Chapter 2

Positive Definite and Completely Monotone Functions

Below we will first summarize facts about positive definite functions, and closely related completely and multiply monotone functions. Most of these facts are integral characterizations and were established in the 1930s by Bochner and Schoenberg. In the second part of this chapter we will mention the more recent extensions to strictly positive definite and strictly completely/multiply monotone functions. Integral characterizations are an essential ingredient in the theoretical analysis of radial basis functions.

2.1 A Brief Summary of Integral Transforms

Before we get into the details of the integral representations we summarize some formulas for various integral transforms to be used later.

The Fourier transform conventions we will adhere to are laid out in

Definition 2.1.1 *The* Fourier transform *of* $f \in L_1(\mathbb{R}^s)$ *is given by*

$$\hat{f}(\boldsymbol{\omega}) = \frac{1}{\sqrt{(2\pi)^s}} \int_{\mathbb{R}^s} f(\boldsymbol{x}) e^{-i\boldsymbol{\omega}\cdot\boldsymbol{x}} d\boldsymbol{x}, \qquad \boldsymbol{\omega} \in \mathbb{R}^s, \tag{2.1}$$

and its inverse Fourier transform *is given by*

$$\check{f}(\boldsymbol{x}) = \frac{1}{\sqrt{(2\pi)^s}} \int_{\mathbb{R}^s} f(\boldsymbol{\omega}) e^{i\boldsymbol{x}\cdot\boldsymbol{\omega}} d\boldsymbol{\omega}, \qquad \boldsymbol{x} \in \mathbb{R}^s.$$

Remark: This definition of the Fourier transform can be found in Rudin [537]. Another, just as common, definition uses

$$\hat{f}(\boldsymbol{\omega}) = \int_{\mathbb{R}^s} f(\boldsymbol{x}) e^{-2\pi i\boldsymbol{\omega}\cdot\boldsymbol{x}} d\boldsymbol{x}, \tag{2.2}$$

and can be found in Stein and Weiss [589]. The form we use can also be found in Wendland's book [634], whereas (2.2) is used in the book by Cheney and Light [132].

Similarly, we can define the Fourier transform of a finite (signed) measure μ on $\mathbb{R}^s$ by

$$\hat{\mu}(\boldsymbol{\omega}) = \frac{1}{\sqrt{(2\pi)^s}} \int_{\mathbb{R}^s} e^{-i\boldsymbol{\omega}\cdot\boldsymbol{x}} d\mu(\boldsymbol{x}), \qquad \boldsymbol{\omega} \in \mathbb{R}^s .$$

Since we will be interested in positive definite radial functions, we note that the Fourier transform of a radial function is again radial. Indeed,

Theorem 2.1.2 *Let $\Phi \in L_1(\mathbb{R}^s)$ be continuous and radial, i.e., $\Phi(\boldsymbol{x}) = \varphi(\|\boldsymbol{x}\|)$. Then its Fourier transform $\hat{\Phi}$ is also radial, i.e., $\hat{\Phi}(\boldsymbol{\omega}) = \mathcal{F}_s\varphi(\|\boldsymbol{\omega}\|)$ with*

$$\mathcal{F}_s\varphi(r) = \frac{1}{\sqrt{r^{s-2}}} \int_0^\infty \varphi(t) t^{\frac{s}{2}} J_{(s-2)/2}(rt) dt,$$

where $J_{(s-2)/2}$ is the classical Bessel function of the first kind of order $(s-2)/2$.

Remark: The integral transform appearing in Theorem 2.1.2 is also referred to as a *B*essel transform.

A third integral transform to play an important role in the following is the *Laplace transform*. We have

Definition 2.1.3 *The* Laplace transform *of a piecewise continuous function f that satisfies $|f(t)| \le Me^{at}$ for some constants a and M is given by*

$$\mathcal{L}f(s) = \int_0^\infty f(t) e^{-st} dt, \qquad s > a.$$

Similarly, the Laplace transform of a Borel measure μ on $[0, \infty)$ is given by

$$\mathcal{L}\mu(s) = \int_0^\infty e^{-st} d\mu(t).$$

The Laplace transform is continuous at the origin if and only if μ is finite.

2.2 Bochner's Theorem

One of the most celebrated results on positive definite functions is their characterization in terms of Fourier transforms established by Bochner in 1932 (for $s = 1$) and 1933 (for general s).

