A unified approach to time consistency of dynamic risk measures and dynamic performance measures in discrete time

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\textbf{Abstract}: In this paper we provide a flexible framework allowing for a unified study of time consistency of risk measures and performance measures, also known as acceptability indices. The proposed framework integrates existing forms of time consistency. In our approach the time consistency is studied for a large class of maps that are postulated to satisfy only two properties – monotonicity and locality. The time consistency is defined in terms of an update rule – a novel notion introduced in this paper. As an illustration of the usefulness of our approach, we show how to recover almost all concepts of weak time consistency by means of constructing various update rules.

\textbf{Keywords}: time consistency, update rule, dynamic LM-measure, dynamic risk measure, dynamic acceptability index, dynamic performance measure.

\textbf{MSC2010}: 91B30, 62P05, 97M30, 91B06.

\section{Introduction}

In the seminal paper by Artzner et al. (1999), the authors proposed an axiomatic approach to defining risk measures that are meant to give a numerical value of the riskiness of a given financial contract or portfolio. Alternatively, one can view the risk measures as a tool that allows to establish preference orders on the set of cashflows according to their riskiness. Another seminal paper, Cherny and Madan (2009), introduced and studied axiomatic approach to defining performance measures, or acceptability indices, that are meant to provide evaluation of performance of a financial portfolio. In their most native form, performance measures evaluate the trade-off between return on the portfolio and the portfolio’s risk. Both Artzner et al. (1999) and Cherny and Madan (2009) were concerned with measures of risk and measures of performance in static framework.

As shown in one of the first papers that studied risk measures in dynamic framework, Riedel (2004), if one is concerned about making noncontradictory decisions (from the risk point of view) over the time, then an additional axiom, called time consistency, is needed. Over the past decade significant progress has been made towards expanding the theory of dynamic risk measures and their time consistency. For example, so called cocycle condition (for convex risk measures) was studied in Föllmer and Penner (2006), recursive construction was exploited in Cheridito and Kupper (2011),

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relation to acceptance and rejection sets was studied in Delbaen (2006), the concept of prudence was introduced in Penner (2007), connections to g-expectations were studied in Rosazza Gianin (2006), and the relation to Bellman’s principle of optimality was shown in Artzner et al. (2007).

For more details on dynamic cash-additive measures also called dynamic monetary risk measures, we also refer the reader to a comprehensive survey paper Acciaio and Penner (2011) and the references therein.

Let us briefly recall the concept of strong time consistency of monetary risk measures, which is one of the most recognized forms of time consistency. Assume that \( \rho_t(X) \) is the value of a dynamic monetary risk measure at time \( t \in [0, T] \), that corresponds to the riskiness, at time \( t \), of the terminal cashflow \( X \), with \( X \) being an \( \mathcal{F}_T \)-measurable random variable. The monetary risk measure is said to be strongly time consistent if for any \( t < s \leq T \), and any \( \mathcal{F}_T \)-measurable random variables \( X,Y \) we have that

\[
\rho_s(X) = \rho_s(Y) \quad \Rightarrow \quad \rho_t(X) = \rho_t(Y).
\] (1.1)

The financial interpretation of strong time consistency is clear – if \( X \) is as risky as \( Y \) at some future time \( s \), then today, at time \( t \), \( X \) is also as risky as \( Y \). One of the main features of the strong time consistency is its connection to dynamic programming principle. It is not hard to show that in the \( L^\infty \) framework, a monetary risk measure is strongly time consistent if and only if

\[
\rho_t = \rho_t(-\rho_s), \quad 0 \leq t < s \leq T.
\] (1.2)

All other forms of time consistency for monetary risk measures, such as weak, acceptance consistent, rejection consistent, are tied to this connection as well. In Tutsch (2008), the author proposed a general approach to time consistency for cash-additive risk measures by introducing so called ‘test sets’ or ‘benchmark sets.’ Each form of time consistency was associated to a benchmark set of random variables, and larger benchmark sets correspond to stronger forms of time consistency.

The first study of time consistency of dynamic performance measures is due to Bielecki et al. (2014b), where the authors elevated the theory of coherent acceptability indices to dynamic setup in discrete time. It was pointed out that none of the forms of time consistency for risk measures is suitable for acceptability indices. Recursive property similar to (1.2), or the benchmark sets approach essentially can not be applied to scale invariant maps such as acceptability indices. One of the specific features of the acceptability indices, that needed to be accounted for in study of their time consistency, was that these measures of performance can take infinite value. In particular, this required extending the analysis beyond the \( L^\infty \) framework.

