# MATH 461: Fourier Series and Boundary Value Problems

## Chapter III: Fourier Series

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# Outline

## Definition

A function $f$, defined on $[a, b]$, is piecewise continuous if it is continuous on $[a, b]$ except at finitely many points. If both $f$ and $f'$ are piecewise continuous, then $f$ is called piecewise smooth.

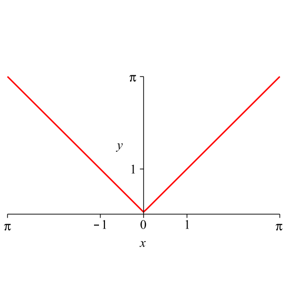
## Remark

*This means that the graphs of $f$ and $f'$ may have only finitely many finite jumps.*

## Example

The function $f(x) = |x|$ defined on $-\pi < x < \pi$ is piecewise smooth since

- $f$ is continuous throughout the interval,
- and $f'$ is discontinuous only at $x = 0$.

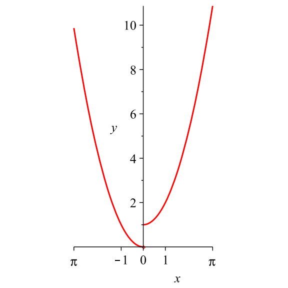



## Example

The function

$$f(x) = \begin{cases} x^2, & -\pi < x < 0 \\ x^2 + 1, & 0 \le x < \pi \end{cases}$$

is piecewise smooth since both $f$ and $f'$ are continuous except at $x = 0$.



## Example

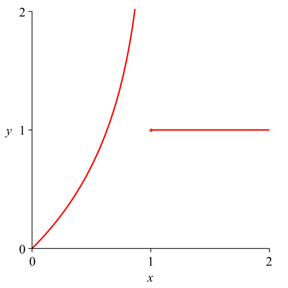
The function

$$f(x) = \begin{cases} -\ln(1-x), & 0 \le x < 1 \\ 1, & 1 \le x < 2 \end{cases}$$

is not piecewise continuous (and therefore also not piecewise smooth) since

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} -\ln(1-x) = \infty,$$

i.e., $f$ has an infinite jump at $x = 1$.



# Periodic Extension

If $f$ is defined on $[-L, L]$, then its periodic extension, defined for all $x$, is given by

$$\bar{f}(x) = \begin{cases} \vdots & \\ f(x+2L), & -3L < x < -L, \\ f(x), & -L < x < L, \\ f(x-2L), & L < x < 3L, \\ f(x-4L), & 3L < x < 5L, \\ \vdots & \end{cases}$$

## Example

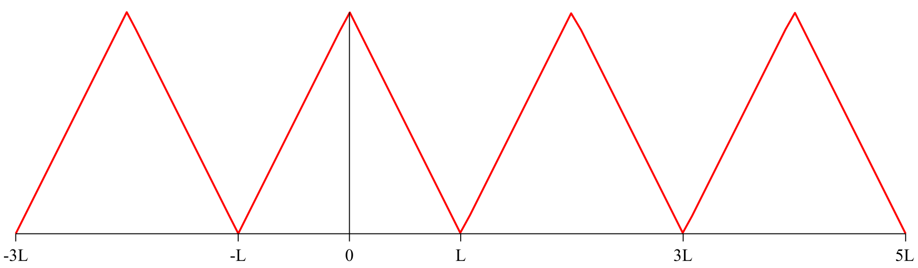



Figure: Plot of $\bar{f}$ for $f(x) = 1 - \left|\frac{x}{L}\right|$.

## Example

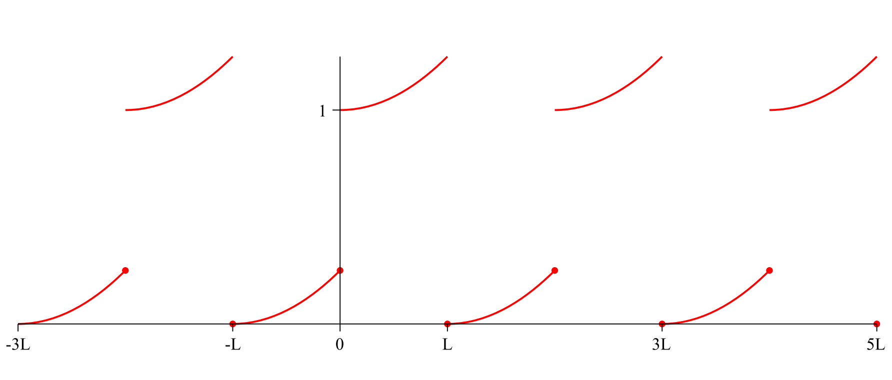



Figure: Plot of $\bar{f}$ for $f(x) = \begin{cases} (x+L)^2, & -L \le x \le 0 \\ x^2+1, & 0 < x < L \end{cases}$.

Even though we have used Fourier series to represent a given function $f$ within our separation of variables approach, we have never made sure that these series actually converge.

Moreover, even if we can assure convergence, how do we know that they converge to the function $f$?

**Remark**

*This should not come as a total surprise, since for power series we also had to determine the interval (or radius) of convergence.*

Using a more precise notation, all we can say is

$$f(x) \sim a_0 + \sum_{n=1}^{\infty} \left[ a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right],$$

i.e., we can

- associate with $f$ this Fourier series,
- but not $f$ is equal to this Fourier series.

The Fourier coefficients of $f$, on the other hand, are never in doubt. They are given by

$$\begin{aligned} a_0 &= \frac{1}{2L} \int_{-L}^{L} f(x)\,\mathrm{d}x \\ a_n &= \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L}\,\mathrm{d}x, \quad n = 1, 2, \ldots \\ b_n &= \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L}\,\mathrm{d}x, \quad n = 1, 2, \ldots \end{aligned}$$

What we need is

**Theorem (Fourier Convergence Theorem)**

*If $f$ is piecewise smooth on $[-L, L]$, then the Fourier series of $f$ converges. Moreover,*

1. *at those points $x$ where the periodic extension $\bar{f}$ of $f$ is continuous, the Fourier series of $f$ converges to $\bar{f}(x)$ and*
2. *at jump discontinuities of the periodic extension, the Fourier series converges to*

$$\frac{1}{2}\left[\bar{f}(x-) + \bar{f}(x+)\right],$$

*i.e., the average of the left and right limits at the jump.*

**Remark**

*Note that (2) actually includes (1) since*

$$\frac{1}{2}\left[\bar{f}(x-) + \bar{f}(x+)\right] = \frac{1}{2}\left[\bar{f}(x) + \bar{f}(x)\right] = \bar{f}(x)$$

**Proof.**

The proof of this theorem is not contained in [Haberman] and goes beyond the scope of this course. It can be found in [Pinsky, Section 1.2] or [Brown & Churchill, Section 19].

The proof requires the Dirichlet kernel

$$D_N(x) = \frac{1}{2} + \sum_{n=1}^{N} \cos nx = \frac{\sin\left(N + \frac{1}{2}\right)x}{2\sin\frac{x}{2}}$$

as well as a careful analysis of one-sided derivatives.

The calculations for Gibbs phenomenon below gives a flavor of this. □

## Remark

*The theorem above is about pointwise convergence of Fourier series.*

*In classical harmonic analysis there are also theorems about other kinds of convergence of Fourier series, such as*

- *uniform convergence or*
- *convergence in the mean.*

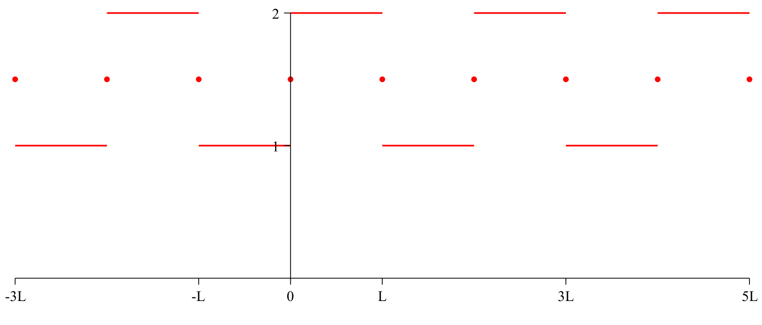
*For these see, e.g., [Brown & Churchill, Pinsky].*

*We will talk about convergence in the mean in Chapter 5, and the Gibbs phenomenon below is evidence that uniform convergence is not guaranteed for general functions $f$.*

## Example

Consider the function $f(x) = \begin{cases} 1, & -L \le x < 0 \\ 2, & 0 < x \le L \end{cases}$

The Fourier series of $f$, $a_0 + \sum_{n=1}^{\infty} \left[ a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right]$, is represented by the following graph:



## Example (cont.)

### Remark

*Even if we know that the series converges, we have*

- $f(x) =$ *its Fourier series only for* $x \in (-L, L)$ *(and provided* $f$ *is continuous at* $x$*).*
- *At all other values of* $x$ *the Fourier series equals the periodic extension of* $f$*,*
- *except at jump discontinuities, where it equals the average jump.*

What are the Fourier coefficients for this example?