Theorem 2.2.1 *(Bochner's Theorem) A (complex-valued) function $\Phi \in C(\mathbb{R}^s)$ is positive definite on $\mathbb{R}^s$ if and only if it is the Fourier transform of a finite non-negative Borel measure μ on $\mathbb{R}^s$, i.e.,*

$$\Phi(\boldsymbol{x}) = \hat{\mu}(\boldsymbol{x}) = \frac{1}{\sqrt{(2\pi)^s}} \int_{\mathbb{R}^s} e^{-i\boldsymbol{x}\cdot\boldsymbol{y}} d\mu(\boldsymbol{y}), \qquad \boldsymbol{x} \in \mathbb{R}^s .$$

Proof: There are many proofs of this theorem. Bochner's original proof can be found in [58], p. 407. Other proofs can be found, e.g., in the books by Cuppens ([147], p. 41) or Gelfand and Vilenkin ([250], p. 155). A nice proof using the Riesz Representation Theorem to interpret the Borel measure as a distribution, and then taking advantage of distributional Fourier transforms can be found in the book by Wendland [634].

We will prove only the one (easy) direction which is important for the application to scattered data interpolation. We assume Φ is the Fourier transform of a finite non-negative Borel measure and show Φ is positive definite. Thus,

$$\begin{aligned}\sum_{j=1}^{N}\sum_{k=1}^{N} c_j\overline{c_k}\Phi(\boldsymbol{x}_j-\boldsymbol{x}_k) &= \frac{1}{\sqrt{(2\pi)^s}}\sum_{j=1}^{N}\sum_{k=1}^{N}\left[c_j\overline{c_k}\int_{\mathbb{R}^s} e^{-i(\boldsymbol{x}_j-\boldsymbol{x}_k)\cdot\boldsymbol{y}}d\mu(\boldsymbol{y})\right] \\ &= \frac{1}{\sqrt{(2\pi)^s}}\int_{\mathbb{R}^s}\left[\sum_{j=1}^{N} c_j e^{-i\boldsymbol{x}_j\cdot\boldsymbol{y}}\sum_{k=1}^{N}\overline{c_k}e^{i\boldsymbol{x}_k\cdot\boldsymbol{y}}\right]d\mu(\boldsymbol{y}) \\ &= \frac{1}{\sqrt{(2\pi)^s}}\int_{\mathbb{R}^s}\left|\sum_{j=1}^{N} c_j e^{-i\boldsymbol{x}_j\cdot\boldsymbol{y}}\right|^2 d\mu(\boldsymbol{y}) \geq 0.\end{aligned}$$

The last inequality holds because of the conditions imposed on the measure μ. □

2.3 Strictly Positive Definite Functions

In order to accomplish our goal of guaranteeing a well-posed interpolation problem, we have to extend (if possible) Bochner's characterization to *strictly* positive definite functions.

We begin with a sufficient condition for a function to be strictly positive definite on $\mathbb{R}^s$.

For this result we require the notion of the *carrier* of a (non-negative) Borel measure defined on some topological space X. This set is given by

$$X \setminus \bigcup\{O : \ O \text{ is open and } \mu(O) = 0\}.$$

Theorem 2.3.1 *Let μ be a non-negative finite Borel measure on $\mathbb{R}^s$ whose carrier is not a set of Lebesgue measure zero. Then the Fourier transform of μ is strictly positive definite on $\mathbb{R}^s$.*

Proof: As in the proof of Bochner's Theorem we have

$$\begin{aligned}\sum_{j=1}^{N}\sum_{k=1}^{N} c_j\overline{c_k}\hat{\mu}(\boldsymbol{x}_j-\boldsymbol{x}_k) &= \frac{1}{\sqrt{(2\pi)^s}}\sum_{j=1}^{N}\sum_{k=1}^{N}c_j\overline{c_k}\left[\int_{\mathbb{R}^s} e^{-i(\boldsymbol{x}_j-\boldsymbol{x}_k)\cdot\boldsymbol{y}}d\mu(\boldsymbol{y})\right] \\ &= \frac{1}{\sqrt{(2\pi)^s}}\int_{\mathbb{R}^s}\left[\sum_{j=1}^{N} c_j e^{-i\boldsymbol{x}_j\cdot\boldsymbol{y}}\sum_{k=1}^{N}\overline{c_k}e^{i\boldsymbol{x}_k\cdot\boldsymbol{y}}\right]d\mu(\boldsymbol{y})\end{aligned}$$

$$= \frac{1}{\sqrt{(2\pi)^s}} \int_{\mathbb{R}^s} \left| \sum_{j=1}^{N} c_j e^{-i\boldsymbol{x}_j \cdot \boldsymbol{y}} \right|^2 d\mu(\boldsymbol{y}) \geq 0.$$