Consequently, one of the main challenge was to find an appropriate form of time consistency of acceptability indices, that would be both financially reasonable and mathematically tractable. For the case of random variables (terminal cashflows), the proposed form of time consistency for a dynamic coherent acceptability index \( \alpha \) reads as follows: for any \( \mathcal{F}_t \)-measurable random variables \( m_t, \; n_t \), and any \( t < T \), the following implications hold

\[
\alpha_{t+1}(X) \geq m_t \quad \Rightarrow \quad \alpha_t(X) \geq m_t,
\]

\[
\alpha_{t+1}(X) \leq n_t \quad \Rightarrow \quad \alpha_t(X) \leq n_t.
\] (1.3)
The financial interpretation is also clear – if tomorrow $X$ is acceptable at least at level $m_t$, then today $X$ is also acceptable at least at level $m_t$; similar interpretation holds true for the second part of (1.3). It is fair to say, we think, that dynamic acceptability indices and their time consistency properties play a critical role in so called conic approach to valuation and hedging of financial contracts Bielecki et al. (2013); Rosazza Gianin and Sgarra (2013).

We recall that both risk measures and performance measures, in the nutshell, put preferences on the set of cashflows. While the corresponding forms of time consistency (1.1) and (1.3) for these classes of maps, as argued above, are different, we note that generally speaking both forms of time consistency are linking preferences between different times. The aim of this paper is to present a unified and flexible framework for time consistency of risk measures and performance measures, that integrates existing forms of time consistency.

We consider a (large) class of maps that are postulated to satisfy only two properties - monotonicity and locality$^1$ - and we study time consistency of such maps. We focus on these two properties, as, in our opinion, these two properties have to be satisfied by any reasonable dynamic risk measure or dynamic performance measure. We introduce the notion of an update rule that is meant to link preferences between different times.$^2$ The time consistency is defined in terms of an update rule.

We should note that this paper is the first step that we made towards a unified theory of time consistency of dynamic risk/performance measures. To illustrate why our approach leads to such unification, we show almost all known concepts of weak time consistency can be reproduced and studied in terms of single concept of an update rule, that is introduced in this paper and that is suitable both for dynamic risk measures and dynamic performance measures. For study of relation of our update rule to other types of time consistency (e.g. middle time consistency, strong time consistency or supermartingale time consistency) and their connections to various update rules as well as new concepts of time consistency, please see our survey paper Bielecki et al. (2015c).

As mentioned earlier, part of this study hinges on some technical results, proved rigorously herein, about conditional expectation and conditional essential infimum/supremum for random variables that may take the values $\pm \infty$.

Finally, we want to mention that traditionally the investigation of dynamic risk measures and dynamic performances indices is accompanied by robust representation type results, which is beyond the scope of this study given the generality of the classes of measures considered. Moreover, usually this is done in the context of convex analysis by exploring convexity (of risk measures) or quasi-concavity (of acceptability indices) properties of some relevant functions. In contrast, we depict time consistency without using convex analysis, and we consider functions that are only local and monotone, which provides for quite a generality of our results.

The paper is organized as follows. In Section 2 we introduce some necessary notations and present the main object of our study – the Dynamic LM-measure. In Section 3 we set forth the main concepts of the paper – the notion of an updated rule and the definition of time consistency.

$^1$See Section 2 for rigorous definitions along with a detailed discussion of each property.

$^2$It needs to be stressed that our notion of the update rule is different from the notion of update rule used in Tutsch (2008).
of a dynamic LM-measure. We prove a general result about time consistency, that can be viewed as counterpart of dynamic programming principle (1.2). Additionally, we show that there is a close relationship between update rule approach to time consistency and the approach based on so called benchmark sets.

Section 4 is devoted to weak time consistency. In Appendix A.1 we provide discussion of extensions of the notion of conditional expectation and of conditional essential infimum/supremum to the case of random variables that take values in \([-\infty, \infty]\). To ease the exposition of the main concepts, all technical proofs are deferred to the Appendix A.2, unless stated otherwise directly below the theorem or proposition.

