MATH 590: Meshfree Methods

Chapter 1 — Part 1: Introduction and a Historical Overview

Greg Fasshauer

Department of Applied Mathematics
Illinois Institute of Technology

Fall 2014

Outline

Outline

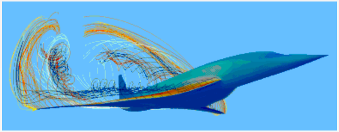
Meshfree Methods

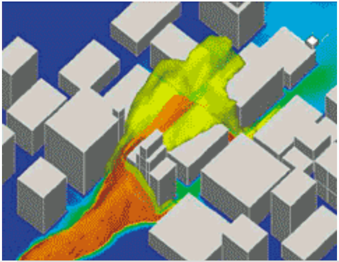
- have gained much attention in recent years
- interdisciplinary field
- many traditional numerical methods (finite differences, finite elements or finite volumes) have trouble with high-dimensional problems
- meshfree methods can often handle changes in the geometry of the domain of interest (e.g., free surfaces, moving particles and large deformations) better
- independence from a mesh is a great advantage since mesh generation is one of the most time consuming parts of any mesh-based numerical simulation
- new generation of numerical tools

Applications

- Original applications were in geodesy, geophysics, mapping, or meteorology
- Later, many other application areas
  - numerical solution of PDEs in many engineering applications,
  - computer graphics,
  - optics,
  - artificial intelligence,
  - machine learning or statistical learning (neural networks or SVMs),
  - signal and image processing,
  - sampling theory,
  - statistics (kriging),
  - response surface or surrogate modeling,
  - finance,
  - optimization.

Complicated Domains

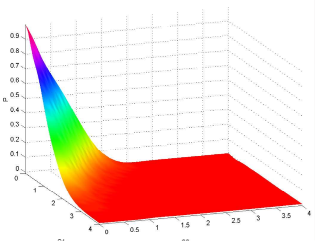




Recent paradigm shift in numerical simulation of fluid flow:

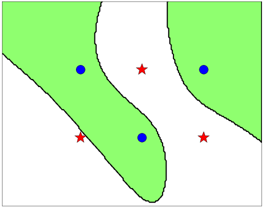
- growing demand to perform numerical simulations associated with complicated domains and geometries
  - complete aircraft configurations
  - turbo-machinery
  - urban domains

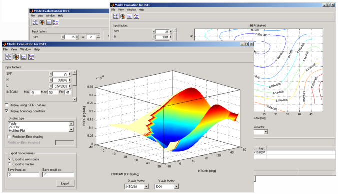
High-Dimensional Domains



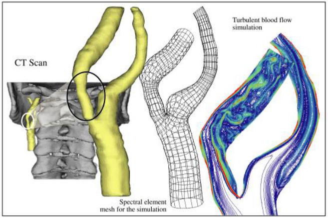
More and more applications “live” in high “space” dimensions

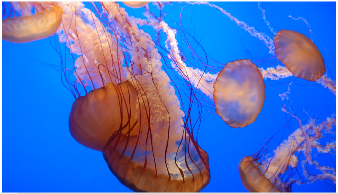
- pricing of basket options in finance
- classification or regression problems in machine learning
- surrogate modeling for computer experiments or other design purposes





Life Sciences





- growing demand for flow simulation in the life sciences
  - blood flow simulations
  - flight dynamics of birds and insects (micro air-vehicles)
  - motion of fish and jellyfish

Special Movie Effects http://www.polhemus.com/

Based on point cloud modeling (also look at [LotR]).

Medical Applications http://www.fastscan3d.com

Waves

- simulation of ocean waves
  - design of boats and ships in turbulent oceans
- splash dynamics
  - spacecraft and oil tankers

Energy

- renewable energy sector
  - study of single wind turbines and whole wind farms

A Long List of Meshfree Methods *Not* Discussed

In this class we do not plan to focus on any of the following methods, also known as meshfree (or meshless) methods:

- element-free Galerkin method (EFG),
- Hp-clouds,
- moving least squares (MLS),
- meshless local Petrov-Galerkin method (MLPG),
- partition of unity finite element method (PUFEM),
- reproducing kernel particle method (RKPM),
- smoothed particle hydrodynamics (SPH),
- extended finite element method (XFEM)
- and many others.

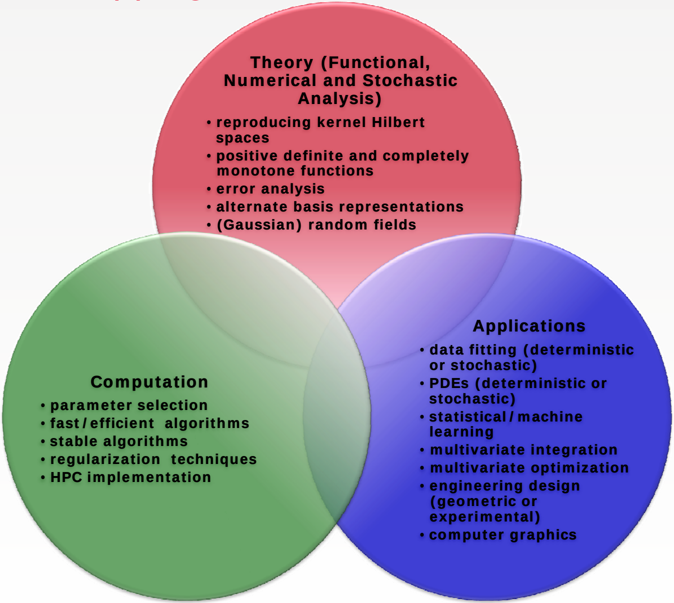
The list on Wikipedia is endless.
Some references are [Atl04, AS02, BC07, LL07, Liu02, LL03].

We will focus mostly on the underlying math of kernel-based methods.
In your project you are encouraged to work on an application.