## Example (cont.)

$$
\begin{aligned}
a_0 &= \frac{1}{2L}\int_{-L}^{L} f(x)\,\mathrm{d}x \\
&= \frac{1}{2L}\left[\int_{-L}^{0} 1\,\mathrm{d}x + \int_{0}^{L} 2\,\mathrm{d}x\right] \\
&= \frac{1}{2L}\left[L + 2L\right] = \frac{3}{2}
\end{aligned}
$$

## Example (cont.)

$$
\begin{aligned}
a_n &= \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,\mathrm{d}x \\
&= \frac{1}{L}\left[\int_{-L}^{0}\cos\frac{n\pi x}{L}\,\mathrm{d}x + 2\int_{0}^{L}\cos\frac{n\pi x}{L}\,\mathrm{d}x\right] \\
&= \frac{1}{L}\left[\underbrace{\int_{-L}^{L}\cos\frac{n\pi x}{L}\,\mathrm{d}x}_{=0} + \int_{0}^{L}\cos\frac{n\pi x}{L}\,\mathrm{d}x\right] \\
&= \frac{L}{n\pi L}\left[\sin\frac{n\pi x}{L}\right]_0^L = 0
\end{aligned}
$$

## Example ((cont.))

$$
\begin{aligned}
b_n &= \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,\mathrm{d}x \\
&= \frac{1}{L}\left[\int_{-L}^{0}\sin\frac{n\pi x}{L}\,\mathrm{d}x + 2\int_{0}^{L}\sin\frac{n\pi x}{L}\,\mathrm{d}x\right] \\
&= \frac{1}{L}\Bigg[\underbrace{\int_{-L}^{L}\sin\frac{n\pi x}{L}\,\mathrm{d}x}_{=0} + \int_{0}^{L}\sin\frac{n\pi x}{L}\,\mathrm{d}x\Bigg] \\
&= -\frac{L}{n\pi L}\left[\cos\frac{n\pi x}{L}\right]_0^L \\
&= -\frac{\cos n\pi}{n\pi} + \frac{\cos 0}{n\pi} \\
&= \frac{1-(-1)^n}{n\pi} = \begin{cases} 0, & n \text{ even} \\ \frac{2}{n\pi}, & n \text{ odd} \end{cases}
\end{aligned}
$$

## Example (cont.)

Summarizing, we have found that the function

$$f(x) = \begin{cases} 1, & -L \leq x < 0 \\ 2, & 0 < x \leq L \end{cases}$$

has Fourier series

$$\begin{aligned} f(x) &\sim \frac{3}{2} + \sum_{n=1}^{\infty} \frac{1-(-1)^n}{n\pi} \sin \frac{n\pi x}{L} \\ &= \frac{3}{2} + \sum_{k=1}^{\infty} \frac{2}{(2k-1)\pi} \sin \frac{(2k-1)\pi x}{L} \end{aligned}$$

## Example (cont.)

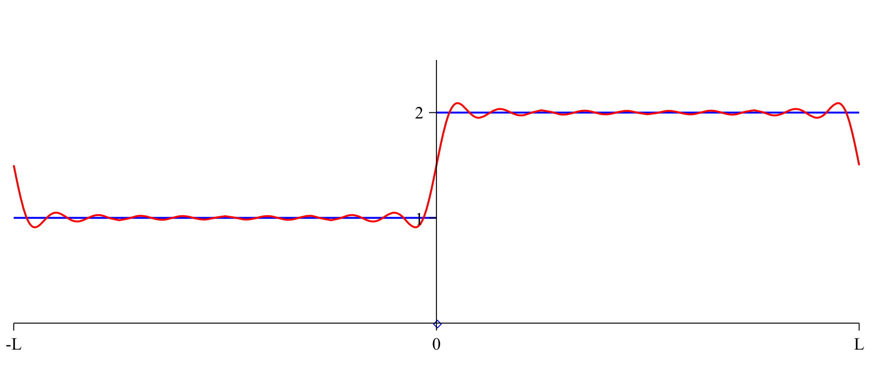



Figure: Plot of 10-term Fourier series approximation $f(x) = \frac{3}{2} + \sum_{k=1}^{10} \frac{2}{(2k-1)\pi} \sin \frac{(2k-1)\pi x}{L}$ (red) together with graph of $f$ (blue).

# The Gibbs Phenomenon

In order to understand the oscillations of the previous plots, and in particular the overshoot, we consider an almost identical function:

$$f(x) = \begin{cases} -1, & -\pi \le x < 0 \\ 1, & 0 < x \le \pi \end{cases}$$

with truncated Fourier series

$$f_{2N}(x) = f_{2N-1}(x) = \sum_{n=1}^{N} \frac{2\left(1-(-1)^n\right)}{n\pi} \sin nx = \sum_{k=1}^{N} \frac{4}{\pi} \frac{\sin(2k-1)x}{2k-1}$$

## Remark

*Compared to the previous example, the sines are simpler since $L = \pi$, and we have a vertical shift by $a_0 = 0$ and a vertical stretching so that*

$$b_n = 2\frac{1-(-1)^n}{n\pi} = \begin{cases} 0, & n \text{ even} \\ \frac{4}{n\pi}, & n \text{ odd} \end{cases}.$$

# Gibbs Phenomenon (cont.)

To find the overshoot at the jump discontinuity we look at the zeros of the derivative of the truncated Fourier series (to locate its maxima), i.e.,

$$f'_{2N-1}(x) = \frac{4}{\pi}\sum_{k=1}^{N}\cos(2k-1)x = \frac{4}{\pi}\left[\cos x + \cos 3x + \ldots + \cos(2N-1)x\right]$$

To find the zeros of this function we multiply both sides by $\sin x$, i.e.,

$$\sin x f'_{2N-1}(x) = \frac{4}{\pi}\left[\sin x\cos x + \sin x\cos 3x + \ldots + \sin x\cos(2N-1)x\right]$$

Using the trigonometric identity $\sin x\cos kx = \frac{\sin(k+1)x - \sin(k-1)x}{2}$ we get

$$\begin{aligned}\sin x f'_{2N-1}(x) &= \frac{2}{\pi}\left[(\sin 2x - \sin 0) + (\sin 4x - \sin 2x) + \ldots + (\sin 2Nx - \sin(2N-2)x)\right]\\ &= \frac{2}{\pi}\sin 2Nx.\end{aligned}$$

# Gibbs Phenomenon (cont.)

Now,

$$\sin 2Nx = 0 \quad \text{if } 2Nx = \pm\pi, \pm 2\pi, \ldots, \pm 2N\pi.$$

The maximum overshoot occurs at $x = \frac{\pi}{2N}$ and its value is

$$\begin{aligned} f_{2N-1}\left(\frac{\pi}{2N}\right) &= \frac{4}{\pi}\sum_{k=1}^{N} \frac{\sin\frac{(2k-1)\pi}{2N}}{2k-1} \\ &= \frac{2}{\pi}\frac{2N}{\pi}\frac{\pi}{N}\sum_{k=1}^{N} \frac{\sin\frac{(2k-1)\pi}{2N}}{2k-1} \\ &= \frac{2}{\pi}\sum_{k=1}^{N} \frac{\sin\frac{(2k-1)\pi}{2N}}{\frac{(2k-1)\pi}{2N}}\frac{\pi}{N}. \end{aligned}$$

# Gibbs Phenomenon (cont.)

If we interpret

$$\sum_{k=1}^{N} \frac{\sin \frac{(2k-1)\pi}{2N}}{\frac{(2k-1)\pi}{2N}} \frac{\pi}{N}$$

as a partial Riemann sum with $\Delta x = \frac{\pi}{N}$ and midpoints $x^* = \frac{\pi}{2N}, \frac{3\pi}{2N}, \ldots, \frac{(2N-1)\pi}{2N}$ for the partition $0, \frac{\pi}{N}, \frac{2\pi}{N}, \ldots, \frac{(N-1)\pi}{N}, \pi$ of $[0, \pi]$ then

$$f_{2N-1}\left(\frac{\pi}{2N}\right) \to \frac{2}{\pi} \int_0^{\pi} \frac{\sin x}{x} \mathrm{d}x \quad \text{for } N \to \infty.$$

This integral can be evaluated numerically to get

$$f_{2N-1}\left(\frac{\pi}{2N}\right) \approx 1.178979744472167\ldots.$$

# Gibbs Phenomenon (cont.)

Since the actual size of the jump discontinuity is 2, we have an approximately 9% overshoot. This is true in general [Pinsky, p. 60]:

**Theorem**

*If $f$ is piecewise smooth on $(-\pi, \pi)$ then the overshoot of the truncated Fourier series of $f$ at a discontinuity $x_0$ (the Gibbs phenomenon) is approximately 9% of the jump, i.e.,*

$$0.09\left[f(x_0+) - f(x_0-)\right].$$

**Remark**

*The "Gibbs phenomenon" was actually discovered by Henry Wilbraham in 1848. Gibbs was just more famous, published in a better journal (50 years later), and built in some mistakes – perhaps drawing more attention to his work (for further discussion see [Trefethen, Chapter 9]).*

We begin by reviewing the concepts of odd and even functions:

**Definition**

$f$ is an odd function if $f(-x) = -f(x)$ for all $x$ in the domain of $f$.