2 Preliminaries

Let \((\Omega, \mathcal{F}, \mathbb{F} = \{\mathcal{F}_t\}_{t \in \mathbb{T}}, P)\) be a filtered probability space, with \(\mathcal{F}_0 = \{\Omega, \emptyset\}\), and \(\mathbb{T} = \{0, 1, \ldots, T\}\), for fixed and finite time horizon \(T \in \mathbb{N}\).

For \(\mathcal{G} \subseteq \mathcal{F}\) we denote by \(L^p(\Omega, \mathcal{G}, P)\), and \(L^0(\Omega, \mathcal{G}, P)\) the sets of all \(\mathcal{G}\)-measurable random variables with values in \((-\infty, \infty)\), and \([-\infty, \infty]\), respectively. In addition, we will use the notation \(L^p(\mathcal{G}) := L^p(\Omega, \mathcal{G}, P)\), \(L^p_t := L^p(\mathcal{F}_t)\), and \(L^p := L^p_T\), for \(p \in \{0, 1, \infty\}\). Analogous definitions will apply to \(L^0\). We will also use the notation \(\mathcal{X} := \{(V_t)_{t \in \mathbb{T}} : V_t \in L^p_t\}\), for \(p \in \{0, 1, \infty\}\).

Throughout this paper, \(\mathcal{X}\) will denote either the space of random variables \(L^p\), or the space of adapted processes \(\mathcal{X}^p\), for \(p \in \{0, 1, \infty\}\). If \(\mathcal{X} = L^p\), for \(p \in \{0, 1, \infty\}\), then the elements \(X \in \mathcal{X}\) are interpreted as discounted terminal cash-flows. On the other hand, if \(\mathcal{X} = \mathcal{X}^p\), for \(p \in \{0, 1, \infty\}\), then the elements of \(\mathcal{X}\), are interpreted as discounted dividend processes. It needs to be remarked, that all concepts developed for \(\mathcal{X} = \mathcal{X}^p\) can be easily adapted to the case of cumulative discounted value processes. The case of random variables can be viewed as a particular case of stochastic processes by considering cash-flows with only the terminal payoff, i.e. stochastic processes such that \(V = (0, \ldots, 0, V_T)\). Nevertheless, we treat this case separately for transparency. For both cases we will consider standard pointwise order, understood in the almost sure sense. In what follows, we will also make use of the multiplication operator denoted as \(\cdot\) and defined by:

\[
\begin{align*}
    m \cdot V &= (V_0, \ldots, V_{t-1}, mV_t, mV_{t+1}, \ldots), \\
    m \cdot X &= mX,
\end{align*}
\]

for \(V \in \{(V_t)_{t \in \mathbb{T}} : V_t \in L^p_t\}\), \(X \in L^0\) and \(m \in L^0_T\). In order to ease the notation, if no confusion arises, we will drop \(\cdot\) from the above product, and we will simply write \(mV\) and \(mX\) instead of \(m \cdot V\) and \(m \cdot X\), respectively.

Remark 2.1. We note that the space \(\mathcal{X}^p\), \(p \in \{0, 1, \infty\}\), endowed with multiplication \((\cdot, \cdot)\) does not define a proper \(L^0\)-module Filipovic et al. (2009) (e.g. \(0 \cdot V \neq 0\) for some \(V \in \mathcal{X}^p\)). However, in what follows, we will adopt some concepts from \(L^0\)-module theory which naturally fit into our study. Moreover, as in many cases we consider, if one additionally assume independence of the past, for brevity, we will omit the discussion of this case here.

\(^3\)Most of the results hold true or can be adjusted respectively, to the case of infinite time horizon. For sake of brevity, we will omit the discussion of this case here.
and replaces $V_0, \ldots, V_{t-1}$ with 0s in (2.1), then $X$ becomes an $L^0$-module. We refer the reader to Bielecki et al. (2015a,b) for a thorough discussion on this matter.

Throughout, we will use the convention that $\infty - \infty = -\infty + \infty = -\infty$ and $0 \cdot \pm \infty = 0$.

