshev</td>
<td>4</td>
<td>16</td>
<td>29</td>
<td>41</td>
<td>54</td>
<td>67</td>
</tr>
</tbody>
</table>

Although Chebyshev basis is also some kind of polynomial basis, it makes a big difference. It seems that Chebyshev basis has done a good job comparing to the other two bases.
Analytic solution for Chebyshev basis

So far, Chebyshev basis has done a good job for us. However, finding the analytic solutions for the integral with Chebyshev basis is not easy. We need to solve the integral

$$\int_{0}^{1} (\min(x, z) - xz) \, T_j(2z - 1) \, dz$$

The analytic solution for this integral is

$$\frac{1}{16} \left[ \frac{2T_{j+1}(u)}{j + 1} - \frac{2T_{j-1}(u)}{j - 1} - \frac{4(-1)^j}{j^2 - 1} + \frac{T_{j+2}(u)}{j + 2} - \frac{T_{j-2}(u)}{j - 2} + \frac{4(-1)^j}{j^2 - 4} \right] + \frac{1}{4^x} \left[ \frac{(1 - (-1)^j)}{j^2 - 4} - \frac{T_{j+1}(u)}{j + 1} + \frac{T_{j-1}(u)}{j - 1} - \frac{1 - (-1)^j}{j^2 - 1} \right]$$

$$u = 2x - 1$$
Quadrature method

In the event that the integrals of concern cannot be evaluated analytically, we would need to use some sort of quadrature method so that we have

$$\int_{\Omega} f(z) dz \approx \sum_{\ell=1}^{\nu} w_{\ell} f(z_{\ell}).$$

Applying this identity to the entries of the matrix A in matrix equation $Ac = \lambda Hc$ we will have

$$\begin{pmatrix}
\sum_{\ell=1}^{\nu} w_{\ell} K(x_1, z_{\ell}) h_1(z_{\ell}) & \cdots & \sum_{\ell=1}^{\nu} w_{\ell} K(x_1, z_{\ell}) h_N(z_{\ell}) \\
\vdots & \ddots & \vdots \\
\sum_{\ell=1}^{\nu} w_{\ell} K(x_M, z_{\ell}) h_1(z_{\ell}) & \cdots & \sum_{\ell=1}^{\nu} w_{\ell} K(x_M, z_{\ell}) h_N(z_{\ell})
\end{pmatrix}
\begin{pmatrix}
c_1 \\
c_2 \\
\vdots \\
c_M
\end{pmatrix}
= \lambda
\begin{pmatrix}
c_1 \\
c_2 \\
\vdots \\
c_M
\end{pmatrix}.$$
Quadrature method

We also can rewrite this matrix as the product of three matrices

\[
\begin{pmatrix}
K(x_1, z_1) & \cdots & K(x_1, z_\nu) \\
\vdots & \ddots & \vdots \\
K(x_M, z_1) & \cdots & K(x_M, z_\nu)
\end{pmatrix}
\begin{pmatrix}
w_1 \\
\vdots \\
w_\nu
\end{pmatrix}
\begin{pmatrix}
h_1(z_1) & \cdots & h_N(z_1) \\
\vdots & \ddots & \vdots \\
h_1(z_\nu) & \cdots & h_N(z_\nu)
\end{pmatrix}
\]

\[= \lambda \mathbf{H} \mathbf{c}.\]

In more concise notation we have

\[\mathbf{KWHc} = \lambda \mathbf{Hc}.\]

After we get that, we can use different quadrature methods to test our result.
Before using some quadrature methods, matlab already has lots of methods which can help us evaluate the integral

\[ \int_{\Omega} K(x_k, z) h_j(z) dz \]

and these methods are more accurate than some basic quadrature methods. Therefore we first compute these integrals in matrix

\[ A_{ij} = \int_{\Omega} K(x_i, z) h_j(z) dz \]

by matlab command quadgk.
Differences between two methods

However, no matter how good our quadrature methods are, we still approximate these integrals. Therefore, there are differences between solutions obtained from these two different methods. To know how big these differences are, first, we denote the eigenvalues which are found by analytic solutions as $\text{eig}_{\text{analytic}}$ and the eigenvalues which are found by quadrature solutions as $\text{eig}_{\text{quadrature}}$. After that we calculate their relative difference by

$$\frac{|\text{eig}_{\text{analytic}} - \text{eig}_{\text{quadrature}}|}{\text{eig}_{\text{analytic}}}$$
Some results for polynomial basis

Following figures show us relative error of 8th eigenvalues with increasing collocation points $N$ calculated by different methods and there relative difference for polynomial basis.