**Remark**

- *The graph of an odd function is symmetric about the origin.*
- *For an odd function we have* $\int_{-L}^{L} f(x)\,\mathrm{d}x = 0$.

$$\begin{aligned}
\int_{-L}^{L} f(x)\,\mathrm{d}x &= \int_{-L}^{0} \underbrace{f(x)}_{\substack{u=-x \\ du=-\mathrm{d}x}} \,\mathrm{d}x + \int_{0}^{L} f(x)\,\mathrm{d}x \\
&= -\int_{L}^{0} \underbrace{f(-u)}_{=-f(u)} \,du + \int_{0}^{L} f(x)\,\mathrm{d}x \\
&= -\int_{0}^{L} f(u)\,du + \int_{0}^{L} f(x)\,\mathrm{d}x = 0
\end{aligned}$$

## Definition

$f$ is an even function if $f(-x) = f(x)$ for all $x$ in the domain of $f$.

## Remark

- *The graph of an even function is symmetric about the $y$-axis.*
- *For an even function we have*

$$\int_{-L}^{L} f(x)\,dx = 2\int_{0}^{L} f(x)\,dx,$$

*which can be shown similarly to the analogous property for odd functions.*

Let's consider the Fourier series of an odd function

$$f(x) \sim a_0 + \sum_{n=1}^{\infty} \left[ a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right]$$

with

$$a_0 = \frac{1}{2L} \int_{-L}^{L} \underbrace{f(x)}_{\text{odd}} \, dx = 0, \qquad a_n = \frac{1}{L} \int_{-L}^{L} \underbrace{\underbrace{f(x)}_{\text{odd}} \underbrace{\cos \frac{n\pi x}{L}}_{\text{even}}}_{\text{odd}} \, dx = 0$$

$$b_n = \frac{1}{L} \int_{-L}^{L} \underbrace{\underbrace{f(x)}_{\text{odd}} \underbrace{\sin \frac{n\pi x}{L}}_{\text{odd}}}_{\text{even}} \, dx = \frac{2}{L} \int_{0}^{L} f(x) \sin \frac{n\pi x}{L} \, dx (= B_n)$$

Therefore,

$$f(x) \sim \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L},$$

i.e., the Fourier series is automatically a Fourier sine series.

# What does the Fourier sine series of $f$ converge to?

**Theorem**

*If $f$ is piecewise smooth on $[0, L]$, then the Fourier sine series of $f$ converges. Moreover,*

1. *at those points $x$ where the odd periodic extension of $f$ is continuous, the Fourier sine series converges to the odd periodic extension and*
2. *at jump discontinuities of the odd periodic extension, the Fourier sine series converges to the average of the left and right limits at the jump.*

## Example

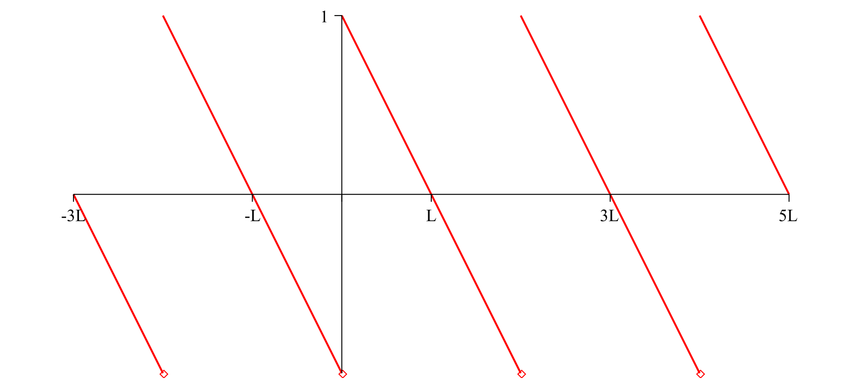



Figure: Plot of odd periodic extension of $f(x) = 1 - \frac{x}{L}$.

## Remark

*Even if $f$ is not an odd function, it may still be necessary to represent it by a Fourier sine series.*

## Example

The heat equation problem

$$\begin{aligned} \frac{\partial u}{\partial t}(x,t) &= k\frac{\partial^2 u}{\partial x^2}(x,t), \qquad 0 < x < L, \quad t > 0 \\ u(0,t) &= u(L,t) = 0 \\ u(x,0) &= \cos\frac{\pi x}{L} \end{aligned}$$

has sines as eigenfunctions, so we need to find the Fourier sine series expansion of $f(x) = \cos\frac{\pi x}{L}$.

## Example (cont.)

We know $u(x,t) = \varphi(x)G(t)$, with eigenvalues $\lambda_n = \left(\frac{n\pi}{L}\right)^2$, $n = 1, 2, 3, \ldots$ and eigenfunctions

$$\varphi_n(x) = \sin\frac{n\pi x}{L}$$

as well as $G_n(t) = e^{-\lambda_n kt}$.
Therefore

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L} e^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

and

$$u(x,0) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L} \stackrel{!}{=} \cos\frac{\pi x}{L}$$

## Example (cont.)

In HW 3.3.2a you should show[a]

$$B_n = \frac{2}{L}\int_0^L \cos\frac{\pi x}{L}\sin\frac{n\pi x}{L}\,\mathrm{d}x = \begin{cases} 0, & n \text{ odd} \\ \frac{4n}{\pi(n^2-1)}, & n \text{ even} \end{cases}$$

Therefore, letting $n = 2k$ (even), we have

$$\cos\frac{\pi x}{L} = \sum_{k=1}^{\infty} \frac{8k}{\pi(4k^2-1)} \sin\frac{2k\pi x}{L}$$

and the equality is true for $0 < x < L$ (since the cosine equals its odd periodic extension there).

For $x = 0$ and $x = L$ the series is zero (which is equal to the average jump of the cosine function there).

---

[a] Remember that we established the orthogonality of sine and cosine only on $[-L, L]$, not on $[0, L]$

Consider the Fourier series of an even function

$$f(x) \sim a_0 + \sum_{n=1}^{\infty} \left[ a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right]$$

with

$$\begin{aligned} a_0 &= \frac{1}{2L} \int_{-L}^{L} \underbrace{f(x)}_{\text{even}} \, dx = \frac{1}{L} \int_0^L f(x) \, dx (= A_0) \\ a_n &= \frac{1}{L} \int_{-L}^{L} \underbrace{f(x) \cos \frac{n\pi x}{L}}_{\text{even}} \, dx = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L} \, dx (= A_n) \\ b_n &= \frac{1}{L} \int_{-L}^{L} \underbrace{f(x) \sin \frac{n\pi x}{L}}_{\text{odd}} \, dx = 0 \end{aligned}$$

Therefore,

$$f(x) \sim A_0 + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{L},$$

i.e., the Fourier series is automatically a Fourier cosine series.

### Theorem

*If $f$ is piecewise smooth on $[0, L]$, then the Fourier cosine series of $f$ converges. Moreover,*

1. *at those points $x$ where the even periodic extension of $f$ is continuous, the Fourier cosine series converges to the even periodic extension and*
2. *at jump discontinuities of the even periodic extension, the Fourier cosine series converges to the average of the left and right limits at the jump.*

### Remark

*Note that jump discontinuities are possible only for $0 < x < L$, i.e., if $f$ itself had jump discontinuities. The even periodic extension cannot have any jumps at $x = 0$ or $x = \pm L$.*

## Example

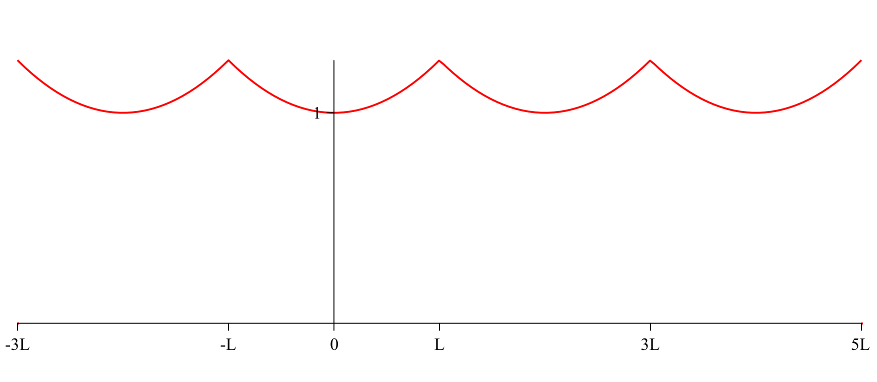


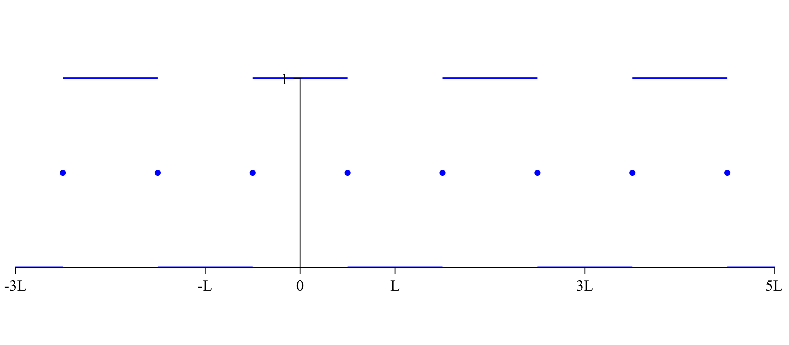

Figure: Plot of even periodic extension of $f(x) = x^2 + 1$.

