Math 532 — Homework 11 — Due: Wednesday, April 22, 2015

1. Construct a 3×3 matrix A with $\sigma(\mathsf{A}) = \{2, 4, 6\}$ and eigenvectors $\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \}$.

2. Work out the details of the proof of (7.2.4) in the textbook. More precisely, consider a matrix $\mathsf{A} \in \mathbb{C}^{n \times n}$ with spectrum $\sigma(\mathsf{A})$ consisting of k distinct eigenvalues $\lambda_1, \ldots, \lambda_k$. Show that if $\mathcal{B}_i$ is a basis for $N(\mathsf{A} - \lambda_i \mathsf{I})$ and $\text{geomult}_{\mathsf{A}}(\lambda_i) = \text{algmult}_{\mathsf{A}}(\lambda_i)$ for each i, then
$$\mathcal{B} = \bigcup_{i=1}^{k} \mathcal{B}_i$$
is a linearly independent set.

3. Assume $\mathsf{U} \in \mathbb{C}^{n \times n}$ is unitary and show that the eigenvectors associated with distinct eigenvalues of U are orthogonal.

4. Assume that $\mathsf{A} \in \mathbb{C}^{n \times n}$ and show that $\lambda \in \sigma(\mathsf{A})$ implies that $\lambda^k \in \sigma(\mathsf{A}^k)$.

5. On slide #102 of Chapter 4 we used the Neumann series of a matrix A to obtain the inverse of $\mathsf{I} - \mathsf{A}$, i.e.,
$$(\mathsf{I} - \mathsf{A})^{-1} = \sum_{k=1}^{\infty} \mathsf{A}^k.$$
Show that if A is diagonalizable then the "sufficiently small" argument used in Chapter 4 to justify convergence of the series amounts to A having small enough eigenvalues. More precisely, show that
$$\rho(\mathsf{A}) < 1 \quad \Longleftrightarrow \quad \lim_{k \to \infty} \mathsf{A}^k = 0,$$
where $\rho(\mathsf{A})$ is the spectral radius of A.