5.5.1 (b) Show $p(x)\left(u(x)v'(x)-v(x)u'(x)\right)\Big|_a^b = 0$

For the SL problem with BCs $\varphi'(0)=0$, $\varphi(L)=0$.

Since u, v also satisfy these BCs we have

$$p(L)\big(\underbrace{u(L)}_{=0}v'(L)-\underbrace{v(L)}_{=0}u'(L)\big)-p(0)\big(u(0)\underbrace{v'(0)}_{=0}-v(0)\underbrace{u'(0)}_{=0}\big)=0.$$

(d) Use BCs $\varphi(a)=\varphi(b)$, $p(a)\varphi'(a)=p(b)\varphi'(b)$

and u, v satisfying same BCs.

$$p(b)\big[\underbrace{u(b)}_{=u(a)}v'(b)-\underbrace{v(b)}_{=v(a)}u'(b)\big]-p(a)\big[u(a)v'(a)-v(a)u'(a)\big]$$

from BCs

So, reordering we get

$$u(a)\big[\underbrace{p(b)v'(b)-p(a)v'(a)}_{=0 \text{ from BCs}}\big]-v(a)\big[\underbrace{p(b)u'(b)-p(a)u'(a)}_{=0 \text{ from BCs}}\big]=0$$

(f) Use BCs $\varphi(L)=0$, and—when $p(0)\neq 0$— $\begin{cases}\varphi(0) \text{ bounded}\\ \lim\limits_{x\to 0} p(x)\varphi'(x)=0\end{cases}$

Again, u, v satisfy the same BCs.

So $p(b)\big[\underbrace{u(L)}_{=0}v'(L)-\underbrace{v(L)}_{=0}u'(L)\big]-p(0)\big[u(0)v'(0)-v(0)u'(0)\big]$

since this could be 0 (fine) or not we use limit here.

$\lim\limits_{x\to 0} p(x)\big[u(x)v'(x)-v(x)u'(x)\big]$, break into parts:

$$\lim_{x\to 0} p(x)u(x)v'(x) = \underbrace{\lim_{x\to 0} p(x)v'(x)}_{=0}\ \underbrace{\lim_{x\to 0} u(x)}_{\text{bounded}} = 0$$

analogously, $\lim\limits_{x\to 0} p(x)v(x)u'(x) = 0$.

5.5.2 Consider the SL eqn

$$\frac{d}{dx}\left[p(x)\varphi'(x)\right] + q(x)\varphi(x) + \lambda\sigma(x)\varphi(x) = 0$$

with BCs $\varphi(1) = 0$

and $\varphi(2) - 2\varphi'(2) = 0 \iff \varphi(2) = 2\varphi'(2)$

From (5.5.12) we know

$$(\lambda_m - \lambda_n)\int_1^2 \varphi_n(x)\varphi_m(x)\sigma(x)\,dx = p(x)\left[\varphi_m(x)\varphi_n'(x) - \varphi_n(x)\varphi_m'(x)\right]_1^2$$

$$= p(2)\left[\underbrace{\varphi_m(2)}_{=2\varphi_m'(2)}\varphi_n'(2) - \underbrace{\varphi_n(2)}_{=2\varphi_n'(2)}\varphi_m'(2)\right] - p(1)\left[\underbrace{\varphi_m(1)}_{=0}\varphi_n'(1) - \underbrace{\varphi_n(1)}_{=0}\varphi_m'(1)\right]$$

$$= 0.$$

The weight function is σ.

5.5.5 Consider $\mathcal{L} = \frac{d^2}{dx^2} + 6\frac{d}{dx} + 9$

(a) $$\mathcal{L}(e^{rx}) = \frac{d^2}{dx}e^{rx} + 6\frac{d}{dx}e^{rx} + 9e^{rx}$$

$$= r^2e^{rx} + 6re^{rx} + 9e^{rx} = e^{rx}(r+3)^2$$

(b) Solve $\mathcal{L}(y) = 0$.