Now let

$$g(\boldsymbol{y}) = \sum_{j=1}^{N} c_j e^{-i\boldsymbol{x}_j \cdot \boldsymbol{y}},$$

and assume that the points $\boldsymbol{x}_j$ are all distinct and $\boldsymbol{c} \neq \boldsymbol{0}$. In this case the functions $\boldsymbol{y} \mapsto e^{-i\boldsymbol{x}_j \cdot \boldsymbol{y}}$ are linearly independent, and thus the zero set of g, i.e., $\{\boldsymbol{y} \in \mathbb{R}^s : g(\boldsymbol{y}) = 0\}$ has Lebesgue measure zero. Therefore, the only remaining way to make the above inequality an equality is if the carrier of μ is contained in the zero set of g, i.e., has Lebesgue measure zero. □

The following corollary gives us a way to *construct* strictly positive definite functions.

Corollary 2.3.2 *Let f be a continuous non-negative function in $L_1(\mathbb{R}^s)$ which is not identically zero. Then the Fourier transform of f is strictly positive definite on $\mathbb{R}^s$.*

Proof: We use the measure μ defined for any Borel set B by

$$\mu(B) = \int_B f(\boldsymbol{x}) d\boldsymbol{x}.$$

Then the carrier of μ is equal to the closed support of f. However, since f is non-negative and not identically equal to zero, its support has positive Lebesgue measure, and hence the Fourier transform of f is strictly positive definite by the preceding theorem. □

Remark: Work toward an analog of Bochner's Theorem, i.e., an integral characterization for functions which are strictly positive definite on $\mathbb{R}^s$, is given in [112] for $s = 1$.

Example: The *Gaussian*

$$\Phi(\boldsymbol{x}) = e^{-\alpha\|\boldsymbol{x}\|^2}, \quad \alpha > 0, \tag{2.3}$$

is strictly positive definite on $\mathbb{R}^s$ for any s. This is essentially due to the fact that the Fourier transform of a Gaussian is again a Gaussian. In particular, for $\alpha = \frac{1}{2}$ we have $\hat{\Phi} = \Phi$ which can be verified by direct calculation. The general statement follows from the properties of the Fourier transform (complete details are given in the book by Wendland on pp. 50 and 69). An easier argument (using completely monotone functions) will become available later.

Remark: Since Gaussians play a central role in statistics, this is a good place to mention that positive definite functions are – up to a normalization $\Phi(0) = 1$ – identical with characteristic functions of distribution functions in statistics.

Finally, a criterion to check whether a given function is strictly positive definite is given in [634].

Theorem 2.3.3 *Let Φ be a continuous function in $L_1(\mathbb{R}^s)$. Φ is strictly positive definite if and only if Φ is bounded and its Fourier transform is non-negative and not identically equal to zero.*

Remark: The proof of Theorem 2.3.3 shows that – if $\Phi \not\equiv 0$ (which implies that $\hat{\Phi} \not\equiv 0$) – we need to ensure only that $\hat{\Phi}$ be non-negative in order for Φ to be strictly positive definite.

Example: Theorem 2.3.3 can be used to show that the so-called *inverse multiquadrics*

$$\Phi(\boldsymbol{x}) = \left(\|\boldsymbol{x}\|^2 + \alpha^2\right)^{-\beta}, \quad \alpha > 0,\ \beta > \frac{s}{2}, \tag{2.4}$$

are strictly positive definite on $\mathbb{R}^s$ (complete details are given in [634]). By using another argument based on completely monotone functions we will be able to show that in fact we need to require only $\beta > 0$, and therefore the inverse multiquadrics are strictly positive definite on any $\mathbb{R}^s$.

2.4 Positive Definite Radial Functions

We now turn our attention to positive definite radial functions. Theorem 2.1.2 can be used to prove the following characterization due to Schoenberg (see [569], p.816).

