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Math 350 — Midterm, March 12, 2008

1. (a) Solve the following linear system by LU factorization/Gauss elimination with pivoting:

$$\begin{aligned} 2x + 6y + 10z &= 0 \\ x + 3y + 3z &= 2 \\ 3x + 14y + 28z &= -8. \end{aligned}$$

Show and explain all the details of your work.

(b) What do the matrices P, L and U look like?

(a) Get first pivot $A_{31} = 3$, i.e. swap rows 1 and 3:

$$\begin{bmatrix} 3 & 14 & 28 \\ 1 & 3 & 3 \\ 2 & 6 & 10 \end{bmatrix} \xrightarrow{\substack{L_{21} = 1/3 \\ L_{31} = 2/3}} \begin{bmatrix} 3 & 14 & 28 \\ 0 & -5/3 & -19/3 \\ 0 & -10/3 & -26/3 \end{bmatrix}$$

swap rows 2 and 3 (also in L!)

$$\begin{bmatrix} 3 & 14 & 28 \\ 0 & -10/3 & -26/3 \\ 0 & -5/3 & -19/3 \end{bmatrix} \xrightarrow{L_{32} = 1/2} \begin{bmatrix} 3 & 14 & 28 \\ 0 & -10/3 & -26/3 \\ 0 & 0 & -2 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2/3 & 1 & 0 \\ 1/3 & 1/2 & 1 \end{bmatrix} \qquad U = \begin{bmatrix} 3 & 14 & 28 \\ 0 & -10/3 & -26/3 \\ 0 & 0 & -2 \end{bmatrix} \qquad P = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Solve $A\underline{x} = \underline{b} \iff PA\underline{x} = P\underline{b} \iff LU\underline{x} = P\underline{b}$

So $L\underline{y} = P\underline{b}$: $\begin{bmatrix} 1 & 0 & 0 \\ 2/3 & 1 & 0 \\ 1/3 & 1/2 & 1 \end{bmatrix} \underline{y} = \begin{bmatrix} -8 \\ 0 \\ 2 \end{bmatrix} \Rightarrow \underline{y} = \begin{bmatrix} -8 \\ 16/3 \\ 2 \end{bmatrix}$

and $U\underline{x} = \underline{y}$: $\begin{bmatrix} 3 & 14 & 28 \\ 0 & -10/3 & -26/3 \\ 0 & 0 & -2 \end{bmatrix} \underline{x} = \begin{bmatrix} -8 \\ 16/3 \\ 2 \end{bmatrix} \Rightarrow \underline{x} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$

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2. Here are a matrix A and its L and U factors:

$$A = \begin{bmatrix} 1 & -1 & 2 \\ -2 & 1 & 1 \\ -1 & 2 & 1 \end{bmatrix}, \quad L = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 1 & -1 & 2 \\ 0 & -1 & 5 \\ 0 & 0 & 8 \end{bmatrix}.$$

(a) Use this information to compute $\det(A)$.

(b) Use this information to compute A^{-1}. Do not compute any matrix inverses – solve linear systems!

(a) Since $A = LU$ and $\det(LU) = \det L \cdot \det U$

we have $\det A = \det L \cdot \det U$

where $\det L = 1$ and $\det U = -8$ (triangular matrices)

$\Rightarrow$ $\underline{\det A = -8}$

(b) Solve $AB = I \Leftrightarrow L\underbrace{UB}_{=Y} = I$

(i) $LY = I \Rightarrow Y = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 1 & 1 \end{bmatrix} (= L^{-1})$ (3 simple simultaneous systems)

(ii) $UB = Y$: $\begin{pmatrix} 1 & -1 & 2 \\ 0 & -1 & 5 \\ 0 & 0 & 8 \end{pmatrix} \underline{b_1} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \Rightarrow \underline{b_1} = \begin{pmatrix} 1/8 \\ -1/8 \\ 3/8 \end{pmatrix}$

$\begin{pmatrix} 1 & -1 & 2 \\ 0 & -1 & 5 \\ 0 & 0 & 8 \end{pmatrix} \underline{b_2} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \Rightarrow \underline{b_2} = \begin{pmatrix} -5/8 \\ -3/8 \\ 1/8 \end{pmatrix}$