Types of math involved:

- approximation theory
- numerical analysis
- linear algebra
- functional analysis
  - positive definite reproducing kernels
  - Sturm–Liouville theory and Green's kernels
  - (generalized) Fourier series
  - Hilbert–Schmidt integral operators
- stochastic analysis and statistics
  - Gaussian random fields
  - maximum likelihood estimation
- harmonic analysis
- special functions
- measure theory
- theoretical computer science

Three Overlapping Domains



Books that contain a significant portion on positive definite kernels:

Analysis: [AM02], [AG93], [Aub00], [BCR84], [Ber50], [Boc32], [Mes62], [Moo35], [RSN55], [Rit00], [Sai88, Sai97], [WW76]

Approximation Theory: [Buh03], [CL99], [Fas07], [FGS98], [GC98], [Isk04], [NW08], [Wen05]

Integral Equations: [Coc72], [Hac89], [Hoc73], [Kre99], [Pog66], [Smi58], [Sta79]

Mathematical Physics: [BS53], [CH53], [MF53]

Engineering and Physics Applications: [AAG00], [Atl04], [AS02], [BC07], [CFC14], [FSK08], [GS03], [LL07], [Liu02], [LL03], [VT01]

Probability Theory and Statistics: [BTA04], [Gu13], [RS05], [Wah90]

Geostatistics: [Cre93], [Kit97], [Mat86], [Mat65], [Ste99]

Statistical/Machine Learning: [Alp09], [Cat04], [CST00], [CZ07], [HTF09], [Her02], [Joa02], [RW06], [SS02], [STC04], [SC08], [SGB$^+$02], [Vap98]

Important (review) papers:

Historical: [Aro50], [Hil72], [Mer09], [Sch08], [Ste76]

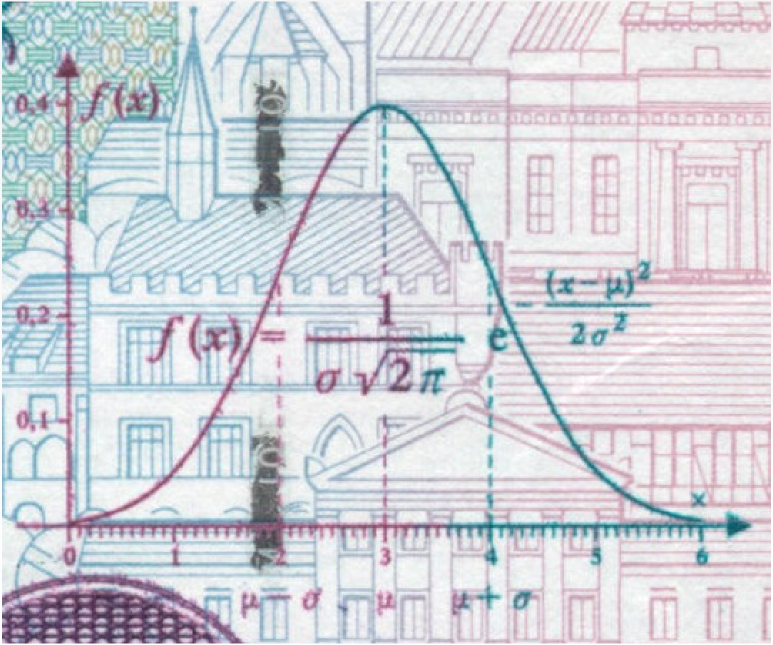
Modern: [Buh00], [CS02], [Dyn87, Dyn89], [Fas11], [FM13], [Pow87, Pow92], [Sch99, Sch00], [SW06], [SSS13].

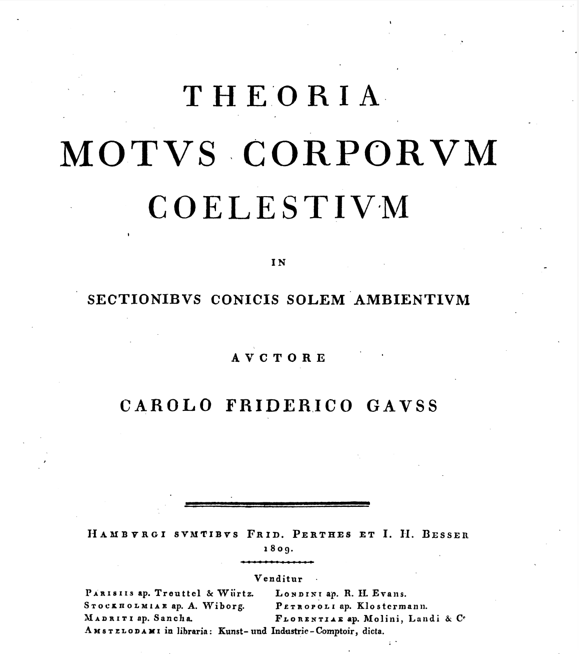

Graduate Texts in Mathematics
Christian Berg
Jens Peter Reus Christensen
Paul Ressel
Harmonic Analysis on Semigroups
Theory of Positive Definite and Related Functions
Springer-Verlag
The Kernel Function and Conformal Mapping
Stefan Bergman
REPRODUCING KERNEL HILBERT SPACES IN PROBABILITY AND STATISTICS
ALAIN BERLINET
CHRISTINE THOMAS-AGNAN
Radial Basis Functions
Martin D. Buhmann
Olivier Catoni
Statistical Learning Theory and Stochastic Optimization
Lecture Notes in Mathematics
Springer
A Course in Approximation Theory
Ward Cheney
Will Light
Support Vector Machines
and other kernel-based learning methods
Learning Theory
An Approximation Theory Viewpoint
Felipe Cucker and Ding Xuan Zhou
CAMBRIDGE
Meshfree Approximation Methods with MATLAB
Gregory E. Fasshauer
Alexander Forrester | András Sóbester | Andy Keane
Engineering Design via Surrogate Modelling
A Practical Guide
WILEY
Constructive Approximation on the Sphere
W. FREEDEN
T. GERVENS
M. SCHREINER
Springer Series in Statistics
Trevor Hastie
Robert Tibshirani
Jerome Friedman
The Elements of Statistical Learning
Data Mining, Inference, and Prediction
Second Edition
Springer
Armin Iske
Multiresolution Methods in Scattered Data Modelling
Springer
LEARNING TO CLASSIFY TEXT USING SUPPORT VECTOR MACHINES
Methods, Theory and Algorithms
Thorsten Joachims
Kluwer Academic Publishers
G.R. Liu
MESH FREE METHODS
Moving beyond the Finite Element Method
CRC PRESS
Gaussian Processes for Machine Learning
Carl Edward Rasmussen and Christopher K. I. Williams
S Saitoh
Integral transforms, reproducing kernels and their applications
LONGMAN
Learning with Kernels
Support Vector Machines, Regularization, Optimization, and Beyond
Bernhard Schölkopf and Alexander J. Smola
Kernel Methods for Pattern Analysis
CAMBRIDGE
Springer Series in Statistics
Michael L. Stein
Interpolation of Spatial Data
Some Theory for Kriging
Springer
Information Science and Statistics
Ingo Steinwart • Andreas Christmann
Support Vector Machines
Springer
Statistical Learning Theory
Vladimir N. Vapnik
59
Spline Models for Observational Data
GRACE WAHBA
Scattered Data Approximation
Holger Wendland