For $t \in T$ and $X \in L^0$ we define the (generalized) $F_t$-conditional expectation of $X$ by

$$E[X|F_t] := \lim_{n \to \infty} E[(X^+ \wedge n)|F_t] - \lim_{n \to \infty} E[(X^- \wedge n)|F_t],$$

where $X^+ = (X \lor 0)$ and $X^- = (-X \lor 0)$. Note that, in view of our convention we have that $(-1)(\infty - \infty) = \infty \neq -\infty + \infty = -\infty$, which, in particular, implies that we might get $-E[X] \neq E[-X]$. Thus, the conditional expectation operator defined above is no longer linear on $L^0$ space (see Proposition A.1 in Appendix A.1). Similarly, for any $t \in T$ and $X \in L^0$, we define the (generalized) $F_t$-conditional essential infimum by $\text{ess inf}_t X := \lim_{n \to \infty} \left[ \text{ess inf}_t (X^+ \wedge n) \right] - \lim_{n \to \infty} \left[ \text{ess sup}_t (X^- \wedge n) \right]$, and respectively we put $\text{ess sup}_t X := -\text{ess inf}_t (-X)$. For basic properties of this operator and the definition of conditional essential infimum on $L^\infty$ see Appendix A.1. In particular, note that for any $X \in L^0$ we get $\text{ess inf}_t X = X$.

Next, we introduce the main object of this study.

**Definition 2.2.** A family $\varphi = \{ \varphi_t \}_{t \in T}$ of maps $\varphi_t : X \to \bar{L}^0$ is a *Dynamic LM-measure* if $\varphi$ satisfies

1) *(Locality)* $1_A \varphi_t(X) = 1_A \varphi_t(1_A \cdot X)$;

2) *(Monotonicity)* $X \leq Y \Rightarrow \varphi_t(X) \leq \varphi_t(Y)$;

for any $t \in T$, $X, Y \in X$ and $A \in F_t$.

We believe that locality and monotonicity are two properties that must be satisfied by any reasonable dynamic measure of performance and/or measure of risk. Monotonicity property is natural for any numerical representation of an order between elements of $X$. The locality property essentially means that the values of the LM-measure restricted to a set $A \in F$ remain invariant with respect to the values of the arguments outside of the same set $A \in F$; in particular, the events that will not happen in the future do not change the value of the measure today.

Dynamic LM-measures contain several important subclasses. Among the most recognized ones are dynamic risk measures and dynamic performance measures (dynamic acceptability indices). These classes of measures have been extensively studied in the literature over the past decade.

Cash additivity is the key property that distinguishes risk measures from all other measures. This property means that adding $m$ to a portfolio today reduces the overall risk by the same amount $m$. From the regulatory perspective, the value of a risk measure is typically interpreted as the minimal capital requirement for a bank. For more details on coherent/covex/monetary

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4Since both sequences $\text{ess inf}_t (X^+ \wedge n)$ and $\text{ess sup}_t (X^- \wedge n)$ are monotone, the corresponding limits exist.

5Formally, we will consider negatives of dynamic risk measures and call them monetary utility measures, as typically risk measures are assumed to be counter monotone, rather than monotone.
risk measures and for formal definition of cash additivity we refer the reader to the survey papers Föllmer and Schied (2010); Acciaio and Penner (2011).

The distinctive property of performance measures is scale invariance - a rescaled portfolio or cashflow is accepted at the same level. Performance and acceptability indices were studied in Cherny and Madan (2009); Bielecki et al. (2014b); Cheridito and Kromer (2013); Bielecki et al. (2015b), and they are meant to provide assessment of how good a financial position is. In particular, Cheridito and Kromer (2013) gives examples of performance indices that are not acceptability indices. It needs to be noted that the theory developed in this paper can also be applied to sub-scale invariant dynamic performance indices studied in Rosazza Gianin and Sgarra (2013); Bielecki et al. (2014a).

3 Time consistency and update rules

In this section we introduce the main concept of this paper - the time consistency of dynamic risk measures and dynamic performance measures, or more generally, the time consistency of dynamic LM-measures introduced in the previous section.

We recall that these dynamic LM-measures are defined on $\mathcal{X}$, where $\mathcal{X}$ either denotes the space $L^p$ of random variables or the space $V^p$ of stochastic processes, for $p \in \{0, 1, \infty\}$, so, our study of time consistency is done relative to such spaces. Nevertheless, the definition of time consistency can be easily adapted to more general spaces, such as Orlicz hearts (as studied in Cheridito and Li (2009)), or, such as topological $L^0$-modules (see for instance Bielecki et al. (2015a)).

Assume that $\varphi$ is a dynamic LM-measure on $\mathcal{X}$. For an arbitrary fixed $X \in \mathcal{X}$ and $t \in T$ the value $\varphi_t(X)$ represents a quantification (measurement) of preferences about $X$ at time $t$. Clearly, it is reasonable to require that any such quantification (measurement) methodology should be coherent as time passes. This is precisely the motivation behind the concepts of time consistency of dynamic LM-measures.

