Math 577 — Homework Assignment 5, due Nov.9, 2006

1. For each of the following statements, prove that it is true or give an example to show it is false. Throughout, $A \in \mathbb{C}^{m\times m}$ unless otherwise noted.

   (a) If A is real and λ is an eigenvalue of A, then so is $-\lambda$.

   (b) If A is real and λ is an eigenvalue of A, then so is $\overline{\lambda}$.

   (c) If λ is an eigenvalue of A and A is nonsingular, then λ^{-1} is an eigenvalue of A^{-1}.

   (d) If A is Hermitian and λ is an eigenvalue of A, then $|\lambda|$ is a singular value of A.

2. Here is *Gerschgorin's theorem*, which holds for any $m\times m$ matrix A, symmetric or nonsymmetric:

   Every eigenvalue of A lies in at least one of the m circular disks in the complex plane with centers a_{ii} and radii $\sum_{j\neq i}|a_{ij}|$. Moreover, if n of these disks form a connected domain that is disjoint from the other $m-n$ disks, then there are precisely n eigenvalues of A within this domain.

   (a) Prove the first part of Gerschgorin's theorem. (Hint: Let λ be any eigenvalue of A, and $\boldsymbol{x}$ a corresponding eigenvector normalized so that its largest entry is 1.)

   (b) Give estimates based on Gerschgorin's theorem for the eigenvalues of

$$A = \begin{bmatrix} 8 & 1 & 0 \\ 1 & 4 & \varepsilon \\ 0 & \varepsilon & 1 \end{bmatrix}, \qquad |\varepsilon| < 1.$$

3. Suppose we have a 3×3 matrix and wish to introduce zeros by left- and/or right-multiplications by unitary matrices Q_j such as Householder reflections or Givens rotations. Consider the following three matrix structures:

$$\text{(a)} \begin{bmatrix} x & x & 0 \\ 0 & x & x \\ 0 & 0 & x \end{bmatrix}, \qquad \text{(b)} \begin{bmatrix} x & x & 0 \\ x & 0 & x \\ 0 & x & x \end{bmatrix}, \qquad \text{(c)} \begin{bmatrix} x & x & 0 \\ 0 & 0 & x \\ 0 & 0 & x \end{bmatrix}.$$

   For each one, decide which of the following situations holds, and justify your claim.

   (i) Can be obtained by a sequence of left-multiplications by matrices Q_j;

   (ii) Not (i), but can be obtained by a sequence of left- and right-multiplications by matrices Q_j;

   (iii) Cannot be obtained by any sequence of left- and right-multiplications by matrices Q_j.

4. Let $A \in \mathbb{C}^{m\times m}$ be given, not necessarily Hermitian. Show that a number $z \in \mathbb{C}$ is a Rayleigh quotient of A if and only if it is a diagonal entry of Q^*AQ for some unitary matrix Q. Thus Rayleigh quotients are just diagonal entries of matrices, once you transform orthogonally to the right coordinate system.

5. The preliminary reduction to tridiagonal form would be of little use if the steps of the QR algorithm did not preserve this structure. Fortunately they do.

   (a) In the QR factorization $A = QR$ of a symmetric tridiagonal matrix A, which entries of R are in general nonzero? Which entries of Q? (Remember that in practice we do not form Q explicitly.)

(b) Show that the tridiagonal structure is recovered when the product RQ is formed.

(c) Explain how Givens rotations or 2×2 Householder reflections can be used in the computation of the QR factorization of a tridiagonal matrix, reducing the operation count far below what would be required for a full matrix.