Outline


GU5672972S2
DEUTSCHE BUNDESBANK
Banknote
ZEHN DEUTSCHE MARK
10
1777–1855 Carl Friedr. Gauß
Deutsche Bundesbank
Frankfurt am Main
1. September 1999
GU5672972S2

The Gaussian kernel


0,4
f(x)
0,3
0,2
0,1
$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$
0
1
2
3
4
5
6
x
μ − σ
μ
μ + σ



THEORIA

MOTVS CORPORVM

COELESTIVM

IN

SECTIONIBVS CONICIS SOLEM AMBIENTIVM

AVCTORE

CAROLO FRIDERICO GAVSS

HAMBVRGI SVMTIBVS FRID. PERTHES ET I. H. BESSER
1809.

Venditur

PARISIIS ap. Treuttel & Würtz. LONDINI ap. R. H. Evans.
STOCKHOLMIAE ap. A. Wiborg. PETROPOLI ap. Klostermann.
MADRITI ap. Sancha. FLORENTIAE ap. Molini, Landi & C°
AMSTELODAMI in libraria: Kunst- und Industrie-Comptoir, dicta.

Porro facile perspicitur, k necessario negatiuam esse debere, quo Ω reuera fieri possit maximum, quamobrem statuemus $\frac{1}{2}k = -hh$; et quum per theorema elegans primo ab ill. Laplace inuentum, integrale $\int e^{-hh\Delta\Delta}\,d\Delta$, a $\Delta = -\infty$ vsque ad $\Delta = +\infty$, fiat $= \frac{\sqrt{\pi}}{h}$, (denotando per π semicircumferentiam circuli cuius radius 1), functio nostra fiet

$$\varphi\Delta = \frac{h}{\sqrt{\pi}}\, e^{-hh\Delta\Delta}$$

Moreover, it is readily perceived that k must be negative, in order that Ω may really become a maximum, for which reason we shall put

$$\tfrac{1}{2}k = -hh;$$

and since, by the elegant theorem first discovered by LAPLACE, the integral

$$\int e^{-hh\Delta\Delta}\,d\Delta$$

from $\Delta = -\infty$ to $\Delta = +\infty$ is $\frac{\sqrt{\pi}}{h}$, (denoting by π the semicircumference of the circle the radius of which is unity), our function becomes

$$\varphi\Delta = \frac{h}{\sqrt{\pi}}\, e^{-hh\Delta\Delta}.$$

[Gau09, Sect.177], Engl. translation [Gau57]

Porro facile perspicitur, k necessario negatiuam esse debere, quo Ω reuera fieri possit maximum, quamobrem statuemus $\frac{1}{2}k = -hh$; et quum per theorema elegans primo ab ill. Laplace inuentum, integrale $\int e^{-hh\Delta\Delta}\,d\Delta$, a $\Delta = -\infty$ vsque ad $\Delta = +\infty$, fiat $= \frac{\sqrt{\pi}}{h}$, (denotando per π semicircumferentiam circuli cuius radius 1), functio nostra fiet

$$\varphi\Delta = \frac{h}{\sqrt{\pi}} e^{-hh\Delta\Delta}$$

Moreover, it is readily perceived that k must be negative, in order that Ω may really become a maximum, for which reason we shall put

$$\tfrac{1}{2}k = -hh;$$

and since, by the elegant theorem first discovered by Laplace, the integral

$$\int e^{-hh\Delta\Delta}\,d\Delta$$

from $\Delta = -\infty$ to $\Delta = +\infty$ is $\frac{\sqrt{\pi}}{h}$, (denoting by π the semicircumference of the circle the radius of which is unity), our function becomes

$$\varphi\Delta = \frac{h}{\sqrt{\pi}} e^{-hh\Delta\Delta}.$$

[Gau09, Sect.177], Engl. translation [Gau57]

Nachman Aronszajn (1907-1980) and Stefan Bergman (1895-1977)

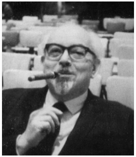
- Aronszajn was at the University of Kansas, Bergman was at Harvard and later Stanford.
- They are credited with independently introducing reproducing kernels [Aro50, Ber50]. Aronszajn had some earlier work in [Aro43]. Bergman already had some bits in his Ph.D. thesis [Ber21].
- The kernel K is a symmetric function of two variables

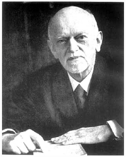
$$K : \Omega \times \Omega \to \mathbb{R} \quad \text{s.t.} \quad K(\boldsymbol{x}, \boldsymbol{z}) = K(\boldsymbol{z}, \boldsymbol{x}), \quad \text{where } \Omega \subseteq \mathbb{R}^d.$$

- The kernel is reproducing in the following sense:

$$\langle K(\boldsymbol{x}, \cdot), f \rangle_{\mathcal{H}_K(\Omega)} = f(\boldsymbol{x}),$$

where $\mathcal{H}_K(\Omega)$ is a Hilbert space of functions on Ω.





http://en.wikipedia.org/wiki/Nachman_Aronszajn

http://en.wikipedia.org/wiki/Stefan_Bergman

James Mercer (1883-1932) and Erhard Schmidt (1876-1959)

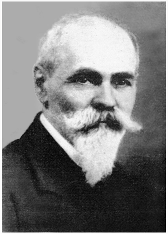
- Mercer was at Cambridge University, Schmidt (a student of Hilbert's) was at the University of Berlin (now called Humboldt U.).
- Prompted by Hilbert's ground-breaking [Hil04], they independently introduced series expansions of positive definite kernels in [Mer09, Sch07, Sch08]:
  - Mercer's theorem (used heavily in machine learning)
  - Hilbert-Schmidt theory (also Gram-Schmidt orthogonalization)

http://en.wikipedia.org/wiki/James_Mercer_(mathematician)

http://en.wikipedia.org/wiki/Erhard_Schmidt

E.H. Moore (1862-1932), Stanislaw Zaremba (1863-1942), Maximilian Mathias (1895-?)