There are various forms of time consistency proposed in the literature, some of them suitable for one class of measures, other for a different class of measures. For example, for dynamic convex (or coherent) risk measures various version of time consistency surveyed in Acciaio and Penner (2011) can be seen as versions of the celebrated dynamic programming principle. On the other hand, as shown in Bielecki et al. (2014b), dynamic programming principle essentially is not suited for scale invariant measures such as dynamic acceptability indices, and the authors introduce a new type of time consistency tailored for these measures and provide a robust representation of them. Nevertheless, in all these cases the time consistency property connects, in a noncontradictory way, the measurements done at different times.

Next, we will introduce the notion of update rule that serves as the main tool in relating the measurements of preferences at different times, and also, it is the main building block of our unified theory of time consistency property.

**Definition 3.1.** We call a family $\mu = \{\mu_{t,s} : t, s \in T, s > t\}$ of maps $\mu_{t,s} : \bar{L}^0_s \times \mathcal{X} \rightarrow \bar{L}^0_t$ an update rule if for any $s > t$, the map $\mu_{t,s}$ satisfies the following conditions:

1) (Locality) $1_A \mu_{t,s}(m, X) = 1_A \mu_{t,s}(1_A m, X)$;
2) (Monotonicity) if \( m \geq m' \), then \( \mu_{t,s}(m, X) \geq \mu_{t,s}(m', X) \); for any \( X \in X, A \in \mathcal{F}_t \) and \( m, m' \in \bar{L}_s^0 \).

Since LM-measures are local and monotone, properties with clear financial interpretations, the update rules are naturally assumed to be local and monotone too.

The first argument \( m \in \bar{L}_s^0 \) in \( \mu_{t,s} \) serves as a benchmark to which the measurement \( \varphi_s(X) \) is compared. The presence of the second argument, \( X \in X \), in \( \mu_{t,s} \), allows the update rule to depend on the objects (the \( X \)s), which the preferences are applied to. However, as we will see in next section, there are natural situations when the update rules are independent of \( X \in X \), and sometimes they do not even depend on the future times \( s \in T \).

**Remark 3.2.** As we have mentioned, the update rule is used for updating preferences through time. This, for example, can be achieved in terms of conditional expectation operator, i.e. we can consider an update rule \( \mu \), given by

\[
\mu_{t,s}(m, X) = E[m|\mathcal{F}_t].
\]  

(3.1)

Note that this particular update rule does not depend on \( s \) and \( X \). Update rule might be also used for discounting the preferences. Intuitively speaking, the risk of loss in the far future might be more preferred than the imminent risk of loss (see Cherny (2010) for the more detailed explanation of this idea). For example, the update rule \( \mu \) of the form

\[
\mu_{t,s}(m, X) = \begin{cases}
\alpha^{s-t}E[m|\mathcal{F}_t] & \text{on } \{ E[m|\mathcal{F}_t] \geq 0 \}, \\
\alpha^{t-s}E[m|\mathcal{F}_t] & \text{on } \{ E[m|\mathcal{F}_t] < 0 \}.
\end{cases}
\]  

(3.2)

for a fixed \( \alpha \in (0, 1) \) would achieve this goal. Note that ‘discounting’ proposed here has nothing to do with the ordinary discounting, as we act on discounted values already.

Next, we define several particular classes of update rules, suited for our needs.

**Definition 3.3.** Let \( \mu \) be an update rule. We will say that \( \mu \) is:

1) \( X \)-invariant, if \( \mu_{t,s}(m, X) = \mu_{t,s}(m, 0) \);

2) \( sX \)-invariant, if there exists a family \( \{ \mu_t \}_{t \in T} \) of maps \( \mu_t : \bar{L}_0^t \rightarrow \bar{L}_0^t \), such that \( \mu_{t,s}(m, X) = \mu_t(m) \);

3) Projective, if it is \( sX \)-invariant and \( \mu_t(m_t) = m_t \);

for any \( s, t \in T, s > t, X \in X, m \in \bar{L}_s^0 \) and \( m_t \in \bar{L}_t^0 \).

Examples of update rules satisfying 1) and 3) are given by (3.2) and (3.1), respectively. The update rule, which satisfies 2), but not 3) can be constructed by substituting \( \alpha^{t-s} \) with a constant in (3.2). Generally speaking update rules for stochastic processes will not satisfy 1) as the information about the process in the time interval \( (t, s) \) will affect \( \mu_{t,s} \); see Subsection 4.2 for details.