## Example

Find the Fourier cosine series expansion of the function

$$f(x) = \begin{cases} 1, & 0 \le x < \frac{L}{2} \\ 0, & \frac{L}{2} \le x \le L \end{cases}$$

and sketch its graph.



## Example (cont.)

Let's compute the Fourier cosine coefficients:

$$
\begin{aligned}
A_0 &= \frac{1}{L}\int_0^L f(x)\,\mathrm{d}x \\
&= \frac{1}{L}\left[\int_0^{L/2} 1\,\mathrm{d}x + \int_{L/2}^{L} 0\,\mathrm{d}x\right] \\
&= \frac{1}{L}\frac{L}{2} = \frac{1}{2}
\end{aligned}
$$

## Example (cont.)

$$\begin{aligned} A_n &= \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,\mathrm{d}x \\ &= \frac{2}{L}\left[\int_0^{L/2}\cos\frac{n\pi x}{L}\,\mathrm{d}x + \int_{L/2}^{L} 0\,\cos\frac{n\pi x}{L}\,\mathrm{d}x\right] \\ &= \frac{2}{L}\frac{L}{n\pi}\left[\sin\frac{n\pi x}{L}\right]_0^{L/2} \\ &= \frac{2}{n\pi}\sin\frac{n\pi}{2} = \begin{cases} 0, & n = 2k \text{ even} \\ \frac{2}{(2k-1)\pi}(-1)^{k+1}, & n = 2k-1 \text{ odd} \end{cases} \end{aligned}$$

Therefore

$$f(x) \sim \frac{1}{2} + \sum_{k=1}^{\infty} \frac{2(-1)^{k+1}}{(2k-1)\pi}\cos\frac{(2k-1)\pi x}{L}$$

Note that "$\sim$" equals "$=$" for all $x \in [0, L]$ except $x = \frac{L}{2}$.

# Even and odd parts of functions

## Theorem

*Any function $f$ can be written as the sum of an even and an odd function:*

$$f(x) = \underbrace{\frac{1}{2}\left[f(x) + f(-x)\right]}_{:=f_e(x)} + \underbrace{\frac{1}{2}\left[f(x) - f(-x)\right]}_{:=f_o(x)}$$

## Proof.

Indeed, $f_e$ is even

$$f_e(-x) = \frac{1}{2}\left[f(-x) + f(x)\right] = f_e(x),$$

and $f_o$ is odd

$$f_o(-x) = \frac{1}{2}\left[f(-x) - f(x)\right] = -\frac{1}{2}\left[f(x) - f(-x)\right] = -f_o(x).$$

□

Now, the Fourier series for an arbitrary function $f$ is given by

$$f(x) \sim \underbrace{a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}}_{\text{even}} + \underbrace{\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}}_{\text{odd}}$$

with Fourier coefficients

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\mathrm{d}x, \quad a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\mathrm{d}x, \quad b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\mathrm{d}x.$$

On the other hand,

$$\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$$

is a Fourier sine series.

However, it is the Fourier sine series of $f_o$, not of $f$!

Note that the Fourier sine series of $f$ has coefficients

$$B_n = \frac{2}{L}\int_{0}^{L} f(x)\sin\frac{n\pi x}{L}\,\mathrm{d}x.$$

We can make a similar observation for cosine.

Therefore,

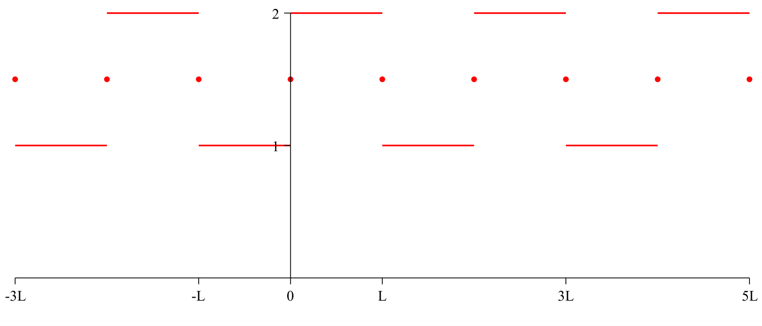
$$\underbrace{a_0 + \sum_{n=1}^{\infty}\left[a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right]}_{\text{Fourier series of } f}$$

$$= \underbrace{a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}}_{\text{cosine series of } f_e} + \underbrace{\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}}_{\text{sine series of } f_o}$$

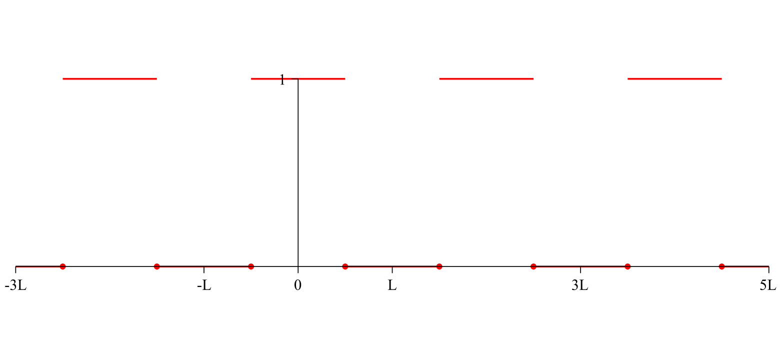
# Summary: Convergence of Fourier series

Let $f$ be piecewise smooth. Then

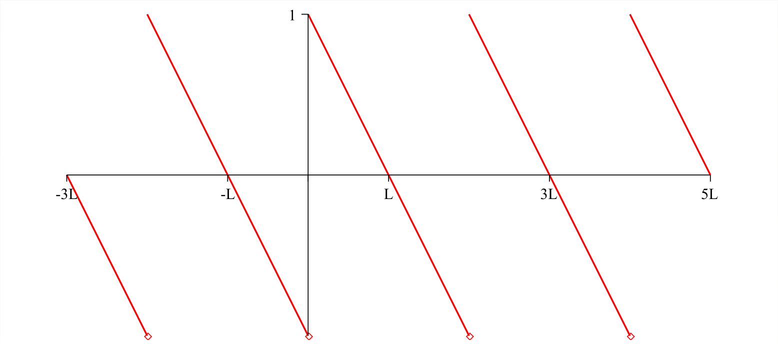
- The Fourier series of $f$ is continuous for all $x$ $\iff$
  - $f$ is continuous on $[-L, L]$ (i.e., no jumps inside) and
  - $f(-L) = f(L)$ (i.e., no jumps at end).



- The Fourier cosine series of $f$ is continuous for all $x$ $\iff$
  - $f$ is continuous on $[0, L]$.



- The Fourier sine series of $f$ is continuous for all $x$ $\iff$
  - $f$ is continuous on $[0, L]$ (i.e., no jumps inside) and
  - $f(0) = f(L) = 0$ (i.e., no jumps at end).



Recall that in HW 2.5.5c we had to deal with the boundary condition

$$\frac{\partial u}{\partial r}(1,\theta) = f(\theta),$$

where

$$u(r,\theta) = \sum_{n=1}^{\infty} B_n r^{2n} \sin 2n\theta.$$

In order to determine the coefficients $b_n$ we needed to differentiate the infinite series, i.e., find

$$\frac{\partial u}{\partial r}(r,\theta) = \sum_{n=1}^{\infty} 2nB_n r^{2n-1} \sin 2n\theta.$$

Was this justified?
Does this new series converge? If so, does it converge to $\frac{\partial u}{\partial r}(r,\theta)$?

## Example

Consider the function $f(x) = x$, and find its Fourier sine series.
Then, compare the termwise derivative of the series with the "correct" derivative $f'(x) = 1$.
We know

$$x \sim \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L}$$

with

$$B_n = \frac{2}{L} \int_0^L x \sin \frac{n\pi x}{L} \, dx.$$

## Example (cont.)

Using integration by parts we have

$$\begin{aligned} B_n &= \frac{2}{L}\int_0^L x \sin\frac{n\pi x}{L}\,\mathrm{d}x \\ &= \frac{2}{L}\left[-x\frac{L}{n\pi}\cos\frac{n\pi x}{L}\Big|_0^L + \frac{L}{n\pi}\int_0^L \cos\frac{n\pi x}{L}\,\mathrm{d}x\right] \\ &= \frac{2}{L}\left[-\frac{L^2}{n\pi}\cos n\pi + \frac{L^2}{(n\pi)^2}\sin\frac{n\pi x}{L}\Big|_0^L\right] \\ &= \frac{2}{L}\left[-\frac{L^2}{n\pi}\cos n\pi\right] = -\frac{2L}{n\pi}(-1)^n \end{aligned}$$

Therefore,

$$x \sim \frac{2L}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\frac{n\pi x}{L} \qquad (1)$$

for which we know that "$\sim$" equals "$=$" for $0 \le x < L$.

## Example (cont.)

Now we consider the termwise derivative of the Fourier sine series

$$x \sim \frac{2L}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin \frac{n\pi x}{L},$$

i.e.,

$$\begin{aligned} & \frac{2L}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \frac{n\pi}{L} \cos \frac{n\pi x}{L} \\ = \quad & 2 \sum_{n=1}^{\infty} (-1)^{n+1} \cos \frac{n\pi x}{L}. \end{aligned}$$

## Remark

*Note that this is a divergent series since the terms in the sequence do not approach zero for $n \to \infty$, and therefore the series diverges by the standard test for divergence from calculus.*

Example (cont.)

