Math 350 — Final, May 13, 2008

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1. Short-answer questions:

   (a) If L is a nonsingular lower triangular matrix, P is a permutation matrix, and $\boldsymbol{b}$ is a given vector, how would you solve the linear system $LP\boldsymbol{x} = \boldsymbol{b}$?

   (b) How is the condition number of a matrix A defined for a given matrix norm?

   (c) True or false: There are arbitrarily many different mathematical functions that interpolate a given set of data.

   (d) True or false: When interpolating a continuous function by a polynomial at equally spaced points on a given interval, the polynomial interpolant always converges to the function as the number of interpolation points increases.

   (e) What condition ensures that the bisection method will find a zero of a continuous nonlinear function f in the interval $[a, b]$?

   (f) List one advantage and one disadvantage of the secant method compared with Newton's method for solving a nonlinear equation in one variable.

   (g) In general, does a differential equation, by itself, determine a unique solution? If so, why, and if not, what additional information must be specified to determine a solution uniquely?

   (h) What is the basic difference between an explicit and an implicit method for solving IVPs numerically? List at least one advantage for each type. Give an example (name is enough) of a method for each type.

   (i) List two applications for the FFT.

(a) $LPx = b$. Solve $Ly = b$ by forward substitution
2 Then compute $x = P^T y$.

1 (b) $\text{cond}(A) = \|A\| \|A^{-1}\|$

1 (c) T

1 (d) F

1 (e) $f(a)f(b) < 0$ (different sign)

2 (f) advantage: does not require derivatives
disadvantage: slower convergence

2 (g) No, you need IC(s)

(h) For explicit methods y_{n+1} is explicitly given, for implicit
2 methods not. Implicit methods stable, explicit simple.
Explicit: Euler, implicit: backward Euler

2 (i) Noise filtering, spectral approximation of derivatives

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2. (a) Find the LU factorization of the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 8 & 11 \\ 3 & 22 & 35 \end{bmatrix}.$$

(b) Find the inverse of A using the LU factorization computed in (a).

(c) Find the determinant of A using the LU factorization computed in (a).

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(a) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 8 & 11 \\ 3 & 22 & 35 \end{bmatrix} \xrightarrow{2,3} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 16 & 26 \end{bmatrix} \xrightarrow{4} \underbrace{\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}}_{=U}, \quad L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 4 & 1 \end{bmatrix}$

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(b) $A = LU$. To find A^{-1} solve $AB = I$

$\iff LUB = I$

(i) Solve $LY = I$: $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 4 & 1 \end{bmatrix} Y = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\Rightarrow Y = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 5 & -4 & 1 \end{bmatrix}$

(ii) Solve $UB = Y$: $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix} B = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 5 & -4 & 1 \end{bmatrix}$

$\Rightarrow B = \begin{bmatrix} \frac{19}{12} & -\frac{1}{6} & -\frac{1}{12} \\ -\frac{37}{24} & \frac{13}{12} & -\frac{5}{24} \\ \frac{5}{6} & -\frac{2}{3} & \frac{1}{6} \end{bmatrix} = A^{-1}$

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(c) $\det(A) = \det(LU) = \det L \cdot \det U$

$= 1 \cdot 24$ (triangular matrices)

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3. Consider the function $f(x) = x^3 - a$, where a is any real number.

(a) Use Newton's method to derive the cube root iteration formula

$$x_{n+1} = \frac{2x_n}{3} + \frac{a}{3x_n^2}, \quad k = 0, 1, 2, \ldots$$

(b) Perform two iterations of this algorithm starting with $x_0 = 1$ (i.e., compute x_2) to approximate $\sqrt[3]{2}$.

(a) Newton: $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$

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there $x_{n+1} = x_n - \dfrac{x_n^3 - a}{3x_n^2} = \dfrac{3x_n^3 - x_n^3 + a}{3x_n^2}$

$$= \frac{2x_n^3}{3} + \frac{a}{3x_n^2}$$

(b) $x_1 = \dfrac{2(1)}{3} + \dfrac{2}{3(1)^2} = \dfrac{4}{3} \approx 1.3333$

a=2

4 $x_2 = \dfrac{2\left(\frac{4}{3}\right)}{3} + \dfrac{2}{3\left(\frac{4}{3}\right)^2} = \dfrac{8}{9} + \dfrac{6}{16} = \dfrac{91}{72} \approx 1.2639$

"Exact": 1.2599

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4. Suppose you are fitting a straight line to the three data points $(0, 1)$, $(1, 2)$, and $(3, 3)$.

   (a) Set up the overdetermined linear system $A\boldsymbol{x} = \boldsymbol{b}$ for the least squares problem.

   (b) Set up the corresponding normal equations.

   (c) Compute the least squares solution.

Line : $y = mx + b$

(a) $Ax = b \iff \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$

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(b) $A^T A x = A^T b$

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$\iff \begin{pmatrix} 0 & 1 & 3 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 0 & 1 & 3 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$

$\iff \begin{pmatrix} 10 & 4 \\ 4 & 3 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 11 \\ 6 \end{pmatrix}$

(c)

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$\left(\begin{array}{cc|c} 10 & 4 & 11 \\ 4 & 3 & 6 \end{array}\right) \to \left(\begin{array}{cc|c} 10 & 4 & 11 \\ 0 & \frac{7}{5} & \frac{8}{5} \end{array}\right)$

$\Rightarrow \underline{\underline{b}} = \frac{8}{5} \cdot \frac{5}{7} = \underline{\underline{\frac{8}{7}}}$

$\underline{\underline{m}} = \frac{1}{10}\left(11 - 4\frac{8}{7}\right) = \underline{\underline{\frac{9}{14}}}$

