# MATH 590: Meshfree Methods

## Chapter 36: Generalized Hermite Interpolation

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# Outline

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- This problem was not further investigated in the RBF literature until [Wu (1992)].
- Since then, the interest in this topic has increased significantly.
- In particular, since there is a close connection between the generalized Hermite interpolation approach and symmetric collocation for elliptic partial differential equations (see Chapter 38).

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- Hermite interpolation with conditionally positive definite functions is also discussed in [Iske (1995)].
- A number of authors have also considered the Hermite interpolation setting on spheres (see, e.g., [F. (1999), Freeden (1982), Freeden (1987), Ron and Sun (1996)]) or even general Riemannian manifolds [Dyn *et al.* (1999), Narcowich (1995)].

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We now consider data $\{\boldsymbol{x}_i, \lambda_i f\}$, $i = 1, \ldots, N$, $\boldsymbol{x}_i \in \mathbb{R}^s$, where $\Lambda = \{\lambda_1, \ldots, \lambda_N\}$ is a linearly independent set of continuous linear functionals and $f$ is some (smooth) data function.

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### Remark

*We stress that there is no assumption that requires the derivatives to be in consecutive order as is usually the case for polynomial or spline-type Hermite interpolation problems.*

We try to find an interpolant of the form

$$\mathcal{P}_f(\boldsymbol{x}) = \sum_{j=1}^{N} c_j \psi_j(\|\boldsymbol{x}\|), \qquad \boldsymbol{x} \in \mathbb{R}^s, \tag{1}$$

with appropriate (radial) basis functions $\psi_j$ so that $\mathcal{P}_f$ satisfies the generalized interpolation conditions

$$\lambda_i \mathcal{P}_f = \lambda_i f, \qquad i = 1, \ldots, N.$$

To keep the discussion that follows as transparent as possible we now introduce the notation $\xi_1, \ldots, \xi_N$ for the centers of the radial basis functions.

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As we will see below, it is natural to let

$$\psi_j(\|\boldsymbol{x}\|) = \lambda_j^{\boldsymbol{\xi}} \varphi(\|\boldsymbol{x} - \boldsymbol{\xi}\|)$$

with the same functionals $\lambda_j$ that generated the data and $\varphi$ one of the usual radial basic functions.

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However, the notation $\lambda^{\xi}$ indicates that the functional $\lambda$ now acts on $\varphi$ viewed as a function of its second argument $\xi$.
We will not add any superscript if $\lambda$ acts on a single variable function or on the kernel $\varphi$ as a function of its first variable.

Therefore, we assume the generalized Hermite interpolant to be of the form

$$\mathcal{P}_f(\boldsymbol{x}) = \sum_{j=1}^{N} c_j \lambda_j^{\boldsymbol{\xi}} \varphi(\|\boldsymbol{x} - \boldsymbol{\xi}\|), \qquad \boldsymbol{x} \in \mathbb{R}^s, \tag{2}$$

and require it to satisfy

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The linear system $\mathsf{A}\boldsymbol{c} = \boldsymbol{f}_\lambda$ which arises in this case has matrix entries

$$\mathsf{A}_{ij} = \lambda_i \lambda_j^{\boldsymbol{\xi}} \varphi, \qquad i, j = 1, \ldots, N, \tag{3}$$

and right-hand side $\boldsymbol{f}_\lambda = [\lambda_1 \boldsymbol{f}, \ldots, \lambda_N \boldsymbol{f}]^T$.

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- *Since the entries of the interpolation matrix* A *are* $\mathsf{A}_{ij} = \lambda_i \lambda_j^{\xi} \varphi$ *we need to use* $C^{2k}$ *functions in order to interpolate* $C^k$ *data.*

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- *The formulation in (2) is very general and goes considerably beyond the standard notion of Hermite interpolation (which refers to interpolation of successive derivative values only).*
    - *Any kind of linear functionals are allowed as long as the set $\Lambda$ is linearly independent.*
    - *In Chapter 38 we apply this formulation to the solution of PDEs.*

One could also envision use of a simpler RBF expansion of the form

$$\mathcal{P}_f(\boldsymbol{x}) = \sum_{j=1}^{N} c_j \varphi(\|\boldsymbol{x} - \boldsymbol{\xi}_j\|), \qquad \boldsymbol{x} \in \mathbb{R}^s.$$

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**Remark**

*Nevertheless, this approach is frequently used for the solution of elliptic PDEs (see Kansa's method in Chapter 38), and it is known that for certain configurations of the collocation points and certain differential operators the system matrix does indeed become singular.*