Obviously, we must conclude that

$$1 \neq 2\sum_{n=1}^{\infty}(-1)^{n+1}\cos\frac{n\pi x}{L}.$$

In fact, the Fourier cosine series of $f'(x) = 1$ is given by

$$f'(x) = 1 \sim 1,$$

i.e.,

$$a_0 = 1, \ a_n = 0 \text{ for } n \geq 1.$$

Obviously, we could replace "$\sim$" by "$=$".

## Example (cont.)

Why did this not work?
What caused the trouble?

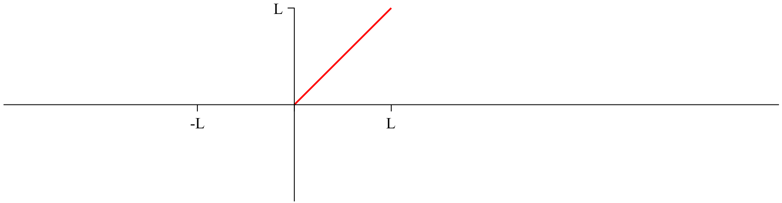



Figure: Plot of $f(x) = x$ for $0 < x < L$.

## Example (cont.)

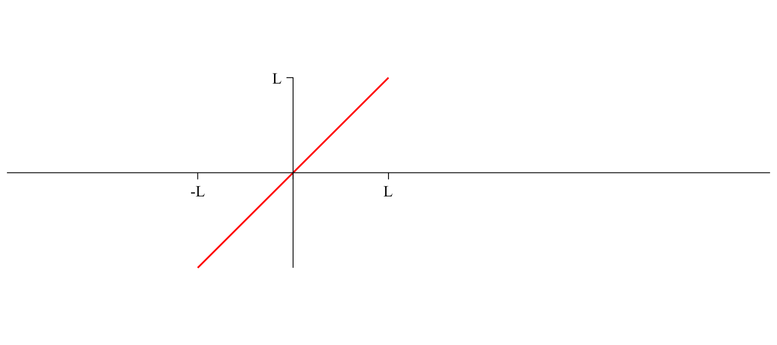



Figure: Plot of odd extension of $f(x) = x$.

## Example (cont.)

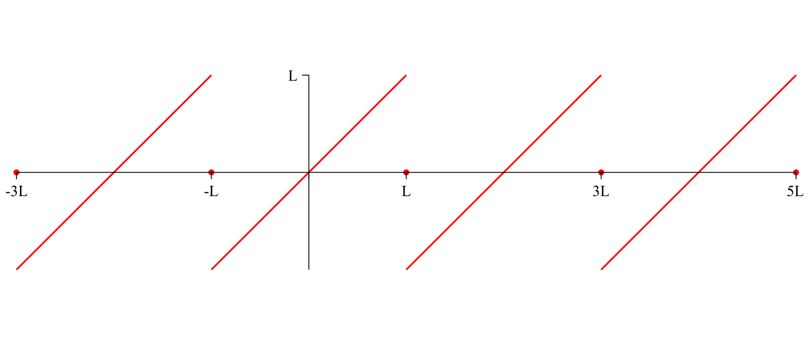



Figure: Plot of odd periodic extension (actually, Fourier sine series) of $f(x) = x$.

The jumps in the Fourier sine series at odd multiples of $L$ prevent the series from being differentiable.

**Theorem (Differentiation of Fourier Series)**

*A continuous Fourier series can be differentiated term-by-term provided $f'$ is piecewise smooth.*

**Remark**

- *In other words, the Fourier series of a continuous function $f$ which satisfies $f(-L) = f(L)$ can be differentiated term-by-term provided $f'$ is piecewise smooth.*
- *Piecewise smoothness of $f'$ ensures that its Fourier series converges.*

# Proof

The Fourier series of $f$ is

$$f(x) \sim a_0 + \sum_{n=1}^{\infty} \left[ a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right] \tag{2}$$

with

$$a_0 = \frac{1}{2L} \int_{-L}^{L} f(x) \mathrm{d}x, \quad a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L} \mathrm{d}x, \quad b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L} \mathrm{d}x.$$

Since $f'$ is piecewise smooth, it has a convergent Fourier series of the form

$$f'(x) \sim A_0 + \sum_{n=1}^{\infty} \left[ A_n \cos \frac{n\pi x}{L} + B_n \sin \frac{n\pi x}{L} \right] \tag{3}$$

with

$$A_0 = \frac{1}{2L} \int_{-L}^{L} f'(x) \mathrm{d}x, \quad A_n = \frac{1}{L} \int_{-L}^{L} f'(x) \cos \frac{n\pi x}{L} \mathrm{d}x, \quad B_n = \frac{1}{L} \int_{-L}^{L} f'(x) \sin \frac{n\pi x}{L} \mathrm{d}x.$$

If allowed, term-by-term differentiation of the Fourier series (2), i.e.,

$$f(x) \sim a_0 + \sum_{n=1}^{\infty} \left[ a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right]$$

would yield

$$f'(x) \sim \sum_{n=1}^{\infty} \left[ -\frac{n\pi}{L} a_n \sin \frac{n\pi x}{L} + \frac{n\pi}{L} b_n \cos \frac{n\pi x}{L} \right].$$

Therefore, comparing with (3), we need to show that

$$A_0 = 0, \quad A_n = \frac{n\pi}{L} b_n, \quad B_n = -\frac{n\pi}{L} a_n.$$

Let's actually compute the Fourier coefficients of $f'$ based on the information we have:

$$\begin{aligned} A_0 &= \frac{1}{2L}\int_{-L}^{L} f'(x)\,\mathrm{d}x \\ &= \frac{1}{2L}\left[f(x)\right]_{-L}^{L} \\ &= \frac{1}{2L}\left[f(L) - f(-L)\right] \end{aligned}$$

Since we assumed that the Fourier series of $f$ is continuous, i.e., in particular, that $f(L) = f(-L)$, we have

$$A_0 = 0.$$

$$
\begin{aligned}
A_n &= \frac{1}{L}\int_{-L}^{L} f'(x)\cos\frac{n\pi x}{L}\,\mathrm{d}x \overset{\text{parts}}{=} \left[\begin{array}{ll} u=\cos\frac{n\pi x}{L}, & du=-\frac{n\pi}{L}\sin\frac{n\pi x}{L}\mathrm{d}x \\ dv=f'(x)\mathrm{d}x, & v=f(x) \end{array}\right] \\
&= \frac{1}{L}\left[f(x)\cos\frac{n\pi x}{L}\Big|_{-L}^{L} + \underbrace{\int_{-L}^{L} f(x)\frac{n\pi}{L}\sin\frac{n\pi x}{L}\,\mathrm{d}x}_{n\pi b_n}\right] \\
&= \frac{1}{L}\left[f(L)\cos n\pi - f(-L)\underbrace{\cos(-n\pi)}_{=\cos n\pi} + n\pi b_n\right] \\
&= \frac{1}{L}\left[\Big(\underbrace{f(L)-f(-L)}_{=0,\text{ since F.S. of } f \text{ cont.}}\Big)\cos n\pi + n\pi b_n\right] \\
&= \frac{n\pi}{L} b_n.
\end{aligned}
$$

$B_n$ is treated similarly. □

If the Fourier series of $f$ is not continuous, i.e., if $f(-L) \neq f(L)$, then the proof above shows us that

$$\begin{aligned} A_0 &= \frac{1}{2L}\left[f(L) - f(-L)\right], \\ A_n &= \frac{1}{L}(-1)^n \left[f(L) - f(-L)\right] + \frac{n\pi}{L} b_n, \\ B_n &= -\frac{n\pi}{L} a_n. \end{aligned}$$

Therefore, even if the Fourier series of $f$ itself is not continuous, the Fourier series of the derivative of a continuous function $f$ is given by

$$\begin{aligned} f'(x) \sim \frac{1}{2L}\left[f(L) - f(-L)\right] + \sum_{n=1}^{\infty} \left( \frac{(-1)^n}{L}\left[f(L) - f(-L)\right] + \frac{n\pi}{L} b_n \right) \cos \frac{n\pi x}{L} \\ - \frac{n\pi}{L} a_n \sin \frac{n\pi x}{L}. \end{aligned}$$

**Theorem (Differentiation of Fourier Cosine Series)**

*A continuous Fourier cosine series can be differentiated term-by-term provided $f'$ is piecewise smooth.*

**Remark**

- *In other words, the Fourier cosine series of a continuous function $f$ can be differentiated term-by-term provided $f'$ is piecewise smooth.*
- *No additional end conditions are required for $f$ since $f(-L) = f(L)$ is automatically satisfied due to even extension.*

**Proof.**

HW 3.4.4b □

## Example

Consider again the function $f(x) = x$, but now find its Fourier cosine series.
Can we apply term-by-term differentiation?
If so, what is the derivative?
We know

$$x \sim A_0 + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{L}$$

with

$$\begin{aligned} A_0 &= \frac{1}{L}\int_0^L x \, \mathrm{d}x = \frac{1}{L}\frac{L^2}{2} = \frac{L}{2}, \\ A_n &= \frac{2}{L}\int_0^L x \cos \frac{n\pi x}{L} \, \mathrm{d}x. \end{aligned}$$

