REVIEWS

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A Course in Approximation Theory. By Ward Cheney and Will Light. Brooks/Cole, Pacific Grove, CA, 1999, xiv + 359 pp., ISBN 0-534-36224-9, $107.95.

Reviewed by G. E. Fasshauer

Finally, someone who was up to the challenge. Ward Cheney and Will Light, the authors of *A Course in Approximation Theory*, have managed to fill three major gaps in the existing literature on multivariate approximation by daring to become pioneers and writing a very special book. They have produced the first book surveying such a large variety of topics in this vast and active area of mathematical research, and the first one with an extensive treatment of meshfree approximation. Moreover, they wrote it as a textbook rather than a research monograph.

Approximation theory is a rather specialized area of mathematics that is practiced as a stand-alone subject by only a small number of mathematicians. However, the issues investigated in approximation theory are of fundamental importance for a wide array of applications ranging from numerical analysis and the solution of PDEs in many areas of science and engineering to computer graphics, data mining, and artificial intelligence. Therefore, approximation theory is actually practiced to some extent by a wide variety of mathematicians, scientists, and engineers.

The dramatic advances in technology over the last few decades have enabled us to use computers to model ever more complex problems. One of the ways in which the complexity has increased is in terms of the number of parameters involved in a computer simulation of a specific problem. "Real world" problems in science and engineering are usually posed in three-dimensional space. However, for applications in artificial intelligence or mathematical finance the "space dimension" can easily go into the hundreds or even thousands. Computer modelling of such problems requires a sound understanding of multivariate approximation theory.

Traditional numerical methods (motivated mostly by early low-dimensional simulations of engineering problems via PDEs) were based on finite element, finite difference, or finite volume methods. All of these methods require some sort of underlying mesh (e.g., a triangulation of the region of interest) for the computation. Creation of these meshes (and possible re-meshing) becomes a rather difficult task in three dimensions, and virtually impossible for higher-dimensional problems. This is where *meshfree* methods enter the picture. Meshfree methods are often radially symmetric in nature. This is achieved by composing some univariate basis function with a (Euclidean) norm, and therefore turning a problem involving many space dimensions into one that is virtually one-dimensional. These *radial basis functions* are one of the main topics discussed in the Cheney/Light book.

While several books on some specialized aspect of multivariate approximation have been available for some time (see, for example, [**4**], [**5**]), no one has made the effort to provide us with a (text)book covering such a wide range of topics. Only in the last two years (following the publication of *A Course in Approximation Theory*) have a few books on meshfree methods appeared on the market (see [**1**], [**2**], [**6**], [**8**], [**10**]). However, again, the focus of these books is rather narrow. None of them are textbooks, and two of them [**1**], [**10**] are intended mostly for an engineering audience.

The book contains thirty-six chapters that start with a description of interpolation (mostly by multivariate polynomials), then move via the discussion of interpolation by translates of a single function to interpolation using strictly positive definite functions and radial basis functions. This constitutes roughly the first half of the book. The second half consists of a collection of more specialized topics such as approximation by convolution, ridge functions, neural networks, optimal recovery and the Golomb-Weinberger theory, reproducing kernel Hilbert spaces, box splines, wavelets, and quasi-interpolation. Most of this material concerns multivariate approximation; the principal exception is the final two chapters on wavelets. The 28-page bibliography containing over five hundred references is a good indication of how many different aspects of modern approximation theory have been addressed in this book.

Some of the problems associated with pioneering work. One of the difficulties associated with writing any kind of book on topics in an area of active research is that, inevitably, new results will be available by the time the book appears. That is certainly true in this case (especially since four years have passed since the date of publication). For example, recent developments for radial basis functions include a complete variational theory for all (conditionally) positive definite functions within a reproducing kernel Hilbert space framework (the latter being very nicely covered in the text). Will Light himself was a contributor to this theory until his sad and much too early death in December 2002 at the age of fifty-two. In the Cheney/Light book the variational theory is discussed only in the context of interpolation by radial functions on spheres. On page 246 the authors remark that "Historically this is not the way the variational theory of radial basis functions developed. Early work of Duchon focused on subspaces of $C(\mathbb{R}^s)$. This theory is best understood with the aid of techniques from distribution theory. The corresponding theory for $\mathcal{S}^{s-1}$ [the unit sphere in $\mathbb{R}^s$] is more elementary." Thanks to the recent advances just mentioned, this is no longer true.