Using (a) and the Ansatz $y = e^{rx}$ we get

$$\mathcal{L}(y) = \mathcal{L}(e^{rx}) = \underbrace{e^{rx}}_{\neq 0}(r+3)^2 = 0 \implies r = -3$$

Thus, one solution is $y = c_1e^{-3x}$

(c) Assume $z = z(x,r)$ and show $\frac{\partial}{\partial r}\mathcal{L}(z) = \mathcal{L}\left(\frac{\partial z}{\partial r}\right)$

$$\frac{\partial}{\partial r}\mathcal{L}(z) = \frac{\partial}{\partial r}\frac{\partial^2}{\partial x^2}z(x,r) + \frac{\partial}{\partial r}6\left(\frac{\partial}{\partial x}z(x,r)\right) + \frac{\partial}{\partial r}9z(x,r)$$

$$= \frac{\partial^2}{\partial x^2}\frac{\partial}{\partial r}z(x,r) + 6\frac{\partial}{\partial x}\frac{\partial}{\partial r}z(x,r) + 9\frac{\partial}{\partial r}z(x,r)$$

$$= \mathcal{L}\left(\frac{\partial z}{\partial r}\right)$$

(d) Find $\mathcal{L}\left(\frac{\partial z}{\partial r}\right)$ for $z = e^{rx}$

$$\mathcal{L}\left(\frac{\partial z}{\partial r}\right) \overset{(c)}{=} \frac{\partial}{\partial r}\mathcal{L}(z) \overset{(a)}{=} \frac{\partial}{\partial r}\left(e^{rx}(r+3)^2\right)$$

$$= xe^{rx}(r+3)^2 + 2(r+3)e^{rx}$$

$$= e^{rx}(r+3)\left[x(r+3)+2\right]$$

(e) Solve for another solution of $\mathcal{L}(y) = 0$.

Now use the Ansatz $y = xe^{rx}$. Then

$$\mathcal{L}(xe^{rx}) = \mathcal{L}\left(\frac{\partial}{\partial r}e^{rx}\right) \overset{(c)}{=} \frac{\partial}{\partial r}\mathcal{L}(e^{rx}) \overset{(d)}{=} e^{rx}(r+3)\left[x(r+3)+2\right]$$

$$\overset{e^{rx}\neq 0}{\Longrightarrow} (r+3)\left[x(r+3)+2\right] = 0$$

So $r = -3$ or $x(r+3)+2 = 0$ or $r = \frac{-2}{x} - 3$

Therefore $y = c_2 x e^{-3rx}$

5.5.7 Let $\mathcal{L} = \frac{d}{dx}\left(p\frac{d}{dx}\right)+q$, p, q real and show

$$\overline{\mathcal{L}(\varphi)} = \mathcal{L}(\bar{\varphi}).$$

$$\overline{\mathcal{L}(\varphi)} = \overline{\frac{d}{dx}\left(p\left(\frac{d\varphi}{dx}\right)\right)+q\varphi} = \overline{\frac{d}{dx}(p\varphi')} + \underbrace{\bar{q}}_{=q}\,\bar{\varphi}$$

$$= \overline{p'\varphi' + p\varphi''} + q\,\bar{\varphi}$$

$$= \overline{p'\varphi'} + \overline{p\varphi''} + q\bar{\varphi}$$

$$= \underbrace{\overline{p'}}_{\substack{\text{real,}\\ =p'}}\,\overline{\varphi'} + \underbrace{\bar{p}}_{\substack{\text{real}\\ =p}}\,\overline{\varphi''} + q\,\bar{\varphi} = p'\,\overline{\varphi'} + p\,\overline{\varphi''} + q\bar{\varphi} = \mathcal{L}(\bar{\varphi})$$

5.5.9 Consider $\varphi^{(4)} + \lambda e^x \varphi = 0$

with BCs $\varphi(0) = \varphi(1) = 0$, $\varphi'(0) = 0$, $\varphi''(1) = 0$

Show $\lambda \leq 0$.

Think Rayleigh quotient, i.e. multiply by φ and integrate:

$$\Rightarrow \int_0^1 \varphi(x)\varphi^{(4)}(x)\,dx + \lambda\int_0^1 e^x \varphi^2(x)\,dx = 0 \qquad (*)$$

Since (prod. rule) $\varphi\varphi^{(4)} = \frac{d}{dx}\left(\varphi\varphi^{(3)}\right) - \varphi'\varphi^{(3)} = \frac{d}{dx}\left(\varphi\varphi^{(3)}\right) - \frac{d}{dx}\left(\varphi'\varphi''\right) + \left(\varphi''\right)^2$

The first integral turns into

$$\int_0^1 \varphi(x)\varphi^{(4)}(x)\,dx = \underbrace{\varphi(x)\varphi^{(3)}(x)\Big|_0^1}_{=0 \text{ from BCs}} - \underbrace{\varphi'(x)\varphi''(x)\Big|_0^1}_{=0 \text{ from BCs}} + \int_0^1 \left[\varphi''(x)\right]^2 dx$$

and we get from $(*)$: $\lambda = -\dfrac{\int_0^1 \left[\varphi''(x)\right]^2 dx}{\int_0^1 e^x \varphi^2(x)\,dx} \geq 0$

$\lambda = 0$ is not possible since then $\varphi''(x) = 0$