$\begin{pmatrix} 1 & -1 & 2 \\ 0 & -1 & 5 \\ 0 & 0 & 8 \end{pmatrix} \underline{b_3} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \Rightarrow \underline{b_3} = \begin{pmatrix} 3/8 \\ 5/8 \\ 1/8 \end{pmatrix}$

$\Rightarrow$

$$\underline{A^{-1} = \frac{1}{8}\begin{pmatrix} 1 & -5 & 3 \\ -1 & -3 & 5 \\ 3 & 1 & 1 \end{pmatrix}}$$

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3. Find the Lagrange form of the interpolating polynomial for the data

x	-2	0	2
y	4	2	8

$$p(x) = L_1(x)\,y_1 + L_2(x)\,y_2 + L_3(x)\,y_3$$

with $L_1(x) = \dfrac{x(x-2)}{-2(-2-2)} = \dfrac{1}{8}x(x-2)$

$$L_2(x) = \frac{(x+2)(x-2)}{2(-2)} = \frac{4-x^2}{4}$$

$$L_3(x) = \frac{(x+2)x}{4\cdot 2} = \frac{1}{8}x(x+2)$$

so $p(x) = \dfrac{4}{8}x(x-2) + \dfrac{2}{4}(4-x^2) + \dfrac{8}{8}x(x+2)$

$$= \frac{1}{2}x(x-2) + \frac{1}{2}(4-x^2) + x(x+2)$$

$$= x^2 + x + 2$$

4. Determine whether the function

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$$S(x) = \begin{cases} 10 - 24x + 18x^2 - 4x^3, & 1 \le x \le 2 \\ -54 + 72x - 30x^2 + 4x^3, & 2 \le x \le 3 \end{cases}$$

is a cubic spline on $[1, 3]$.

Clearly, S consists of cubic polynomial pieces.

Need to verify that S, S' and S'' are continuous at the breakpoints:

$$S_1(2) = 10 - 48 + 72 - 32 = 2 \overset{?}{=} -54 + 144 - 120 + 32 = S_2(2) \checkmark$$

$$S_1'(x) = -24 + 36x - 12x^2$$

$$S_2'(x) = 72 - 60x + 12x^2$$

$$S_1'(2) = \underbrace{-24 + 72 - 48}_{=0} = \underbrace{72 - 120 + 48}_{=0} = S_2'(2) \checkmark$$

$$S_1''(x) = 36 - 24x$$

$$S_2''(x) = -60 + 24x$$

$$S_1''(2) = \underbrace{36 - 48}_{=-12} = \underbrace{-60 + 48}_{=-12} = S_2''(2) \checkmark$$

It <u>is</u> a cubic spline.

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5. Let $f(x) = x^3 - 4$.

(a) Apply four iterations of the bisection method on $[a, b] = [0, 2]$ to approximate a root of f.

(b) Use Newton's method with $x_0 = 1$ to compute x_3.

(a) Iteration 1: $a = 0$, $b = 2$, $f(a) = -4$, $f(b) = 4$

$\Rightarrow x = 1$, $f(x) = -3$, drop a

Iteration 2: $a = 1$, $b = 2$

$\Rightarrow x = 3/2$, $f(x) = -5/8$, drop a

Iteration 3: $a = 3/2$, $b = 2$

$\Rightarrow x = 7/4$, $f(x) = \frac{87}{64} \approx 1.359375$, drop b

Iteration 4: $a = 3/2$, $b = 7/4$

$\Rightarrow \underline{x} = \frac{13}{8} = \underline{1.625}$, $\underline{f(x)} = \frac{149}{512} = \underline{0.2910156250}$

(b) $f'(x) = 3x^2$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1 - \frac{f(1)}{f'(1)} = 1 - \frac{-3}{3} = 2$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 2 - \frac{4}{12} = \frac{5}{3}$$

$$\underline{x_3} = x_2 - \frac{f(x_2)}{f'(x_2)} = \frac{5}{3} - \frac{17/27}{25/3} = \frac{358}{225} = \underline{1.59\overline{11}}$$

$$\underline{f(x_3) = 0.02811}$$

6. Recall that Newton's method for systems of nonlinear equations is given by

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Input $\boldsymbol{f}$, J, $\boldsymbol{x}^{(0)}$

for $n = 0, 1, 2, \ldots$ do

  Solve $J(\boldsymbol{x}^{(n)})\boldsymbol{h} = -\boldsymbol{f}(\boldsymbol{x}^{(n)})$ for $\boldsymbol{h}$