## Example (cont.)

$$
\begin{aligned}
A_n &= \frac{2}{L}\int_0^L x\cos\frac{n\pi x}{L}\,\mathrm{d}x \\
&\overset{\text{parts}}{=} \frac{2}{L}\left[x\frac{L}{n\pi}\sin\frac{n\pi x}{L}\Big|_0^L - \frac{L}{n\pi}\int_0^L \sin\frac{n\pi x}{L}\,\mathrm{d}x\right] \\
&= \frac{2}{n\pi}\frac{L}{n\pi}\cos\frac{n\pi x}{L}\Big|_0^L \\
&= \frac{2L}{(n\pi)^2}\left[(-1)^n - 1\right] = \begin{cases} 0, & \text{for } n \text{ even} \\ -\frac{4L}{(n\pi)^2}, & \text{for } n \text{ odd} \end{cases}
\end{aligned}
$$

Therefore, with $n = 2k-1$,

$$
x \sim \frac{L}{2} - \frac{4L}{\pi^2}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2}\cos\frac{(2k-1)\pi x}{L} \tag{4}
$$

## Example (cont.)

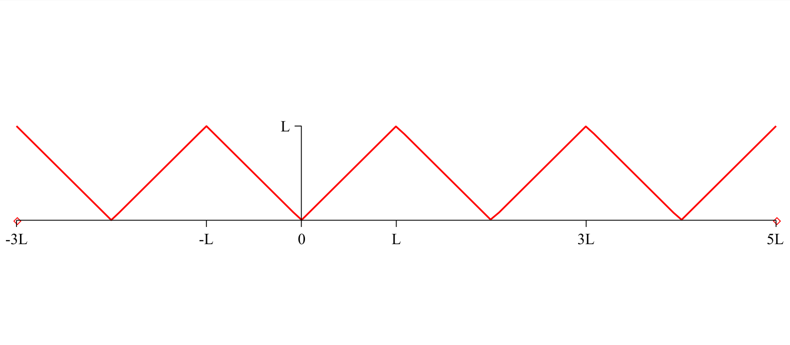



Figure: Plot of odd periodic extension (i.e., Fourier cosine series) of $f(x) = x$.

From the plots it is clear that "$\sim$" equals "$=$" in (4) for $0 \leq x \leq L$.

## Example (cont.)

Since $f$ and its Fourier cosine series are continuous we can now perform the term-by-term derivative of the Fourier cosine series from (4)

$$x \sim \frac{L}{2} - \frac{4L}{\pi^2} \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} \cos \frac{(2k-1)\pi x}{L},$$

i.e.,

$$\begin{aligned} 1 &\sim \frac{4L}{\pi^2} \sum_{k=1}^{\infty} \frac{\pi}{(2k-1)L} \sin \frac{(2k-1)\pi x}{L} \\ &= \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{1}{(2k-1)} \sin \frac{(2k-1)\pi x}{L}. \end{aligned} \tag{5}$$

## Remark

*Note that this is the Fourier sine series of $f'(x) = 1$, for $0 < x < L$.*

## Example (cont.)

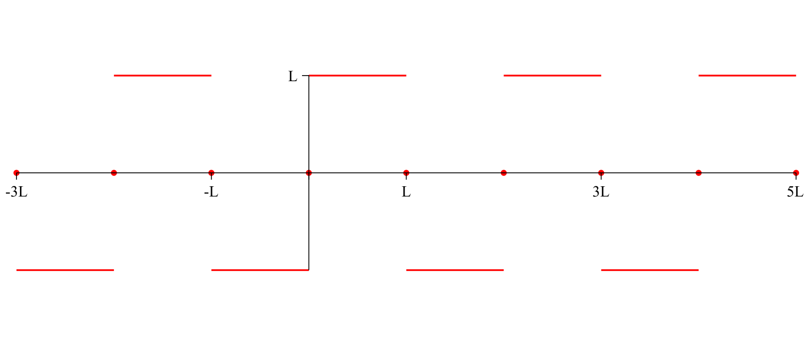



Figure: Plot of Fourier sine series of $f'(x) = 1$.

Note that due to the jumps in the graph of the Fourier sine series "$\sim$" equals "$=$" in (5) only for $0 < x < L$.

**Theorem (Differentiation of Fourier Sine Series)**

*A continuous Fourier sine series can be differentiated term-by-term provided $f'$ is piecewise smooth.*

**Remark**

*In other words, the Fourier sine series of a continuous function $f$ which satisfies $f(0) = f(L) = 0$ can be differentiated term-by-term provided $f'$ is piecewise smooth.*

**Proof.**

See the textbook on pages 120-121. □

From the proof of the theorem we get that if $f$ is continuous, but does not satisfy $f(0) = f(L) = 0$, with Fourier sine series

$$f(x) \sim \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L}$$

then, provided $f'$ is piecewise smooth, we get the Fourier cosine series

$$f'(x) \sim \frac{1}{L}\left[f(L) - f(0)\right] + \sum_{n=1}^{\infty} \left( \frac{n\pi}{L} B_n + \frac{2}{L}\left[(-1)^n f(L) - f(0)\right] \right) \cos \frac{n\pi x}{L}. \tag{6}$$

## Example

We saw earlier that for the function $f(x) = x$ we have

$$x = \frac{2L}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\frac{n\pi x}{L}, \quad 0 < x < L.$$

Since $f(0) = 0 \neq f(L) = L$ we can't expect term-by-term differentiation to yield $f'(x)$ (and we observed this in the earlier example).

However, (6) does provide the expected (and correct) answer

$$\begin{aligned} f'(x) &\sim \frac{1}{L}\left[f(L) - f(0)\right] + \sum_{n=1}^{\infty}\left(\frac{n\pi}{L}B_n + \frac{2}{L}\left[(-1)^n f(L) - f(0)\right]\right)\cos\frac{n\pi x}{L} \\ &= \frac{1}{L}\left[L - 0\right] + \sum_{n=1}^{\infty}\left(\frac{n\pi}{L}\frac{2L(-1)^{n+1}}{n\pi} + \frac{2}{L}\left[(-1)^n L - 0\right]\right)\cos\frac{n\pi x}{L} \\ &= 1 + \sum_{n=1}^{\infty}\left[\underbrace{2(-1)^{n+1} + 2(-1)^n}_{=0}\right]\cos\frac{n\pi x}{L} = 1 \end{aligned}$$

# Another Look at Separation of Variables: The Eigenfunction Perspective

By starting our discussion of the solution of the heat equation with an eigenfunction expansion we are able to obtain a justification for why the separation of variables approach works.

**Remark**

*The main advantage of taking this different point of view is that it can be applied to nonhomogeneous problems as well (see HW 3.4.9 and 3.4.12).*

## Example

Let's once more solve the 1D heat equation

$$\begin{aligned}\frac{\partial u}{\partial t} &= k\frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < L,\ t > 0\\ u(0,t) &= u(L,t) = 0\\ u(x,0) &= f(x).\end{aligned}$$

We know that the eigenfunctions for this problem are

$$\left\{\sin\frac{\pi x}{L}, \sin\frac{2\pi x}{L}, \sin\frac{3\pi x}{L}, \ldots\right\}$$

and therefore we make the *Ansatz*

$$u(x,t) = \sum_{n=1}^{\infty} B_n(t)\sin\frac{n\pi x}{L}.$$

Note the time-dependence of the Fourier sine coefficients.

## Example (cont.)

The first thing to do is to enforce the initial condition $u(x,0) = f(x)$, i.e.,

$$f(x) \sim \sum_{n=1}^{\infty} B_n(0) \sin \frac{n\pi x}{L}$$

with

$$B_n(0) = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} \mathrm{d}x, \qquad n = 1, 2, 3, \ldots$$

## Remark

*Note that now we are approaching the problem from a different angle, and so we don't know yet whether u satisfies the heat equation.*

## Example (cont.)

To check whether $u$ satisfies the heat equation we compute all the required partial derivatives using term-by-term differentiation:

$$u(x,t) \sim \sum_{n=1}^{\infty} B_n(t) \sin \frac{n\pi x}{L}$$

$$\Longrightarrow \quad \frac{\partial u}{\partial x}(x,t) \sim \sum_{n=1}^{\infty} \frac{n\pi}{L} B_n(t) \cos \frac{n\pi x}{L} \tag{7}$$

$$\Longrightarrow \quad \frac{\partial^2 u}{\partial x^2}(x,t) \sim \sum_{n=1}^{\infty} -\left(\frac{n\pi}{L}\right)^2 B_n(t) \sin \frac{n\pi x}{L} \tag{8}$$

$$\text{and} \quad \frac{\partial u}{\partial t}(x,t) \sim \sum_{n=1}^{\infty} B_n'(t) \sin \frac{n\pi x}{L} \tag{9}$$

**Example (cont.)**

Were all of these differentiations justified?

- (7) was OK since we differentiated the sine series of a continuous function (for fixed $t$) which satisfies $u(0,t) = u(L,t) = 0$.
- (8) was OK since we differentiated the cosine series of a continuous function (for fixed $t$).
- (9) was questionable. So far we have no theorem covering this case – see below.

Example (cont.)