Thin plate splines are one of the best known and most widely used radial basis functions, but they receive little mention in this book. The term "(spherical) thin plate spline" appears in the title of chapter 32, but its use is a little misleading as it seems to refer to any kind of spherical radial basis function covered by the variational theory. The specific spherical thin plate spline studied originally by Wahba and Freeden is never presented, nor is the classical Duchon thin plate spline

$$\sum_{j=1}^{n} c_j \|x - x_j\|^2 \log \|x - x_j\|.$$

This is a consequence of the decision not to go into the distributional framework for conditionally positive definite functions. (Luckily, the special case of conditionally positive definite functions of order one—which includes multiquadrics—is covered by the Micchelli Interpolation Theorem of chapter 16.)

Another area that has seen much progress since publication of the book is fast computation with radial basis functions, using methods based on multipole expansions and domain decompositions. (Again, Will Light was one of the people who helped expand this frontier.) Figure 1 is a beautiful illustration of what can be accomplished with these recent advances. It shows a so-called minimum energy radial basis function approximation along with the underlying point data. The size of the problem is remarkable (the original data set contains more than three million points). Another benefit of the

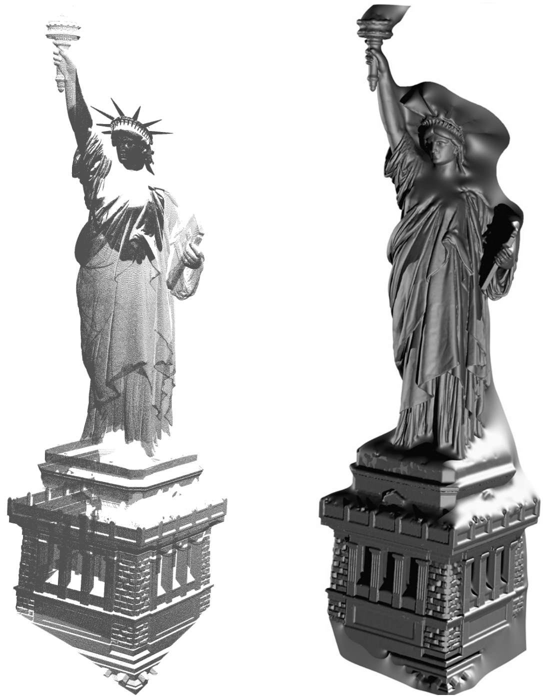

Figure 1. Cloud of 3,360,000 points from a Lidar scan of the Statue of Liberty (left); minimum energy radial basis function implicit surface fit (right).

minimum energy radial basis function is its ability to approximate accurately and also to cover holes smoothly (notice the data-free regions in the left part of the figure; these holes occur because regions of the statue were obscured from the limited number of scanning positions). The exact procedure for obtaining a minimum energy implicit surface fit with radial basis functions is described in the recent paper [**3**]. (I am grateful to Cyra Technologies for providing the Lidar scan data, and Applied Research Associates NZ Limited for the fitted radial basis function implicit surface.)

Some other topics that could have been included. The authors point out in the preface that the choice of topics was influenced by their own interests. However, it would have been nice to at least have pointers to other important topics that were omitted. Given that Ward Cheney has authored two successful textbooks on numerical analysis together with David Kincaid, it is somewhat surprising that the computational aspects of the theory were mostly ignored. It has been known for quite some time that the matrices associated with radial basis function interpolation based on the functions covered in the text (see, for example, the examples listed in chapters 13 and 16) are notoriously ill-conditioned. The theoretical analysis of this fact (which depends on the separation distance, i.e., the minimal separation of any two data locations) could have been included, although real progress on preconditioning interpolation matrices has been made only since the time of publication. Another topic intimately tied to the ill-conditioning problem by the so-called trade-off principle is the order of approximation (specified in terms of the fill distance, i.e., the radius of the largest ball containing no data sites that can be placed among the underlying data). Such results come naturally out of the variational theory based on reproducing kernel Hilbert spaces.