Theorem 2.4.1 *A continuous function $\varphi : [0,\infty) \to \mathbb{R}$ is positive definite and radial on $\mathbb{R}^s$ if and only if it is the Bessel transform of a finite non-negative Borel measure μ on $[0,\infty)$, i.e.,*

$$\varphi(r) = \int_0^\infty \Omega_s(rt)\,d\mu(t),$$

where

$$\Omega_s(r) = \begin{cases} \cos r & \text{for } s = 1, \\ \Gamma\left(\frac{s}{2}\right)\left(\frac{2}{r}\right)^{(s-2)/2} J_{(s-2)/2}(r) & \text{for } s \geq 2, \end{cases}$$

and $J_{(s-2)/2}$ is the classical Bessel function of the first kind of order $(s-2)/2$.

Since any function which is positive definite and radial on $\mathbb{R}^{s_1}$ is also positive definite and radial on $\mathbb{R}^{s_2}$ as long as $s_2 \leq s_1$, those functions which are positive definite and radial on $\mathbb{R}^s$ *for all* s are of particular interest. This latter class of functions was also characterized by Schoenberg ([569], pp. 817–821.). We saw above that the Gaussians and inverse multiquadrics provide examples of such functions.

Theorem 2.4.2 *A continuous function $\varphi : [0,\infty) \to \mathbb{R}$ is positive definite and radial on $\mathbb{R}^s$ for all s if and only if it is of the form*

$$\varphi(r) = \int_0^\infty e^{-r^2t^2}\,d\mu(t),$$

where μ is a finite non-negative Borel measure on $[0,\infty)$.

Remark: Schoenberg referred to the functions which are positive definite and radial on $\mathbb{R}^s$ for all s as positive definite radial functions on ℓ_2.

We end this section with examples of functions that are strictly positive definite and radial on $\mathbb{R}^s$ with restrictions on the space dimension s. Moreover, the following functions differ from the previous ones in that they have *compact support*.

Examples:

1. The *truncated power function*

$$\varphi_\ell(r) = (1-r)_+^\ell \tag{2.5}$$

is strictly positive definite and radial on $\mathbb{R}^s$ provided ℓ satisfies $\ell \geq \lfloor \frac{s}{2} \rfloor + 1$. For details see [634]. Here we have used the cutoff function $(\cdot)_+$ which is defined by

$$(x)_+ = \begin{cases} x, & \text{for } x \geq 0, \\ 0, & \text{for } x < 0. \end{cases}$$

2. Let $f \in C[0,\infty)$ be non-negative and not identically equal to zero, and define the function φ by

$$\varphi(r) = \int_0^\infty (1-rt)_+^{k-1} f(t)dt. \tag{2.6}$$

Then φ is strictly positive definite and radial on $\mathbb{R}^s$ provided $k \geq \lfloor \frac{s}{2} \rfloor + 2$. This can been verified by considering the quadratic form

$$\sum_{j=1}^N \sum_{k=1}^N c_j c_k \varphi(\|\boldsymbol{x}_j - \boldsymbol{x}_k\|) = \int_0^\infty \sum_{j=1}^N \sum_{k=1}^N c_j c_k \varphi_{k-1}(t\|\boldsymbol{x}_j - \boldsymbol{x}_k\|) f(t)dt$$

which is non-negative since φ_{k-1} is strictly positive definite by the first example, and f is non-negative. Since f is also assumed to be not identically equal to zero, the only way for the quadratic form to equal zero is if $\boldsymbol{c} = \boldsymbol{0}$.
Note that (2.6) amounts to another integral transform of f with the compactly supported truncated power function as integration kernel. We will take another look at these functions in the context of multiply monotone functions below.

The Schoenberg characterization of positive definite radial functions on $\mathbb{R}^s$ for all s implies that we have a finite non-negative Borel measure μ on $[0,\infty)$ such that

$$\varphi(r) = \int_0^\infty e^{-r^2t^2} d\mu(t).$$

If we want to find a zero r_0 of φ then we have

$$\varphi(r_0) = \int_0^\infty e^{-r_0^2t^2} d\mu(t) = 0.$$

Since the exponential function is positive and the measure is non-negative, it follows that μ must be the zero measure. However, then ϕ is identically equal to zero. Therefore, a non-trivial function φ that is positive definite and radial on $\mathbb{R}^s$ for all s can have no zeros. This implies in particular that *there are no compactly supported univariate continuous functions that are positive definite and radial on* $\mathbb{R}^s$ *for all* s.

2.5 Completely Monotone Functions

We now introduce a class of functions which is very closely related to positive definite radial functions and leads to a simple characterization of such functions.