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5. Assume you have found the reduced SVD of a given $m \times n$ matrix A as $A = \hat{U}\hat{\Sigma}V^T$. We said in class that the pseudoinverse of A is given by $A^\dagger = V\hat{\Sigma}^{-1}\hat{U}^T$. Show that the pseudoinverse satisfies the following *Moore-Penrose condition* which shows that $A^\dagger$ behaves very much like an inverse (Note: there are three other similar Moore-Penrose conditions):

$$AA^\dagger A = A.$$

Assume $A = \hat{U}\hat{\Sigma}V^T$

and $A^\dagger = V\hat{\Sigma}^{-1}\hat{U}^T$

Show

$$\underline{\underline{AA^\dagger A}} = (\hat{U}\hat{\Sigma}\underbrace{V^T)(V}_{=I}\hat{\Sigma}^{-1}\underbrace{\hat{U}^T)(\hat{U}}_{=I}\hat{\Sigma}V^T)$$

$$= \hat{U}\underbrace{\hat{\Sigma}\hat{\Sigma}^{-1}}_{=I}\hat{\Sigma}V^T$$

$$= \underline{\underline{A}}$$

6. Derive an error estimate for the numerical differentiation formula

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$$f'(x) \approx D_h f = \frac{-3f(x) + 4f(x+h) - f(x+2h)}{2h}$$

using appropriate Taylor series expansions.

$$4 \times \left(f(x+h) = f(x) + h f'(x) + \frac{h^2}{2} f''(x) + \frac{h^3}{6} f'''(c_1) \right)$$

$$- \left(f(x+2h) = f(x) + 2h f'(x) + \frac{(2h)^2}{2} f''(x) + \frac{(2h)^3}{6} f'''(c_2) \right)$$

$$4 f(x+h) - f(x+2h) = 3 f(x) + 2h f'(x) + \frac{2}{3} h^3 \left[f'''(c_1) - 2 f'''(c_2) \right]$$

$$\Rightarrow f'(x) = \frac{-3f(x) + 4f(x+h) - f(x+2h)}{2h} - \underbrace{\frac{h^2}{3} \left[f'''(c_1) - 2 f'''(c_2) \right]}_{\text{error term}}$$

is $O(h^2)$

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7. Write each of the following ODEs as an equivalent system of first-order ODEs:

(a) $y''(t) = t + y(t) + y'(t)$,

(b) $y'''(t) = y''(t) + ty(t)$.

(a) 6

$$\underline{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} y \\ y' \end{bmatrix}$$

So

$$\underline{y}' = \begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} y' \\ y'' \end{bmatrix} = \begin{bmatrix} y_2 \\ t + y_1 + y_2 \end{bmatrix}$$

(b) 6

$$\underline{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} y \\ y' \\ y'' \end{bmatrix}$$

So

$$\underline{y}' = \begin{bmatrix} y_1' \\ y_2' \\ y_3' \end{bmatrix} = \begin{bmatrix} y' \\ y'' \\ y''' \end{bmatrix} = \begin{bmatrix} y_2 \\ y_3 \\ y_3 + t\,y_1 \end{bmatrix}$$

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8. Consider the initial-value problem

$$y'(t) = \frac{1}{1+t^2} - 2\,(y(t))^2, \quad y(0) = 0.$$

Choose $h = 0.1$ and compute $y_1 \approx y(0.1)$ by

(a) Euler's method, and

(b) the explicit Runge-Kutta method

$$y_{n+1} = y_n + h\,[\gamma_1 s_1 + \gamma_2 s_2]$$

$$\begin{aligned} s_1 &= f(t_n, y_n) \\ s_2 &= f(t_n + \alpha_2 h, y_n + h\beta_{21} s_1), \end{aligned}$$

with Butcher tableaux

$$\begin{array}{c|cc} 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ \hline & 0 & 1. \end{array}$$

(a) Euler: $y_{n+1} = y_n + h\, f(t_n, y_n)$

6 So $\underline{y(0.1)} \approx y_1 = y_0 + (0.1)\, f(0, y_0)$

$$= 0 + (0.1)\left[\frac{1}{1+0^2} - 2(0)^2\right] = \underline{0.1}$$

(b) RK2: $y_{n+1} = y_n + h\, s_2$

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$$s_1 = f(t_n, y_n)$$

$$s_2 = f(t_n + \tfrac{1}{2}h,\ y_n + \tfrac{1}{2}h s_1)$$

So $s_1 = \frac{1}{1+0^2} - 2(0)^2 = 1$

$$s_2 = \frac{1}{1+0.05^2} - 2(0.05)^2 = 0.9925$$

and $\underline{y_1} = 0 + 0.1\,(0.9925) = \underline{0.09925}$

9. (a) What is the result of the MATLAB function call `[x y] = problem1([1 2 3], [4 5 8])` if we're given

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```
function [x y] = problem1(x,y)
while sum(x)~=max(y)
    x = x.^2;
    y = y+x;
end
```

(b) What is the result of the MATLAB function call `F = problem2([1 2 3], [4 5 8])` if we're given

```
function F = problem2(x,y)
x = x(:); y = y(:);    % just ensures we're working with column vectors
n = length(x);
F = zeros(n);
F(:,1) = y;
for i = 2:n
    F(i:n,i) = diff(F(i-1:n,i-1))./(x(i:n)-x(1:n+1-i));
end
```

(a) $x = [1\ 2\ 3] \rightarrow [1\ 4\ 9] \rightarrow [1\ 16\ 81]$

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$y = [4\ 5\ 8] \rightarrow [5\ 9\ 17] \rightarrow [6\ 25\ 98]$

(b) $F = \begin{bmatrix} 4 & 0 & 0 \\ 5 & 1 & 0 \\ 6 & 3 & 1 \end{bmatrix}$

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