- Moore and Zaremba were professors of mathematics at Chicago, and Jagiellonian University in Kraków, respectively. Mathias was a PhD student of Erhard Schmidt in Berlin.
- Moore called his version of reproducing kernels positive Hermitian matrices [Moo16, Moo35].
- Zaremba introduced the reproducing property in [Zar07, Zar09].
- Mathias wrote his thesis [Mat23] on positive definite functions.

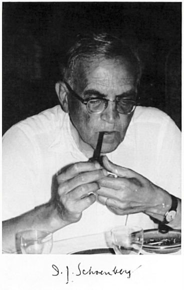



http://en.wikipedia.org/wiki/E_H_Moore

http://en.wikipedia.org/wiki/Stanislaw_Zaremba_(mathematician)

Salomon Bochner (1899-1982) and Iso Schoenberg (1903-1990)

- Bochner (also a student of Schmidt's) was at Princeton and then at Rice University, Schoenberg was at the University of Wisconsin-Madison.
- One of Bochner's fundamental results is Bochner's theorem, which characterizes positive definite functions in terms of Fourier transforms [Boc32, Boc33].
- Schoenberg is known as the father of splines. He also characterized positive definite functions, especially radial ones [Sch37, Sch38a, Sch38b].



http://en.wikipedia.org/wiki/Salomon_Bochner

http://en.wikipedia.org/wiki/Isaac_Jacob_Schoenberg

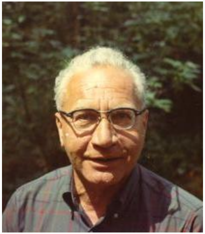
Marc Atteia

- Was at the Université Joseph Fourier in Grenoble in the 60s, later at the Université Paul Sabatier in Toulouse.
- Possibly the first to use the term spline in the general multivariate setting.
- Established the foundations of a variational theory via Green's kernels interpreted as reproducing kernels in the sense of Aronszajn and Bergman in [Att66].

Aleksandr Khinchin (1894-1959), Kari Karhunen (1915-1992), Michel Loève (1907-1979)

- Khinchin was a professor of mathematics at Moscow State University, Karhunen was in Helsinki and Loève at Berkeley.
- Khinchin established the foundation for stationary stochastic processes in [Khi34] using Bochner's theorem.
- Karhunen and Loève established the Karhunen–Loève theorem, a series representation for stochastic processes [Kar47, Loè77].





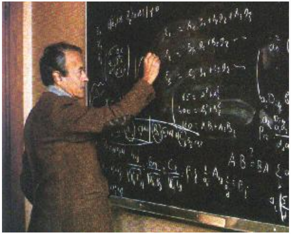
http://en.wikipedia.org/wiki/Aleksandr_Khinchin

http://en.wikipedia.org/wiki/Kari_Karhunen

http://en.wikipedia.org/wiki/Michel_Loeve

Danie Krige (1919-2013) and Georges Matheron (1930-2000)

- Krige was a professor of geostatistics at the University of Witwatersrand.
- Invented the so-called kriging method in [Kri51].
- Matheron was a professor of mathematics and geostatistics at the Paris School of Mines in Fontainebleau.
- One of the founders of geostatistics who established the mathematical theory for kriging in his Ph.D. thesis [Mat65].



http://en.wikipedia.org/wiki/Danie_Krige

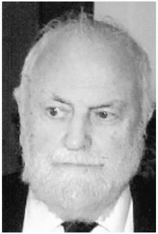
http://en.wikipedia.org/wiki/Georges_Matheron

Rolland L. Hardy (1920-2009) and Richard Franke

- Hardy was a professor of civil and construction engineering at Iowa State University (retired 1988). Franke was a mathematician at the Naval Postgraduate School in Monterey, California (retired 2001).
- Hardy introduced multiquadrics (MQs) in the early 1970s (see, e.g., [Har71]).
- His work was primarily concerned with applications in geodesy and mapping.
- Franke compared various scattered data interpolation methods [Fra79, Fra82], and concluded MQs and TPSs were the best. Conjectured that the interpolation matrix for MQs is invertible.

Robert L. Harder and Robert N. Desmarais (1930-2008)

- Aerospace engineers at MacNeal-Schwendler Corporation (MSC Software), and NASA's Langley Research Center.
- Introduced thin plate splines (TPSs) in 1972 [HD72].
- Work was concerned mostly with aircraft design.



http://en.wikipedia.org/wiki/Thin_plate_spline

Jean Duchon and Jean Meinguet

- Duchon is a senior researcher in mathematics at the Université Joseph Fourier in Grenoble, France. Meinguet was a mathematics professor at Université Catholique de Louvain, Belgium (retired 1996).
- Duchon provided the foundation for a variational approach minimizing the integral of $\nabla^2 f$ in $\mathbb{R}^2$ [Duc76, Duc77, Duc78, Duc80]. This also leads to thin plate splines.
- Meinguet introduced surface splines in the late 1970s [Mei79c, Mei79b, Mei79a, Mei84].
- Surface splines and thin plate splines are both considered as polyharmonic splines.

http://en.wikipedia.org/wiki/Polyharmonic_spline

Wolodymyr (Wally) Madych and Stuart Alan Nelson

- Both professors of mathematics. Madych at the University of Connecticut, and Nelson at Iowa State University (now retired).
- Proved Franke's conjecture (and much more) based on a variational approach in [MN83]. This manuscript was never published.

Charles Micchelli

- Used to be a mathematician at IBM Watson Research Center. Now a professor at the State University of New York, Albany.
- Published [Mic86] in which he also proved Franke's conjecture. His proofs are rooted in the work of Bochner and Schoenberg on positive definite and completely monotone functions.
- His approach is followed throughout much of [Fas07].