Definition 2.5.1 *A function* $\varphi : [0, \infty) \to \mathbb{R}$ *which is in* $C[0, \infty) \cap C^\infty(0, \infty)$ *and which satisfies*

$$(-1)^\ell \varphi^{(\ell)}(r) \geq 0, \qquad r > 0, \; \ell = 0, 1, 2, \ldots,$$

is called completely monotone on $[0, \infty)$.

Example: Some examples of completely monotone functions are

1. $\varphi(r) = \alpha, \qquad \alpha \geq 0,$
2. $\varphi(r) = e^{-\alpha r}, \qquad \alpha \geq 0,$
3. $\varphi(r) = \dfrac{\alpha}{r^{1-\alpha}}, \quad \alpha \leq 1,$
4. $\varphi(r) = \dfrac{1}{(r + \alpha^2)^\beta}, \qquad \alpha > 0, \, \beta \geq 0.$

The following theorem gives an integral characterization of completely monotone functions.

Theorem 2.5.2 *(Hausdorff-Bernstein-Widder Theorem) A function* $\varphi : [0, \infty) \to \mathbb{R}$ *is completely monotone on* $[0, \infty)$ *if and only if it is the Laplace transform of a finite non-negative Borel measure* μ *on* $[0, \infty)$*, i.e.,* φ *is of the form*

$$\varphi(r) = \mathcal{L}\mu(r) = \int_0^\infty e^{-rt} d\mu(t).$$

Remark: Widder's proof of this theorem can be found in [644], p. 160, where he reduces the proof of this theorem to another theorem by Hausdorff on completely monotone sequences. A detailed proof can also be found in the books by Cheney and Light [132] and Wendland [634]. □

Remark: Some properties of completely monotone functions are:

1. A non-negative finite linear combination of completely monotone functions is completely monotone.
2. The product of two completely monotone functions is completely monotone.

The following connection between positive definite radial and completely monotone functions was first pointed out by Schoenberg in 1938.

Theorem 2.5.3 *A function* φ *is completely monotone on* $[0, \infty)$ *if and only if* $\Phi = \varphi(\|\cdot\|^2)$ *is positive definite and radial on* $\mathbb{R}^s$ *for all* s.

Remark: Note that the function Φ is now defined via the *square* of the norm. This is different from our earlier definition of radial functions (see Definition 1.2.8).

Proof: One possibility is to use a change of variables to combine Schoenberg's characterization of functions that are positive definite and radial on any $\mathbb{R}^s$, Theorem 2.4.2, with the Hausdorff-Bernstein-Widder characterization of completely monotone functions. To get more insight we present an alternative proof of the claim that the completely monotone function φ gives rise to a Φ that is positive definite and radial on any $\mathbb{R}^s$. Details for the other direction can be found, e.g., in [634].

The Hausdorff-Bernstein-Widder Theorem implies that we can write φ as

$$\varphi(r) = \int_0^\infty e^{-rt} d\mu(t)$$

with a finite non-negative Borel measure μ. Therefore, $\Phi(\boldsymbol{x}) = \varphi(\|\boldsymbol{x}\|^2)$ has the representation

$$\Phi(\boldsymbol{x}) = \int_0^\infty e^{-\|\boldsymbol{x}\|^2 t} d\mu(t).$$

To see that this function is positive definite on any $\mathbb{R}^s$ we consider the quadratic form

$$\sum_{j=1}^N \sum_{k=1}^N c_j c_k \Phi(\boldsymbol{x}_j - \boldsymbol{x}_k) = \int_0^\infty \sum_{j=1}^N \sum_{k=1}^N c_j c_k e^{-t\|\boldsymbol{x}_j - \boldsymbol{x}_k\|^2} d\mu(t).$$

Since we saw earlier that the Gaussians are strictly positive definite and radial on any $\mathbb{R}^s$ it follows that the quadratic form is non-negative. $\square$.

We can see from the previous proof that if the measure μ is not concentrated in the origin, then Φ is even strictly positive definite and radial on any $\mathbb{R}^s$. This condition on the measure is equivalent with ϕ not being constant. With this additional restriction on φ we can apply the notion of a completely monotone function to the scattered data interpolation problem. The following *interpolation theorem* was already proved by Schoenberg in 1938 ([569], p. 823).