Using (8) and (9), *u satisfies the heat equation if*

$$\sum_{n=1}^{\infty} B_n'(t) \sin \frac{n\pi x}{L} = k \sum_{n=1}^{\infty} \left[ -\left(\frac{n\pi}{L}\right)^2 B_n(t) \sin \frac{n\pi x}{L} \right].$$

Comparing coefficients of sines of like frequencies we get an ODE for the coefficients $B_n$:

$$B_n'(t) = -k \left(\frac{n\pi}{L}\right)^2 B_n(t), \qquad n = 1, 2, 3, \ldots$$

This ODE is easily solved and yields

$$B_n(t) = B_n(0) e^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

which is the same answer we had earlier using separation of variables.

We close the section with the theorem that justifies the derivation of (9) above.

**Theorem**

*If $u = u(x,t)$ is a continuous function of $t$ with time-dependent Fourier series*

$$u(x,t) = a_0(t) + \sum_{n=1}^{\infty} \left[ a_n(t) \cos \frac{n\pi x}{L} + b_n(t) \sin \frac{n\pi x}{L} \right]$$

*then*

$$\frac{\partial u}{\partial t}(x,t) = a_0'(t) + \sum_{n=1}^{\infty} \left[ a_n'(t) \cos \frac{n\pi x}{L} + b_n'(t) \sin \frac{n\pi}{L} \right]$$

*provided $\frac{\partial u}{\partial t}$ is piecewise smooth.*

## Theorem

*The Fourier series of a piecewise smooth function $f$ can always be integrated term-by-term.*
*Moreover, the result is a continuous infinite series (but not necessarily a Fourier series) which converges to the integral of $f$ on the interval $[-L, L]$, i.e., if $f$ has the Fourier series*

$$f(x) \sim a_0 + \sum_{n=1}^{\infty} \left[ a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right], \qquad -L \leq x \leq L$$

*then, for all $x \in [-L, L]$, we have*

$$\int_{-L}^{x} f(t)\mathrm{d}t = a_0(x+L) + \sum_{n=1}^{\infty} \left[ \frac{a_n L}{n\pi} \sin \frac{n\pi x}{L} + \frac{b_n L}{n\pi} \left( \cos n\pi - \cos \frac{n\pi x}{L} \right) \right]. \quad (10)$$

## Remark

*The integrals in the theorem need not be from $-L$ to $x$.*
*They can also be from $a$ to $b$, $a, b \in [-L, L]$, since we can always write*

$$\int_a^b \ldots = \int_a^{-L} \ldots + \int_{-L}^b \ldots = -\int_{-L}^a \ldots + \int_{-L}^b \ldots,$$

*and the latter two integrals are covered by the formula in the theorem.*

Before we prove the theorem we note the following facts:

$$\begin{aligned}
\int_{-L}^{x} a_0 \, \mathrm{d}t &= a_0 t \Big|_{-L}^{x} = a_0(x+L) \\
\int_{-L}^{x} \cos\frac{n\pi t}{L} \, \mathrm{d}t &= \frac{L}{n\pi} \sin\frac{n\pi t}{L} \Big|_{-L}^{x} = \frac{L}{n\pi} \sin\frac{n\pi x}{L} \\
\int_{-L}^{x} \sin\frac{n\pi t}{L} \, \mathrm{d}t &= -\frac{L}{n\pi} \cos\frac{n\pi t}{L} \Big|_{-L}^{x} = \frac{L}{n\pi} \left( \cos n\pi - \cos\frac{n\pi x}{L} \right)
\end{aligned}$$

This shows that the coefficients in formula (10) indeed are likely candidates for term-by-term integration of the Fourier series.

# Proof.

We begin by defining the function $F$ as

$$F(x) = \int_{-L}^{x} f(t)\,dt.$$

Since we assumed $f$ to be piecewise smooth, $F$ is continuous.
Its Fourier series is continuous if and only if $F(-L) = F(L)$.
However,

$$\begin{aligned} F(-L) &= \int_{-L}^{-L} f(t)\,dt = 0 \\ F(L) &= \int_{-L}^{L} f(t)\,dt = 2a_0 L, \end{aligned}$$

where $a_0$ is one of the Fourier coefficients of $f$. These are in general not the same. Therefore, the Fourier series of $F$ is not continuous in general, and we cannot assume that $F(x)$ equals its Fourier series for $-L \le x \le L$.

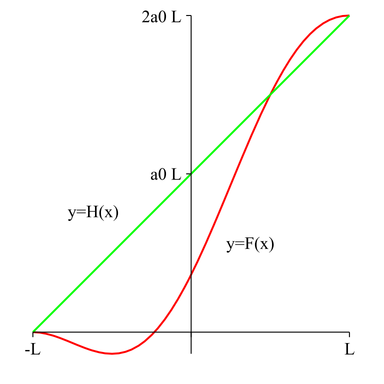
# Proof (cont.)

In addition to $F(x) = \int_{-L}^{x} f(t)\,\mathrm{d}t$ we now also define

$$H(x) = a_0(x+L)$$

and

$$G(x) = F(x) - H(x).$$




Clearly, $H$ denotes the line passing through $(-L, 0)$ and $(L, 2a_0 L)$.

As a consequence we have

- $G(-L) = G(L) = 0$ (since $F(-L) = 0$ and $F(L) = 2a_0 L$),
- $G$ is continuous (since $F$ and $H$ are)

so that $G(x)$ equals its Fourier series on $[-L, L]$.

# Proof (cont.)

Let's write the Fourier series of $G$ in the form

$$G(x) = A_0 + \sum_{n=1}^{\infty} \left[ A_n \cos \frac{n\pi x}{L} + B_n \sin \frac{n\pi x}{L} \right]$$

with (remember that $G(x) = F(x) - H(x) = F(x) - a_0(x+L)$)

$$\begin{aligned} A_0 &= \frac{1}{2L} \int_{-L}^{L} [F(x) - a_0(x+L)] \, \mathrm{d}x \\ A_n &= \frac{1}{L} \int_{-L}^{L} [F(x) - a_0(x+L)] \cos \frac{n\pi x}{L} \, \mathrm{d}x \\ B_n &= \frac{1}{L} \int_{-L}^{L} [F(x) - a_0(x+L)] \sin \frac{n\pi x}{L} \, \mathrm{d}x \end{aligned}$$

and let's compute $A_0$, $A_n$ and $B_n$.

# Proof (cont.)

$$
\begin{aligned}
A_n &= \frac{1}{L}\int_{-L}^{L}\left[F(x)-a_0(x+L)\right]\cos\frac{n\pi x}{L}\,\mathrm{d}x \\
&= \frac{1}{L}\int_{-L}^{L}\underbrace{\left[F(x)-a_0L\right]}_{=u,\ du=f(x)\mathrm{d}x}\ \underbrace{\cos\frac{n\pi x}{L}}_{=dv,\ v=\frac{L}{n\pi}\sin\frac{n\pi x}{L}}\ \mathrm{d}x-\underbrace{\frac{1}{L}\int_{-L}^{L}a_0x\cos\frac{n\pi x}{L}\,\mathrm{d}x}_{=0,\ \text{odd}} \\
&= \frac{1}{L}\left[\left(F(x)-a_0L\right)\frac{L}{n\pi}\underbrace{\sin\frac{n\pi x}{L}}_{\to 0}\Big|_{-L}^{L}-\frac{L}{n\pi}\underbrace{\int_{-L}^{L}f(x)\sin\frac{n\pi x}{L}\,\mathrm{d}x}_{=Lb_n}\right] \\
&= -\frac{L}{n\pi}b_n
\end{aligned}
$$

$$B_n = \frac{L}{n\pi}a_n$$ is computed similarly (see HW 3.5.5)

# Proof (cont.)

To compute $A_0$ we note that

$$0 = G(L) = A_0 + \sum_{n=1}^{\infty} \left[ A_n \cos \frac{n\pi L}{L} + B_n \underbrace{\sin \frac{n\pi L}{L}}_{=0} \right]$$

Therefore

$$A_0 = -\sum_{n=1}^{\infty} A_n \cos n\pi = \sum_{n=1}^{\infty} \frac{L}{n\pi} b_n \cos n\pi.$$

Putting everything together we get

$$\begin{aligned} F(x) &= H(x) + G(x) \\ &= a_0(x+L) + A_0 + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{L} + B_n \sin \frac{n\pi x}{L} \end{aligned}$$

which matches the claim of the theorem if we use the representations of $A_0$, $A_n$ and $B_n$. □

## Example

Integrate the following Fourier cosine series (see (4)) from 0 to $x$:

$$x = \frac{L}{2} - \frac{4L}{\pi^2} \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} \cos \frac{(2k-1)\pi x}{L}$$

## Solution

We immediately have

$$\int_0^x t \, \mathrm{d}t = \frac{x^2}{2}$$

and

$$\int_0^x \frac{L}{2} \mathrm{d}t = \frac{Lx}{2}$$

**Solution** (cont.)

The remaining part becomes

$$\frac{4L}{\pi^2}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2}\int_0^x \cos\frac{(2k-1)\pi t}{L}\mathrm{d}t = \frac{4L^2}{\pi^3}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^3}\sin\frac{(2k-1)\pi t}{L}\Big|_0^x$$
$$= \frac{4L^2}{\pi^3}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^3}\sin\frac{(2k-1)\pi x}{L}$$

Putting all three parts together we have

$$x^2 = Lx - \frac{8L^2}{\pi^3}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^3}\sin\frac{(2k-1)\pi x}{L}.$$

Note that this is in agreement with the statement of the theorem. Due to the presence of the linear term $Lx$ this is not a Fourier (sine) series.