Grace Wahba

- I. J. Schoenberg Professor of statistics at the University of Wisconsin-Madison.
- Studied the use of thin plate splines for statistical purposes in the context of smoothing splines for noisy data and data on spheres.
- Introduced ANOVA and cross validation approaches to the radial basis function setting (e.g., [Wah79, Wah81, Wah82, WW80]).
- One of the first monographs on the subject is [Wah90].

http://en.wikipedia.org/wiki/Grace_Wahba

Nira Dyn and David Levin

- Professors of applied mathematics at Tel-Aviv University.
- They first used the term radial basis function in [DL83], and were one of the first proponents of radial basis function methods in approximation theory (see, e.g., the surveys [Dyn87, Dyn89]).
- Dyn collaborated early with Grace Wahba on connections between numerical analysis and statistics via smoothing splines (see [DWW79, DW82]).
- Both have also contributed to many other areas of approximation theory (including subdivision, MLS and computer graphics).

Robert Schaback and Holger Wendland

- Professor of mathematics at University of Göttingen (retired 2011).
- Introduced compactly supported RBFs [Sch95].
- Another popular family of CSRBFs was presented by Holger Wendland (professor of mathematics at University of Bayreuth) in his Ph.D. thesis at Göttingen (see also [Wen95]).
- Both have contributed extensively to the field of radial basis functions (see, e.g., the many surveys [Sch99, Sch00, SW06, SSS13] and the book [Wen05]).

Ed Kansa

- Physicist at Lawrence Livermore National Laboratory, California (retired).
- First proposed the use of radial basis functions for the solution of PDEs [Kan86].
- Later papers [Kan90a, Kan90b] proposed “Kansa’s method” (or non-symmetric collocation).

http://en.wikipedia.org/wiki/Kansa_method

Bengt Fornberg

- Professor of applied mathematics at the University of Colorado at Boulder.
- Has done lots of seminal work (together with a number of co-workers) on stable evaluation of RBFs and RBF solution of PDEs.
- Has also written an important book on (polynomial) pseudospectral methods [For98].

Vladimir Vapnik

- Professor of computer science at Columbia University.
- First used reproducing kernels in machine learning as support vector machines in [BGV92] while at Bell Labs. This was based on [VL63].
- Author of the first book of statistical learning theory [Vap98].

http://en.wikipedia.org/wiki/Vladimir_Vapnik

References I

[AAG00] S. T. Ali, J.-P. Antoine, and J.-P. Gazeau, *Coherent States, Wavelets and Their Generalizations*, Graduate Texts in Contemporary Physics, Springer, 2000.

[AG93] N. I. Akhiezer and I. M. Glazman, *Theory of Linear Operators in Hilbert Space*, Dover Publications, 1993, Originally published in 1961.

[Alp09] Ethem Alpaydin, *Introduction to Machine Learning*, 2nd ed., The MIT Press, December 2009.

[AM02] J. Agler and J. E. McCarthy, *Pick Interpolation and Hilbert Function Spaces*, American Mathematical Soc., 2002 (en).

[Aro43] N. Aronszajn, *La théorie des noyaux reproduisants et ses applications, Prèmiere Partie*, Mathematical Proceedings of the Cambridge Philosophical Society **39** (1943), no. 03, 133–153.

[Aro50] ______, *Theory of reproducing kernels*, Transactions of the American Mathematical Society **68** (1950), no. 3, 337–404.

References II

[AS02] S. N. Atluri and S. Shen, *The Meshless Local Petrov-Galerkin (MLPG) Method*, Tech Science Press (Encino, CA), 2002.

[Atl04] S. N. Atluri, *The Meshless Method (MLPG) for Domain & BIE Discretizations*, Tech Science Press, Encino, CA, 2004.

[Att66] Marc Atteia, *Existence et détermination des fonctions “spline” à plusieurs variables*, C. R. Acad. Sci. Paris **262** (1966), 575—578.

[Aub00] Jean-Pierre Aubin, *Applied Functional Analysis*, 2nd ed., Wiley-Interscience, New York, 2000.

[BC07] T. Belytschko and J. S. Chen, *Meshfree and Particle Methods*, Wiley, 2007.

[BCR84] C. Berg, J. P. R. Christensen, and P. Ressel, *Harmonic Analysis on Semigroups: Theory of positive definite and related functions*, Graduate Texts in Mathematics, Springer, 1984.

References III

[Ber21] Stefan Bergmann, *über die Entwicklung der harmonischen Funktionen der Ebene und des Raumes nach Orthogonalfunktionen*, Ph.D. thesis, University of Berlin, 1921.

[Ber50] S. Bergman, *The Kernel Function and Conformal Mapping*, 2nd ed., American Mathematical Society, 1950.

[BGV92] Bernhard E. Boser, Isabelle M. Guyon, and Vladimir N. Vapnik, *A training algorithm for optimal margin classifiers*, Proceedings of the Fifth Annual Workshop on Computational Learning Theory (New York, NY, USA), COLT '92, ACM, 1992, pp. 144–152.

[Boc32] S. Bochner, *Vorlesungen über Fouriersche Integrale*, Mathematik und ihre Anwendungen, vol. 12, Akademische Verlagsgesellschaft, Leipzig, 1932, Translated as: Lectures on Fourier Integrals. (AM-42) Princeton University Press, 1959.

[Boc33] ______, *Monotone Funktionen, Stieltjes Integrale und harmonische Analyse*, Math. Ann. **108** (1933), 378–410.

References IV

[BS53] Stefan Bergman and Menahem Schiffer, *Kernel Functions and Elliptic Differential Equations in Mathematical Physics*, Academic Press, 1953.

[BTA04] A. Berlinet and C. Thomas-Agnan, *Reproducing Kernel Hilbert Spaces in Probability and Statistics*, Kluwer Academic Publishers, Dordrecht, 2004.

[Buh00] M. D. Buhmann, *Radial basis functions*, Acta Numerica **9** (2000), 1–38.

[Buh03] ______, *Radial Basis Functions: Theory and Implementations*, Cambridge Monographs on Applied and Computational Mathematics, Cambridge University Press, 2003.

[Cat04] O. Catoni, *Statistical Learning Theory and Stochastic Optimization*, Springer, 2004.

[CFC14] Wen Chen, Zhuo-Jia Fu, and C. S Chen, *Recent Advances on Radial Basis Function Collocation Methods*, Springer Briefs in Applied Sciences and Technology, Springer, 2014.

References V

[CH53] R. Courant and D. Hilbert, *Methods of Mathematical Physics, Vol. 1*, Wiley, 1953, Reprinted 1989.