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### Theorem

*Suppose that $K \in L_1(\mathbb{R}^s) \cap C^{2k}(\mathbb{R}^s)$ is a strictly positive definite kernel. If the functionals $\lambda_j = \delta_{\boldsymbol{x}_j} \circ D^{\boldsymbol{\alpha}^{(j)}}$, $j = 1, \ldots, N$, with multi-indices $|\boldsymbol{\alpha}^{(j)}| \leq k$ are pairwise distinct, meaning that $\boldsymbol{\alpha}^{(j)} \neq \boldsymbol{\alpha}^{(\ell)}$ if $\boldsymbol{x}_j = \boldsymbol{x}_\ell$ for different $j \neq \ell$, then they are also linearly independent over the native space $\mathcal{N}_K(\mathbb{R}^s)$.*

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# Outline

We now illustrate the Hermite interpolation approach with a simple 2D example using first-order partial derivative functionals.

### Example

Given: data $\{\boldsymbol{x}_i, f(\boldsymbol{x}_i)\}_{i=1}^{n}$ and $\{\boldsymbol{x}_i, \frac{\partial f}{\partial x}(\boldsymbol{x}_i)\}_{i=n+1}^{N}$ with $\boldsymbol{x} = (x, y) \in \mathbb{R}^2$.

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$$\lambda_i = \begin{cases} \delta_{\boldsymbol{x}_i}, & i = 1, \ldots, n, \\ \delta_{\boldsymbol{x}_i} \circ \frac{\partial}{\partial x}, & i = n+1, \ldots, N. \end{cases}$$

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After enforcing the interpolation conditions the system matrix is given by

$$\mathsf{A} = \begin{bmatrix} \tilde{\mathsf{A}} & \tilde{\mathsf{A}}_{\xi} \\ \tilde{\mathsf{A}}_{x} & \tilde{\mathsf{A}}_{x\xi} \end{bmatrix}$$

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\end{aligned}$$

The blocks $\tilde{\mathsf{A}}_\xi$ and $\tilde{\mathsf{A}}_x$ are identical if the data sites and centers coincide since the sign change due to differentiation with respect to the second variable in $\tilde{\mathsf{A}}_\xi$ is cancelled by the interchange of the roles of $\boldsymbol{x}_i$ and $\boldsymbol{\xi}_j$ when compared to $\tilde{\mathsf{A}}_x$.

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## Remark

*Note that the partial derivative of $\varphi$ with respect to the coordinate $x$ will always contain a linear factor in $x$, i.e., (for the 2D example considered here)* $\varphi(\|\boldsymbol{x}\|) = \varphi(r) = \varphi(\sqrt{x^2 + y^2})$, *so that by the chain rule*

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$$
\begin{aligned}
\frac{\partial}{\partial x}\varphi(\|\boldsymbol{x}\|) &= \frac{d}{dr}\varphi(r)\frac{\partial}{\partial x}r(x,y) \\
&= \frac{d}{dr}\varphi(r)\frac{x}{\sqrt{x^2+y^2}} \\
&= \frac{d}{dr}\varphi(r)\frac{x}{r} \qquad (4)
\end{aligned}
$$

*since $r = \|\boldsymbol{x}\| = \sqrt{x^2 + y^2}$.*

## Remark

*Note that the partial derivative of $\varphi$ with respect to the coordinate $x$ will always contain a linear factor in $x$, i.e., (for the 2D example considered here) $\varphi(\|\boldsymbol{x}\|) = \varphi(r) = \varphi(\sqrt{x^2+y^2})$, so that by the chain rule*

$$\begin{aligned}
\frac{\partial}{\partial x}\varphi(\|\boldsymbol{x}\|) &= \frac{d}{dr}\varphi(r)\frac{\partial}{\partial x}r(x,y) \\
&= \frac{d}{dr}\varphi(r)\frac{x}{\sqrt{x^2+y^2}} \\
&= \frac{d}{dr}\varphi(r)\frac{x}{r} \qquad (4)
\end{aligned}$$

*since $r = \|\boldsymbol{x}\| = \sqrt{x^2+y^2}$.*
*This argument generalizes for any odd-order derivative.*

Note that the matrix A is also symmetric for even-order derivatives. For example, one can easily verify that

$$\frac{\partial^2}{\partial x^2}\varphi(\|\boldsymbol{x}\|) = \frac{1}{r^2}\left(x^2\frac{d^2}{dr^2}\varphi(r) + \frac{y^2}{r}\frac{d}{dr}\varphi(r)\right),$$

so that now the interchange of $\boldsymbol{x}_i$ and $\xi_j$ does not cause a sign change. On the other hand, two derivatives of $\varphi$ with respect to the second variable $\xi$ do not lead to a sign change, either.

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**Remark**

*A catalog of RBFs and their derivatives is provided in Appendix D.*

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