Math 577 — Homework Assignment 3, due Oct.12, 2006

1. Prove that if $A \in \mathbb{C}^{m\times n}$ with $\text{rank}(A) = n$, then A^*A is Hermitian and positive definite.

2. Consider the matrices
$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$
used on the previous homework.

   (a) Using any method you like, determine (on paper) a reduced QR factorization $A = \hat{Q}\hat{R}$ and a full QR factorization $A = QR$.

   (b) Again using any method you like, determine reduced and full QR factorizations $B = \hat{Q}\hat{R}$ and $B = QR$.

3. Let A be an $m \times n$ matrix. Determine the *exact* number of floating point additions, subtractions, multiplications and divisions involved in performing the classical and modified Gram-Schmidt algorithms as listed in the classnotes.

4. Consider the 2×2 orthogonal matrices
$$F = \begin{bmatrix} -c & s \\ s & c \end{bmatrix}, \quad J = \begin{bmatrix} c & s \\ -s & c \end{bmatrix},$$
where $s = \sin\theta$ and $c = \cos\theta$ for some θ. The first matrix has $\det F = -1$ and is a reflector — the special case of a Householder reflector in dimension 2. The second has $\det J = 1$ and effects a rotation instead of a reflection. Such a matrix is called a *Givens rotation*.

   (a) Describe exactly what geometric effects left-multiplications by F and J have on the plane $\mathbb{R}^2$. (J rotates the plane by an angle θ, for example, but is the rotation clockwise or counterclockwise?)

   (b) Describe an algorithm for QR factorization that is analogous to the Householder QR algorithm listed in the classnotes but based on Givens rotations instead of Householder reflections.