Figure: Relative error, Relative difference
Some results for Connect-the-dot basis

Following figures show us relative error of 8th eigenvalue with increasing collocation points $N$ calculated by different methods and their relative difference for Connect-the-dot basis.

Figure: Relative error, Relative difference
Some results for Chebyshev basis

Following figures show us relative error of 8th eigenvalue with increasing collocation points N calculated by different methods and their relative difference for Chebyshev basis.

Figure: Relative error, Relative difference
From previous figures, it's not hard to see that there are some penalties for using quadrature methods. It depends on number of collocation points and the basis we are using. However, the tolerance for quadrature methods may also affect the results.
Effect of tolerance

The following table shows us the relative difference for 8th eigenvalues between two methods with different tolerance for quadgk and the basis is Connect-the-dot basis.

<table>
<thead>
<tr>
<th>Relative difference</th>
<th>AbsTel</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$10^{-8}$</td>
</tr>
<tr>
<td>RelTol</td>
<td></td>
</tr>
<tr>
<td>$10^{-3}$</td>
<td>0.0023</td>
</tr>
<tr>
<td>$10^{-4}$</td>
<td>0.0018</td>
</tr>
<tr>
<td>$10^{-5}$</td>
<td>0.0001</td>
</tr>
<tr>
<td>$10^{-6}$</td>
<td>0.0003</td>
</tr>
</tbody>
</table>
Effect of tolerance

The following table shows us the relative difference for 8th eigenvalues between two methods with different tolerance for quadgk and the basis is Chebyshev basis.

<table>
<thead>
<tr>
<th>Relative difference ($\times 10^{-3}$)</th>
<th>AbsTel</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$10^{-8}$</td>
</tr>
<tr>
<td>RelTol</td>
<td></td>
</tr>
<tr>
<td>$10^{-3}$</td>
<td>0.2158</td>
</tr>
<tr>
<td>$10^{-4}$</td>
<td>0.0280</td>
</tr>
<tr>
<td>$10^{-5}$</td>
<td>0.0033</td>
</tr>
<tr>
<td>$10^{-6}$</td>
<td>0.0013</td>
</tr>
</tbody>
</table>

From previous two tables, we can know, the relative tolerance may be the main reason which effects the error caused by quadrature methods.
Some conclusions for quadrature methods

- Quadgk may give us good results
- Number of collocation points may and probably should effect our results
- The results which are found by quadrature method may be affected by different bases
- Decreasing the relative tolerance for quadrature method may give us better results
So far, we measure how well we have done by comparing the result with the exact solutions we already know. However, that’s not enough. Our goal is to figure out the real solution. We cannot use it as a factor to help us determine our solution.
One way to help us determine the result is to compare the results obtained from different numbers of collocation points.

Suppose $\lambda_i^{(N)}$ represents the $i$th eigenvalue which is calculated by $N$ collocations points. Then our approximate error for $i$th eigenvalue with $N$ collocation points $e_i^{(N)}$ has the following relation:

$$e_i^{(N)} = \left| \frac{\lambda_i^{(N)} - \lambda_i^{(N-1)}}{\lambda_i^{(N)}} \right|$$
Determine error without using real solutions

Following figure show us the real relative error and our approximate error of the 8th eigenvalue with increasing collocation points $N$ and the basis is connect-the-dot basis.
Determine error without using real solutions

Following figure show us the real relative error and our approximate error of the 8th eigenvalue with increasing collocation points N and the basis is Chebyshev basis.
Determine error without using real solutions

So far, the behaviour of our approximate error is similar to relative error. Therefore, we may learn the behaviour of relative of eigenvalue from that, but that’s not good enough. There may be some better methods that can approximate error more accurately than this method.

What should I do?
Current/Future work

1. Try more bases
2. Study more about the penalty we will have for quadrature method
3. Using some other quadrature methods but not matlab command
4. Find better method to determine error without using real solutions
5. Get more information about error for eigenfunction
Thank You!