Theorem 2.5.4 *If the function $\varphi : [0, \infty) \to \mathbb{R}$ is completely monotone but not constant, then $\varphi(\|\cdot\|^2)$ is strictly positive definite and radial on $\mathbb{R}^s$ for any s.*

Proof: Very similar to earlier proofs. We obtain strictness by using the measure condition, i.e., the property that φ is not constant. $\square$

Example: The following functions are completely monotone and not constant. Therefore, they lead to strictly positive definite radial functions on any $\mathbb{R}^s$, and can be used as basic functions to generate bases for (1.5).

1. The functions $\varphi(r) = (r + \alpha^2)^{-\beta}$, $\alpha,\ \beta > 0$, are completely monotone and not constant since

   $$(-1)^\ell \varphi^{(\ell)}(r) = (-1)^{2\ell} \beta(\beta+1)\cdots(\beta+\ell-1)(r+\alpha^2)^{-\beta-\ell} \geq 0, \qquad \ell = 0, 1, 2, \ldots.$$

   Thus

   $$\mathcal{P}f(\boldsymbol{x}) = \sum_{j=1}^N c_j \left(\|\boldsymbol{x} - \boldsymbol{x}_j\|^2 + \alpha^2\right)^{-\beta}, \qquad \boldsymbol{x} \in \mathbb{R}^s,$$

can be used to solve the scattered data interpolation problem. The associated interpolation matrix is guaranteed to be positive definite. These functions are the inverse multiquadrics encountered earlier. Now it is clear that the earlier restriction $\beta > \frac{s}{2}$ is no longer required.

2. The functions $\varphi(r) = e^{-\alpha r}$, $\alpha > 0$, are completely monotone and not constant since
$$(-1)^\ell \varphi^{(\ell)}(r) = \alpha^\ell e^{-\alpha r} \geq 0, \qquad \ell = 0, 1, 2, \ldots.$$
Thus
$$\mathcal{P}f(\boldsymbol{x}) = \sum_{j=1}^{N} c_j e^{-\alpha\|\boldsymbol{x}-\boldsymbol{x}_j\|^2}, \qquad \boldsymbol{x} \in \mathbb{R}^s,$$
corresponds to interpolation with Gaussian radial basis functions.

Remarks:

1. A complete characterization of strictly positive definite functions in terms of completely monotone functions, i.e., the converse of Schoenberg's Theorem 2.5.4, is given in Wendland's book [634].

2. We just saw (for the second time) that Gaussians are strictly positive definite and radial on all $\mathbb{R}^s$. Also, Theorem 1.2.6 stating basic properties of positive definite functions shows us that (positive) linear combinations of (strictly) positive definite functions are (strictly) positive definite. The Schoenberg characterization of functions that are (strictly) positive definite and radial on any $\mathbb{R}^s$, Theorem 2.4.2, shows that *all* such functions are given as linear combinations of Gaussians.

2.6 Multiply Monotone Functions

As we will see below, another interesting class of functions is given by

Definition 2.6.1 *A function $\varphi : (0, \infty) \to \mathbb{R}$ which is in $C^{k-2}(0, \infty)$ ($k \geq 2$), and for which $(-1)^l \varphi^{(l)}(r)$ is non-negative, non-increasing, and convex for $l = 0, 1, 2, \ldots, k-2$ is called* k-times monotone *on $(0, \infty)$. In case $k = 1$ we only require $\varphi \in C(0, \infty)$ to be non-negative and non-increasing.*

Since convexity of φ means that $\varphi(\frac{r_1+r_2}{2}) \leq \frac{\varphi(r_1)+\varphi(r_2)}{2}$, or simply $\varphi''(r) \geq 0$ if φ'' exists, a multiply monotone function is in essence just a completely monotone function whose monotonicity is "truncated".

Examples:

1. The truncated power function
$$\varphi_\ell(r) = (1 - r)_+^\ell$$
is ℓ-times monotone for any ℓ.

2. If we define the integral operator I by

$$(If)(r) = \int_r^\infty f(s)ds, \qquad r \geq 0,$$

and f is ℓ-times monotone, then If is $\ell+1$-times monotone.

Remark: The operator I plays an important role in the construction of compactly supported radial basis functions (more later).

An integral representation for the class of multiply monotone functions was given by Williamson [645] but apparently already known to Schoenberg in 1940.