[CL99] E. W. Cheney and W. A. Light, *A Course in Approximation Theory*, Brooks/Cole (Pacific Grove, CA), 1999.

[Coc72] James Alan Cochran, *The Analysis of Linear Integral Equations*, McGraw-Hill, 1972.

[Cre93] N. Cressie, *Statistics for Spatial Data*, revised ed., Wiley-Interscience, 1993.

[CS02] Felipe Cucker and Steve Smale, *On the mathematical foundations of learning*, Bulletin of the American Mathematical Society **39** (2002), no. 1, 1–49.

[CST00] N. Cristianini and J. Shawe-Taylor, *An introduction to support vector machines and other kernel-based learning methods*, Cambridge University Press, 2000.

References VI

[CZ07] F. Cucker and D. X. Zhou, *Learning Theory: An Approximation Theory Viewpoint*, Cambridge Monographs on Applied and Computational Mathematics, Cambridge University Press, 2007.

[DL83] N. Dyn and D. Levin, *Iterative solution of systems originating from integral equations and surface interpolation*, SIAM Journal on Numerical Analysis **20** (1983), no. 2, 377–390.

[Duc76] J. Duchon, *Interpolation des fonctions de deux variables suivant le principe de la flexion des plaques minces*, Rev. Francaise Automat. Informat. Rech. Opér., Anal. Numer. **10** (1976), 5–12.

[Duc77] ______, *Splines minimizing rotation-invariant semi-norms in Sobolev spaces*, Constructive Theory of Functions of Several Variables, Oberwolfach 1976 (W. Schempp and K. Zeller, eds.), Springer-Verlag (Berlin), 1977, Springer Lecture Notes in Math. 571, pp. 85–100.

[Duc78] ______, *Sur l'erreur d'interpolation des fonctions de plusieurs variables par les d^m-splines*, RAIRO Analyse Numerique **12** (1978), no. 4, 325–334.

References VII

[Duc80] ______, *Fonctions splines homogènes à plusiers variables*, Tech. report, Université de Grenoble, 1980.

[DW82] N. Dyn and G. Wahba, *On the estimation of functions of several variables from aggregated data*, SIAM J. Math. Anal. **13** (1982), 134–152.

[DWW79] N. Dyn, G. Wahba, and W. H. Wong, *Smooth pycnophylactic interpolation for geographical regions: Comment*, J. Amer. Stat. Assoc. **74** (1979), 530–535.

[Dyn87] N. Dyn, *Interpolation of scattered data by radial functions*, Topics in Multivariate Approximation (C. K. Chui, L. L. Schumaker, and F. Utreras, eds.), Academic Press (New York), 1987, pp. 47–61.

[Dyn89] ______, *Interpolation and approximation by radial and related functions*, Approximation Theory VI (C. Chui, L. Schumaker, and J. Ward, eds.), Academic Press (New York), 1989, pp. 211–234.

References VIII

[Fas07] G. E. Fasshauer, *Meshfree Approximation Methods with* MATLAB, Interdisciplinary Mathematical Sciences, vol. 6, World Scientific Publishing Co., Singapore, 2007.

[Fas11] ______, *Positive definite kernels: past, present and future*, Dolomites Research Notes on Approximation **4** (2011), 21–63.

[FGS98] W. Freeden, T. Gervens, and M. Schreiner, *Constructive Approximation on the Sphere: With Applications to Geomathematics*, Oxford University Press, USA, 1998.

[FM13] J. C. Ferreira and V. A. Menegatto, *Positive definiteness, reproducing kernel Hilbert spaces and beyond*, Ann. Funct. Anal. **4** (2013), no. 1, 64–88.

[For98] Bengt Fornberg, *A Practical Guide to Pseudospectral Methods*, Cambridge Monographs on Applied and Computational Mathematics, Cambridge University Press, October 1998.

References IX

[Fra79] Richard Franke, *A critical comparison of some methods for interpolation of scattered data*, NPS-53-79-003, Naval Postgraduate School, Monterey, CA, 1979.

[Fra82] ______, *Scattered data interpolation: Tests of some method*, Mathematics of Computation **38** (1982), no. 157, 181–200.

[FSK08] A. I. J. Forrester, A. Sobester, and A. J. Keane, *Engineering Design via Surrogate Modelling*, Wiley, Chichester, 2008.

[Gau09] Carl Friedrich Gauss, *Theoria motus corporum coelestium in sectionibus conicis solem ambientium*, Friedrich Perthes and I.H. Besser, Hamburg, 1809.

[Gau57] ______, *Theory of the motion of the heavenly bodies moving about the sun in conic sections : a translation of Gauss's 'Theoria motus.'*, Boston, Little, Brown and Company, 1857.

References X

[GC98] M. A. Golberg and C. S. Chen, *The Method of Fundamental Solutions for Potential, Helmholtz and Diffusion Problems*, Computational Engineering, Computational Mechanics Publications, WIT Press, 1998.

[GS03] Roger G. Ghanem and Pol D. Spanos, *Stochastic Finite Elements: A Spectral Approach*, Courier Dover Publications, August 2003.

[Gu13] Chong Gu, *Smoothing Spline ANOVA Models*, 2nd ed., Springer, New York, NY, 2013.

[Hac89] W. Hackbusch, *Integralgleichungen: Theorie und Numerik*, Teubner B.G., 1989.

[Har71] Rolland L. Hardy, *Multiquadric equations of topography and other irregular surfaces*, Journal of Geophysical Research **76** (1971), no. 8, 1905–1915.

[HD72] R. L. Harder and R. N. Desmarais, *Interpolation using surface splines*, Journal of Aircraft **9** (1972), no. 2, 189–191.

References XI

[Her02] Ralf Herbrich, *Learning Kernel Classifiers: Theory and Algorithms*, MIT Press, 2002.

[Hil04] D. Hilbert, *Grundzüge einer allgemeinen Theorie der linearen Integralgleichungen I*, Göttinger Nachrichten, Math.-Phys. Kl. (1904), 49–91.

[Hil72] Einar Hille, *Introduction to general theory of reproducing kernels*, Rocky Mountain Journal of Mathematics **2** (1972), no. 3, 321–368.

[Hoc73] Harry Hochstadt, *Integral Equations*, Wiley, 1973.