Solution (cont.)

We can interpret

$$x^2 = Lx - \frac{8L^2}{\pi^3} \sum_{k=1}^{\infty} \frac{1}{(2k-1)^3} \sin \frac{(2k-1)\pi x}{L}$$

differently. Namely, we do have the following two sine series:

- $$Lx - x^2 = \frac{8L^2}{\pi^3} \sum_{k=1}^{\infty} \frac{1}{(2k-1)^3} \sin \frac{(2k-1)\pi x}{L}$$
- and, using the Fourier sine series of $f(x) = x$ (see (1)),

$$x^2 = L\frac{2L}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{k} \sin \frac{n\pi x}{L} - \frac{8L^2}{\pi^3} \sum_{k=1}^{\infty} \frac{1}{(2k-1)^3} \sin \frac{(2k-1)\pi x}{L}$$

## Example

Use the fact – established earlier (see (5)) – that

$$1 = \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{1}{2k-1} \sin \frac{(2k-1)\pi x}{L}, \qquad 0 < x < L$$

to show that

$$\sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \ldots = \frac{\pi^2}{8}.$$

## Solution

The main idea is to integrate the given identity from 0 to $x$. For the left-hand side we have

$$\int_0^x 1\,\mathrm{d}t = x,$$

while the right-hand side is

$$\begin{aligned}
\int_0^x \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{1}{2k-1}\sin\frac{(2k-1)\pi t}{L}\,\mathrm{d}t &= \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{1}{2k-1}\int_0^x \sin\frac{(2k-1)\pi t}{L}\,\mathrm{d}t \\
&= -\frac{4L}{\pi^2}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2}\cos\frac{(2k-1)\pi t}{L}\Big|_0^x \\
&= \frac{4L}{\pi^2}\sum_{k=1}^{\infty}\left[\frac{1}{(2k-1)^2}-\frac{1}{(2k-1)^2}\cos\frac{(2k-1)\pi x}{L}\right]
\end{aligned}$$

Solution (cont.)

Therefore, splitting into two series,

$$x = \frac{4L}{\pi^2}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2} - \frac{4L}{\pi^2}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2}\cos\frac{(2k-1)\pi x}{L}.$$

This is a cosine series with constant term

$$A_0 = \frac{4L}{\pi^2}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2} = \frac{1}{L}\int_0^L x\,\mathrm{d}x = \frac{L}{2},$$

and we can now conclude that

$$\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2} = \frac{\pi^2}{4L}\frac{L}{2} = \frac{\pi^2}{8}.$$

Solution (cont.)

Alternatively, we could have evaluated the series expansion

$$x = \frac{4L}{\pi^2}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2} - \frac{4L}{\pi^2}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2}\cos\frac{(2k-1)\pi x}{L}$$

for some special value of $x$. For example,

- for $x = L$ we get

$$\begin{aligned} L &= \frac{4L}{\pi^2}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2} - \frac{4L}{\pi^2}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2}\cos\frac{(2k-1)\pi L}{L} \\ &= \frac{4L}{\pi^2}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2} - \frac{4L}{\pi^2}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2}\underbrace{\cos(2k-1)\pi}_{=-1} \end{aligned}$$

  so that

$$L = 2\frac{4L}{\pi^2}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2} \quad\Longleftrightarrow\quad \sum_{k=1}^{\infty}\frac{1}{(2k-1)^2} = \frac{\pi^2}{8}.$$

## Solution (cont.)

- For $x = \frac{L}{2}$ we get

$$\begin{aligned}\frac{L}{2} &= \frac{4L}{\pi^2}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2} - \frac{4L}{\pi^2}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2}\cos\frac{(2k-1)\pi\frac{L}{2}}{L} \\ &= \frac{4L}{\pi^2}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2} - \frac{4L}{\pi^2}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2}\underbrace{\cos(2k-1)\frac{\pi}{2}}_{=0}\end{aligned}$$

so that

$$\frac{L}{2} = \frac{4L}{\pi^2}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2} \iff \sum_{k=1}^{\infty}\frac{1}{(2k-1)^2} = \frac{\pi^2}{8}.$$


## Example

The Basel problem, first proved by Leonhard Euler in 1735:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

One can prove this as we did for the squares of odd integers above. Here we evaluate the Fourier series of $f(x) = x^2$,

$$x^2 = \frac{L^2}{3} + \frac{4L^2}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos \frac{n\pi x}{L},$$

at $x = L$.

See [Proofs from THE BOOK] for three different proofs.

Fourier series are often expressed in terms of complex exponentials instead of sines and cosines.
The main ingredient for understanding this translation in notation is Euler's formula

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

This, of course, implies

$$e^{-i\theta} = \cos\theta - i\sin\theta,$$

and so

$$\begin{aligned} \cos\theta &= \frac{e^{i\theta} + e^{-i\theta}}{2} \\ \sin\theta &= \frac{e^{i\theta} - e^{-i\theta}}{2i}. \end{aligned}$$

We can therefore rewrite the Fourier series

$$f(x) \sim a_0 + \sum_{n=1}^{\infty} \left[ a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right]$$

as

$$\begin{aligned} f(x) &\sim a_0 + \sum_{n=1}^{\infty} \left[ a_n \frac{e^{\mathrm{i}\frac{n\pi x}{L}} + e^{-\mathrm{i}\frac{n\pi x}{L}}}{2} + b_n \frac{e^{\mathrm{i}\frac{n\pi x}{L}} - e^{-\mathrm{i}\frac{n\pi x}{L}}}{2\mathrm{i}} \right] \\ &= a_0 + \frac{1}{2} \sum_{n=1}^{\infty} \left[ \left( a_n + \frac{b_n}{\mathrm{i}} \right) e^{\mathrm{i}\frac{n\pi x}{L}} + \left( a_n - \frac{b_n}{\mathrm{i}} \right) e^{-\mathrm{i}\frac{n\pi x}{L}} \right] \end{aligned}$$

We break this into two series and use $\frac{1}{\mathrm{i}} = -\mathrm{i}$ to arrive at

$$f(x) \sim a_0 + \frac{1}{2}\sum_{n=1}^{\infty}(a_n - \mathrm{i}b_n)\,e^{\mathrm{i}\frac{n\pi x}{L}} + \frac{1}{2}\sum_{n=1}^{\infty}(a_n + \mathrm{i}b_n)\,e^{-\mathrm{i}\frac{n\pi x}{L}}$$

Now we perform an index transformation, $n \to -n$, on the first series to get

$$f(x) \sim a_0 + \frac{1}{2}\sum_{n=-1}^{-\infty}(a_{-n} - \mathrm{i}b_{-n})\,e^{-\mathrm{i}\frac{n\pi x}{L}} + \frac{1}{2}\sum_{n=1}^{\infty}(a_n + \mathrm{i}b_n)\,e^{-\mathrm{i}\frac{n\pi x}{L}}$$

Note that, using the symmetries of cosine and sine,

$$\begin{aligned} a_{-n} &= \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{(-n)\pi x}{L}\,\mathrm{d}x = a_n \\ b_{-n} &= \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{(-n)\pi x}{L}\,\mathrm{d}x = -b_n \end{aligned}$$

We can therefore rewrite

$$f(x) \sim a_0 + \frac{1}{2}\sum_{n=-1}^{-\infty}(a_{-n} - \mathrm{i}b_{-n})\,e^{-\mathrm{i}\frac{n\pi x}{L}} + \frac{1}{2}\sum_{n=1}^{\infty}(a_n + \mathrm{i}b_n)\,e^{-\mathrm{i}\frac{n\pi x}{L}}$$

as

$$f(x) \sim a_0 + \frac{1}{2}\sum_{n=-1}^{-\infty}(a_n + \mathrm{i}b_n)\,e^{-\mathrm{i}\frac{n\pi x}{L}} + \frac{1}{2}\sum_{n=1}^{\infty}(a_n + \mathrm{i}b_n)\,e^{-\mathrm{i}\frac{n\pi x}{L}}$$

If we introduce new coefficients

$$c_0 = a_0 \qquad \text{and} \qquad c_n = \frac{a_n + \mathrm{i}b_n}{2}$$

then we get the exponential form of the Fourier series

$$f(x) \sim \sum_{n=-\infty}^{\infty} c_n e^{-\mathrm{i}\frac{n\pi x}{L}}$$

with Fourier coefficients

$$c_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,\mathrm{d}x$$

and

$$
\begin{aligned}
c_n &= \frac{a_n + \mathrm{i} b_n}{2} \\
&= \frac{1}{2L}\left[\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,\mathrm{d}x + \mathrm{i}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,\mathrm{d}x\right] \\
&= \frac{1}{2L}\int_{-L}^{L} f(x)\left[\cos\frac{n\pi x}{L} + \mathrm{i}\sin\frac{n\pi x}{L}\right]\mathrm{d}x \\
&= \frac{1}{2L}\int_{-L}^{L} f(x) e^{\mathrm{i}\frac{n\pi x}{L}}\,\mathrm{d}x
\end{aligned}
$$

Note that this formula also gives the correct value for $c_0$.

**Remark**

*Sometimes the formula for the Fourier coefficients $c_n$ is referred to as the finite Fourier transform of $f$.*

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