  Update $\boldsymbol{x}^{(n+1)} = \boldsymbol{x}^{(n)} + \boldsymbol{h}$

end

Output $\boldsymbol{x}^{(n+1)}$

where the Jacobian is of the form

$$J = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_m} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_m} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \cdots & \frac{\partial f_m}{\partial x_m} \end{bmatrix}.$$

Perform one iteration of Newton's method starting with $\boldsymbol{x}^{(0)} = [\pi/4, \pi/4]^T$ to get an approximate solution of the system

$$\begin{aligned} 5\cos(\alpha) + 6\cos(\alpha+\beta) &= 10 \\ 5\sin(\alpha) + 6\sin(\alpha+\beta) &= 4. \end{aligned}$$

$$\underline{f}(\underline{x}) = \begin{bmatrix} 5\cos x_1 + 6\cos(x_1+x_2) - 10 \\ 5\sin x_1 + 6\sin(x_1+x_2) - 4 \end{bmatrix}$$

$$J(\underline{x}) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{pmatrix} = \begin{pmatrix} -5\sin x_1 - 6\sin(x_1+x_2) & -6\sin(x_1+x_2) \\ 5\cos x_1 + 6\cos(x_1+x_2) & 6\cos(x_1+x_2) \end{pmatrix}$$

Newton:

Solve $J(\underline{x}^{(0)})\,\underline{h} = -\underline{f}(\underline{x}^{(0)})$ for $\underline{h}$:

$$\begin{bmatrix} -\frac{5\sqrt{2}}{2} - 6 & -6 \\ \frac{5\sqrt{2}}{2} & 0 \end{bmatrix} \underline{h} = -\begin{bmatrix} \frac{5\sqrt{2}}{2} - 10 \\ \frac{5\sqrt{2}}{2} + 2 \end{bmatrix} \Rightarrow \underline{h} = \begin{bmatrix} -\frac{2\sqrt{2}}{5} - 1 \\ -\frac{1}{3} + \frac{57\sqrt{2}}{30} \end{bmatrix} \approx \begin{bmatrix} -1.56568 \\ 1.41086 \end{bmatrix}$$

Update

$$\underline{\underline{x^{(1)}}} = \underline{x}^{(0)} + \underline{h} = \underline{\begin{bmatrix} -0.780287 \\ 2.196262 \end{bmatrix}}$$

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7. Give one possible MATLAB code fragment that would allow you to define the matrix

$$A = \begin{bmatrix} -10 & 1 & 4 & 0 & 0 & 0 \\ 1 & -10 & 0 & 4 & 0 & 0 \\ 4 & 0 & -10 & 1 & 4 & 0 \\ 0 & 4 & 1 & -10 & 0 & 4 \\ 0 & 0 & 4 & 0 & -10 & 1 \\ 0 & 0 & 0 & 4 & 1 & -10 \end{bmatrix}$$

in sparse matrix format.

(i)

```
a = -10 * ones(6,1)
b = [1 0 1 0 1]
c = 4 * ones(4,1)
A = diag(a,0) + diag(b,-1) + diag(b,1) + diag(c,-2) + diag(c,2)
A = sparse(A)
```

OR

(ii)

```
a = -10 * ones(6,1)
b = [1 0 1 0 1 0]'
c = 4 * ones(6,1)
A = spdiags([c b a flipud(b) c], -2:2, 6, 6)
```

OR

(iii)

```
i = [1 1 1 2 2 2 3 3 3 3 4 4 4 4 5 5 5 6 6 6]
j = [1 2 3 1 2 4 1 3 4 5 2 3 4 6 3 5 6 4 5 6]
x = [-10 1 4 1 -10 4 4 -10 1 4 4 1 -10 4 4 -10 1 4 1 -10]

A = sparse(i, j, x, 6, 6)
```

OR (iv) enter A as full matrix and then use sparse(A)