[HTF09] T. Hastie, R. Tibshirani, and J. Friedman, *Elements of Statistical Learning: Data Mining, Inference, and Prediction*, 2nd ed., Springer Series in Statistics, Springer Science+Business Media, Inc., New York, 2009.

[Isk04] A. Iske, *Multiresolution Methods in Scattered Data Modelling*, Lecture Notes in Computational Science and Engineering, vol. 37, Springer, Berlin, 2004.

References XII

[Joa02] T. Joachims, *Learning to Classify Text Using Support Vector Machines: Methods, Theory and Algorithms*, Springer, 2002.

[Kan86] E. J. Kansa, *Application of Hardy's multiquadric interpolation to hydrodynamics*, Proc. 1986 Simul. Conf., 1986, pp. 111–117.

[Kan90a] ______, *Multiquadrics–A scattered data approximation scheme with applications to computational fluid-dynamics–I: Surface approximations and partial derivative estimates*, Computers & Mathematics with Applications **19** (1990), no. 8–9, 127–145.

[Kan90b] E. J. Kansa, *Multiquadrics–A scattered data approximation scheme with applications to computational fluid-dynamics–II solutions to parabolic, hyperbolic and elliptic partial differential equations*, Computers & Mathematics with Applications **19** (1990), no. 8-9, 147–161.

[Kar47] K. Karhunen, *Über lineare Methoden in der Wahrscheinlichkeitsrechnung*, Ann. Acad. Sci. Fennicae. Ser. A. I. Math.-Phys. **37** (1947), 1–79.

References XIII

[Khi34] A. Khintchine, *Korrelationstheorie der stationären stochastischen Prozesse*, Mathematische Annalen **109** (1934), no. 1, 604–615.

[Kit97] P. K. Kitanidis, *Introduction to Geostatistics: Applications in Hydrogeology*, Cambridge University Press, 1997.

[Kre99] Rainer Kress, *Linear Integral Equations*, Springer, 1999.

[Kri51] D. G. Krige, *A statistical approach to some basic mine valuation problems on the Witwatersrand*, J. Chem. Met. & Mining Soc., S. Africa **52** (1951), no. 6, 119–139.

[Liu02] G. R. Liu, *Mesh Free Methods: Moving Beyond the Finite Element Method*, CRC Press, Boca Raton, FL, 2002.

[LL03] G. R. Liu and M. B. Liu, *Smoothed Particle Hydrodynamics: A Meshfree Particle Method*, World Scientific Pub Co Inc, 2003.

[LL07] S. Li and W. K. Liu, *Meshfree Particle Methods*, Springer, 2007.

[Loè77] M. Loève, *Probability Theory I*, Springer, 1977.

References XIV

[LotR] The Lord of the Rings, *Enter "The Prologue" and select "Digital Scanning" from "Orcs"; and "Digital Lands" from "The Battlefield"*, http://www.lordoftherings.net/effects/.

[Mat23] M. Mathias, *Über positive Fourier-Integrale*, Math. Zeit. **16** (1923), 103–125.

[Mat65] G. Matheron, *Les variables régionalisées et leur estimation*, Masson (Paris), 1965.

[Mat86] B. Matérn, *Spatial Variation*, 2nd ed., Lecture Notes in Statistics, vol. 36, Springer-Verlag, Berlin, 1986.

[Mei79a] J. Meinguet, *Basic mathematical aspects of surface spline interpolation*, Numerische Integration (G. Hämmerlin, ed.), Birkhäuser (Basel), 1979, pp. 211–220.

[Mei79b] ______, *An intrinsic approach to multivariate spline interpolation at arbitrary points*, Polynomial and Spline Approximations (N. B. Sahney, ed.), Reidel (Dordrecht), 1979, pp. 163–190.

References XV

[Mei79c] Jean Meinguet, *Multivariate interpolation at arbitrary points made simple*, Zeitschrift für Angewandte Mathematik und Physik (ZAMP) **30** (1979), no. 2, 292–304.

[Mei84] J. Meinguet, *A practical method for obtaining a priori error bounds in pointwise and mean-square approximation problems*, Approximation Theory and Spline Functions (S. P. Singh, J. H. W. Burry, and B. Watson, eds.), Springer, 1984, pp. 97–126.

[Mer09] J. Mercer, *Functions of positive and negative type, and their connection with the theory of integral equations*, Philosophical Transactions of the Royal Society of London. Series A, Containing Papers of a Mathematical or Physical Character **209** (1909), no. 441-458, 415–446.

[Mes62] H. Meschkowski, *Hilbertsche Räume mit Kernfunktion*, Springer, Berlin, 1962.

[MF53] Philip McCord Morse and Herman Feshbach, *Methods of Theoretical Physics*, McGraw-Hill, 1953.

References XVI

[Mic86] C. A. Micchelli, *Interpolation of scattered data: Distance matrices and conditionally positive definite functions*, Constructive Approximation **2** (1986), no. 1, 11–22.

[MN83] W. R. Madych and S. A. Nelson, *Multivariate interpolation: a variational theory*, Unpublished manuscript, 1983.

[Moo16] E. H. Moore, *On properly positive Hermitian matrices*, Bull. Amer. Math. Soc. **23** (1916), no. 2, 59, Paper read at the 23rd Summer Meeting of the AMS.

[Moo35] ______, *General Analysis, Part I*, Memoirs, vol. 1, Amer. Philos. Soc., Philadelphia, 1935.

[NW08] E. Novak and H. Woźniakowski, *Tractability of Multivariate Problems Volume 1: Linear Information*, EMS Tracts in Mathematics, no. 6, European Mathematical Society, 2008.

[Pog66] W. Pogorzelski, *Integral Equations and their Applications*, Pergamon Press, 1966.

References XVII

[Pow87] M. J. D. Powell, *Radial basis functions for multivariable interpolation: a review*, Algorithms for the Approximation of Functions and Data (J. C. Mason and M. G. Cox, eds.), Oxford Univ. Press (Oxford), 1987, pp. 143–167.

[Pow92] ______, *The theory of radial basis functions in 1990*, Advances in Numerical Analysis II: Wavelets, Subdivision, and Radial Basis Functions (W. Light, ed.), Oxford University Press (Oxford), 1992, pp. 105–210.