Another notable omission is compactly supported radial basis functions. They offer a completely different approach to meshfree approximation. In this case the interpolation matrices can be made sparse and well-conditioned. However, in order to achieve good convergence one usually embeds this approach into a multilevel algorithm that iterates on the residuals of the data computed on finer and finer sets (an idea that can be nicely compared to the wavelet approach). It might have been more fitting to replace the wavelet chapters by a discussion of compactly supported radial basis functions, resulting in a book dedicated entirely to multivariate approximation.

Of course multivariate approximation includes methods other than radial basis functions. The authors have included chapters on multivariate polynomials, tensor-product interpolation, ridge functions, and box splines, as well as approximation by convolution and quasi-interpolation. Again, a number of methods are missing from this list. For example, box splines are only one way of generalizing the idea of univariate splines to higher dimensions; other possibilities include splines on triangulations or tetrahedralizations, simplex splines, and polysplines.

Another powerful multivariate approximation technique not discussed is the moving least squares method. This method has become rather popular in the engineering community, and has been featured in various books [**1**], [**9**], [**10**]. The moving least squares method has also been mentioned in the numerical analysis textbook [**7**] by David Kincaid and Ward Cheney since its first edition of 1991.

A few words of caution. I wish to warn potential readers about the degree of difficulty of some of the material presented in this book. From a superficial browsing of the preface one might get the impression that not much background knowledge is required to follow the presentation. However, someone who has never heard of Borel measures, Hausdorff spaces, Hilbert spaces, L^p-spaces, the Hahn-Banach theorem, the Riesz representation theorem, or Fourier transforms and the Plancherel theorem may feel a little lost. A number of sections get rather technical. For example, chapters 13 (on strictly positive definite functions) and 18 (on approximation by positive definite functions) use quite a bit of measure theory, and the Riesz representation theorem plays an important role in chapters 18, 30 (on the Golomb-Weinberger theory), and 31 (on reproducing kernel Hilbert spaces). Chapter 32, on the other hand, asks the reader to delve into spherical harmonics. Most of these topics are not in your standard undergraduate curriculum (at least not the ones I've seen). Therefore, when the authors say in the preface, "we start with relatively elementary matters in a series of about ten short chapters that do not, in general, require more of the reader than undergraduate mathematics (in the American university system). From that point on, the gradient gradually increases and the text becomes more demanding," then that is exactly what they mean. This is definitely not a textbook for beginning graduate students.

The problems at the end of each chapter are a little on the challenging side, with the high point given by the three "research problems" listed at the end of chapter 11 and the problems 10 in chapter 15 and 2 in chapter 22, which are identified as *open* problems in the file `openprobs` posted on Ward Cheney's anonymous FTP site ftp://ftp.ma.utexas.edu/pub/papers/cheney/ATBOOK/. (This FTP site also contains a list of errata.)

What I might have changed. I think some of the material could have been arranged in a more natural way. There are two chapters (17 and 32) that deal with (conditionally) positive definite functions on spheres, but the authors do not seem to have made much of an effort to unify the two. For example, no attempt is made to match the Gegenbauer polynomials of chapter 32 with those of chapter 17. Why wait until chapter 32 to talk about general mth order conditionally positive definite functions when chapter 17 already goes into a lot of details for the $m = 0$ and $m = 1$ case? And then it would be nice to have a discussion of how Theorem 1 of chapter 17 relates to Theorem 4 of chapter 32. There could be more such comments along with examples in the entire book. Such things may not be necessary in a research paper, but a textbook should strive to provide the reader with insights.

I would also consider moving chapters 28, 29, and 36. Chapter 28 ("Algorithmic Orthogonal Projections") should be moved closer to chapter 6 ("Projections"), to which it is closely related. And chapter 29 ("Cardinal B-Splines and the Sinc Function") has connections to chapters 33 ("Box Splines") and 36 ("Quasi-Interpolation"), so it could profitably be moved to a later place in the book. By moving chapters 28 and 29 one would also achieve greater continuity in the discussion on optimal recovery (chapter 27), the Golomb-Weinberger theory (chapters 30), and reproducing kernel Hilbert spaces (chapter 31). Chapter 36 could have been placed before the two chapters on wavelets (34 and 35) putting it closer to the related chapters 29 and 33.

Closing remarks. As indicated at the beginning of this review, I admire and thank Ward Cheney and Will Light for writing this book, especially since they produced not just a research monograph about their own work but a textbook covering a wide array of topics in modern approximation theory. Let's hope that this book will inspire others to follow in their footsteps.

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