Theorem 2.6.2 *A continuous function $\varphi : (0,\infty) \to \mathbb{R}$ is k-times monotone on $(0,\infty)$ if and only if it is of the form*

$$\varphi(r) = \int_0^\infty (1-rt)_+^{k-1} d\mu(t), \tag{2.7}$$

where μ is a non-negative Borel measure on $(0,\infty)$.

Proof: To see that a function of the form 2.7 is indeed multiply monotone we just need to differentiate under the integral (since derivatives up to order $k-2$ of $(1-rt)_+^{k-1}$ are continuous and bounded). The other direction can be found in [645]. □

For $k \to \infty$ this characterization is equivalent to the Hausdorff-Bernstein-Widder characterization Theorem 2.5.2. Williamson also shows that the product of multiply monotone functions is multiply monotone.

We can see from the Examples 1 and 2 of Section 2.4 that certain multiply monotone functions give rise to positive definite radial functions. Such a connection was first noted by Askey [10] using the truncated power functions of Example 1 in Section 2.4 (and in the one-dimensional case by Pólya). In the RBF literature the following theorem was stated in Micchelli's paper [456], and then refined by Buhmann [79]:

Theorem 2.6.3 *Let $k = \lfloor s/2 \rfloor + 2$ be a positive integer. If $\varphi : (0,\infty) \to \mathbb{R}$ is k-times monotone on $(0,\infty)$ but not constant, then $\varphi(\|\cdot\|^2)$ is strictly positive definite and radial on $\mathbb{R}^s$.*

Remark: Most versions of Theorem 2.6.3 contain misprints in the literature. The correct form should be as stated above.

Wu [655] states

Theorem 2.6.4 *A function $\varphi : [0,\infty) \to \mathbb{R}$ is strictly positive definite and radial on $\mathbb{R}^s$ for $s \leq 2k+1$ if and only if $\varphi(r)r^{2k} \in L_1(0,\infty) \cap C[0,\infty)$ and $\mathcal{F}_1\varphi(\|\cdot\|^2/2)$ is k-times monotone.*

Using this theorem he starts with the truncated power function $f_k(r) = (1-2r)_+^k$ (which is k-times monotone) and obtains functions of the form

$$\varphi_k(r) = \mathcal{F}_1 f_k(\cdot^2/2)(r) = \sqrt{\frac{2}{\pi}} \int_0^\infty (1-t^2)_+^k \cos(rt)dt$$

which are strictly positive definite and radial in $\mathbb{R}^{2k+1}$ and for which $\mathcal{F}_1\varphi_k(\|\cdot\|^2/2)$ is multiply monotone, i.e.,

$$(-1)^\ell \frac{d^\ell}{dr^\ell}\left(\mathcal{F}_1\varphi_k(\cdot^2/2)\right)(r) = \frac{2^{k-\ell}k!}{(k-\ell)!}f_{k-\ell}(r) \geq 0, \qquad 0 \leq \ell \leq k.$$

The special case $k=0$ yields

$$\varphi_0(r) = \sqrt{\frac{2}{\pi}}\mathrm{sinc}(r),$$

and the family of functions $\{\varphi_k\}$ generalizes the sinc function used in sampling theory. These functions have a *compactly supported Fourier transform.*

However, if we start with the truncated power function $\varphi(r) = (1-2r)_+^{k+1}$, which we know to be strictly positive definite and radial in $\mathbb{R}^s$ for $s \leq 2k+1$, then (as above)

$$\mathcal{F}_1\varphi(\cdot^2/2)(r) = \sqrt{\frac{2}{\pi}}\int_0^\infty (1-t^2)_+^{k+1}\cos(rt)dt.$$

In fact, Wu gives the explicit formula

$$\sqrt{\frac{2}{\pi}}\int_0^\infty (1-t^2)_+^{k+1}\cos(rt)dt = 2^{k+1}\Gamma(k+2)r^{-k-3/2}J_{k+3/2}(r).$$

Clearly, these functions are *not* monotone. This seems to present a contradict the statement of Theorem 2.6.4.

Remark: As a final remark in this chapter we mention we are a long way from having a complete characterization of (radial) functions for which the scattered data interpolation problem has a unique solution. As we will see later, such a characterization will involve also functions which are not strictly positive definite. For example, we will mention a result of Micchelli's according to which *conditionally* positive definite functions of order one can be used for the scattered data interpolation problem. Furthermore, all of the results dealt with so far involve radial basis functions which are centered at the given data sites. There are only limited results addressing the situation in which the centers for the basis functions and the data sites may differ.