[Rit00] Klaus Ritter, *Average-Case Analysis of Numerical Problems*, Lecture Notes in Mathematics, vol. 1733, Springer, 2000.

[RS05] James Ramsay and B. W. Silverman, *Functional Data Analysis*, 2nd ed., Springer, New York, 2005.

[RSN55] F. Riesz and B. Sz.-Nagy, *Functional Analysis*, Dover Publications (New York), 1955, republished 1990.

[RW06] C. E. Rasmussen and C. Williams, *Gaussian Processes for Machine Learning*, MIT Press, Cambridge, Massachussetts, 2006.

References XVIII

[Sai88] S. Saitoh, *Theory of Reproducing Kernels and Its Applications*, Longman Higher Education, 1988.

[Sai97] ______, *Integral Transforms, Reproducing Kernels and Their Applications*, Chapman and Hall/CRC, 1997.

[SC08] I. Steinwart and A. Christmann, *Support Vector Machines*, Information Science and Statistics, Springer, New York, 2008.

[Sch07] E. Schmidt, *Zur Theorie der linearen und nichtlinearen Integralgleichungen. I. Teil: Entwicklung willkürlicher Funktionen nach Systemen vorgeschriebener*, Math. Ann. **63** (1907), 433–476.

[Sch08] Erhard Schmidt, *Über die Auflösung linearer Gleichungen mit unendlich vielen Unbekannten*, Rendiconti del Circolo Matematico di Palermo **25** (1908), no. 1, 53–77.

[Sch37] I. J. Schoenberg, *On certain metric spaces arising from Euclidean spaces by a change of metric and their imbedding in Hilbert space*, Ann. of Math. **38** (1937), 787–793.

References XIX

[Sch38a] ______, *Metric spaces and completely monotone functions*, The Annals of Mathematics **39** (1938), no. 4, 811–841.

[Sch38b] ______, *Metric spaces and positive definite functions*, Trans. Amer. Math. Soc. **44** (1938), 522–536.

[Sch95] R. Schaback, *Creating surfaces from scattered data using radial basis functions*, Mathematical Methods for Curves and Surfaces (M. Dæhlen, T. Lyche, and L. L. Schumaker, eds.), Vanderbilt University Press (Nashville), 1995, pp. 477–496.

[Sch99] ______, *Native Hilbert spaces for radial basis functions I*, New Developments in Approximation Theory (M. W. Müller, M. D. Buhmann, D. H. Mache, and M. Felten, eds.), Birkhäuser (Basel), 1999, pp. 255–282.

[Sch00] ______, *A unified theory of radial basis functions: Native Hilbert spaces for radial basis functions II*, Journal of Computational and Applied Mathematics **121** (2000), no. 1-2, 165–177.

References XX

[SGB+02] Johan A. K. Suykens, Tony Van Gestel, J. De Brabanter, Bart De Moor, and Joos Vandewalle, *Least Squares Support Vector Machines*, World Scientific, 2002.

[Smi58] F. Smithies, *Integral Equations*, Cambridge University Press, Cambridge, 1958.

[SS02] B. Schölkopf and A. J. Smola, *Learning with Kernels: Support Vector Machines, Regularization, Optimization, and Beyond*, The MIT Press, 2002.

[SSS13] M. Scheuerer, R. Schaback, and M. Schlather, *Interpolation of spatial data — a stochastic or a deterministic problem?*, European Journal of Applied Mathematics **24** (2013), no. 04, 601–629.

[Sta79] I. Stakgold, *Green's Functions and Boundary Value Problems*, Wiley, 1979.

[STC04] J. Shawe-Taylor and N. Cristianini, *Kernel Methods for Pattern Analysis*, Cambridge University Press, 2004.

References XXI

[Ste76] J. Stewart, *Positive definite functions and generalizations, an historical survey*, Rocky Mountain Journal of Mathematics **6** (1976), no. 3, 409–434.

[Ste99] M. L. Stein, *Interpolation of Spatial Data: Some theory for Kriging*, Springer-Verlag, New York, 1999.

[SW06] R. Schaback and H. Wendland, *Kernel techniques: From machine learning to meshless methods*, Acta Numerica **15** (2006), 543–639.

[Vap98] V. N. Vapnik, *Statistical Learning Theory*, Wiley-Interscience, 1998.

[VL63] V Vapnik and A Lerner, *Pattern recognition using generalized portrait method*, Automation and Remote Control **24** (1963), no. 6, 774—780.

[VT01] Harry L. Van Trees, *Detection, Estimation, and Modulation Theory, Part I*, Wiley-Interscience, 2001.

[Wah79] G. Wahba, *Convergence rate of "thin plate" smoothing splines when the data are noisy (preliminary report)*, Springer Lecture Notes in Math. **757** (1979), 233–245.

References XXII

[Wah81] ______, *Spline interpolation and smoothing on the sphere*, SIAM J. Sci. Statist. Comput. **2** (1981), 5–16.

[Wah82] ______, *Erratum: Spline interpolation and smoothing on the sphere*, SIAM J. Sci. Statist. Comput. **3** (1982), 385–386.

[Wah90] Grace Wahba, *Spline Models for Observational Data*, SIAM, Philadelphia, 1990.

[Wen95] H. Wendland, *Piecewise polynomial, positive definite and compactly supported radial functions of minimal degree*, Advances in Computational Mathematics **4** (1995), no. 1, 389–396.

[Wen05] ______, *Scattered Data Approximation*, Cambridge Monographs on Applied and Computational Mathematics, vol. 17, Cambridge University Press, 2005.

[WW76] J. H. Wells and L. R. Williams, *Embeddings and Extensions in Analysis*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Springer, Berlin, 1976.

References XXIII

[WW80] G. Wahba and J. Wendelberger, *Some new mathematical methods for variational objective analysis using splines and cross validation*, Monthly Weather Review **108** (1980), 1122–1143.

[Zar07] Stanislaw Zaremba, *L'équation biharmonique et une class remarquable de functions fondamentales harmoniques*, Bulletin International de l'Acadmie des Sciences de Cracovie (1907), 147–196.

[Zar09] ______, *Sur le calcul numérique des fonctions demandées dans le probléme de dirichlet et le probléme hydrodynamique*, Bulletin International de l'Acadmie des Sciences de Cracovie (1909), 125–195.