**Remark 3.4.** If an update rule \( \mu \) is \( sX \)-invariant, then it is enough to consider only the corresponding family \( \{ \mu_t \}_{t \in T} \). Hence, with slight abuse of notation we will write \( \mu = \{ \mu_t \}_{t \in T} \) and call it an update rule as well.
We are now ready to introduce the general definition of time consistency.

**Definition 3.5.** Let $\mu$ be an update rule. We say that the dynamic LM-measure $\varphi$ is $\mu$-acceptance (resp. $\mu$-rejection) **time consistent** if

$$\varphi_s(X) \geq m_s \quad (\text{resp.} \leq) \implies \varphi_t(X) \geq \mu_{t,s}(m_s, X) \quad (\text{resp.} \leq),$$

(3.3)

for all $s, t \in T$, $s > t$, $X \in \mathcal{X}$ and $m_s \in \bar{L}_s^0$. If property (3.3) is satisfied only for $s, t \in T$, such that $s = t + 1$, then we say that $\varphi$ is **one step $\mu$-acceptance** (resp. **one step $\mu$-rejection**) time consistent.

The financial interpretation of acceptance time consistency is straightforward: if $X \in \mathcal{X}$ is accepted at some future time $s \in T$, at least at level $m$, then today, at time $t \in T$, it is accepted at least at level $\mu_{t,s}(m, X)$. Similarly for rejection time consistency. Essentially, the update rule $\mu$ translates the preference levels at time $s$ to preference levels at time $t$. As it turns out, this simple and intuitive definition of time consistency, with appropriately chosen $\mu$, will cover various cases of time consistency for risk and performance measures that can be found in the existing literature (see Bielecki et al. (2015c) for a survey). Next, we will give an equivalent formulation of time consistency. While the proof of the equivalence is simple, the result itself will be conveniently used in the sequel. Moreover, it can be viewed as a counterpart of dynamic programming principle, which is an equivalent formulation of dynamic consistency for convex risk measures.

**Proposition 3.6.** Let $\mu$ be an update rule and let $\varphi$ be a dynamic LM-measure. Then $\varphi$ is $\mu$-acceptance (resp. $\mu$-rejection) time consistent if and only if

$$\varphi_t(X) \geq \mu_{t,s}(\varphi_s(X), X) \quad (\text{resp.} \leq),$$

(3.4)

for any $X \in \mathcal{X}$ and $s, t \in T$, such that $s > t$.

**Remark 3.7.** It is clear, and also naturally desired, that a monotone transformation of an LM-measure will not change the preference order of the underlying elements. We want to emphasize that a monotone transformation will also preserve the time consistency. In other words, the preference orders will be also preserved in time. Indeed, if $\varphi$ is $\mu$-acceptance time consistent, and $g : \mathbb{R} \to \mathbb{R}$ is a strictly monotone function, then the family $\{g \circ \varphi_t\}_{t \in T}$ is $\tilde{\mu}$-acceptance time consistent, where the update rule $\tilde{\mu}$ is defined by $\tilde{\mu}_{t,s}(m, X) = g(\mu_{t,s}(g^{-1}(m), X))$, for $t, s \in T$, $s > t$, $X \in \mathcal{X}$ and $m \in \bar{L}_s^0$.

In the case of random variables, $\mathcal{X} = L^p$, we we will usually consider update rules that are $X$-invariant. The case of stochastic processes is more intricate. If $\varphi$ is a dynamic LM-measure, and $V \in \mathbb{V}^p$, then in order to compare $\varphi_t(V)$ and $\varphi_s(V)$, for $s > t$, one also needs to take into account the cash-flows between times $t$ and $s$. Usually, for $\mathcal{X} = \mathbb{V}^p$ we consider update rules, such that

$$\mu_{t,t+1}(m, V) = \mu_{t,t+1}(m, 0) + f(V_t),$$

(3.5)

We introduce the concept of time consistency only for LM-measures, as this is the only class of measures used in this paper. However, the definition itself is suitable for any map acting from $\mathcal{X}$ to $\bar{L}_s^0$. For example, traditionally in the literature, the time consistency is defined for dynamic risk measures (negatives of LM-measures), and the above definition of time consistency will be appropriate, although one has to flip ‘acceptance’ with ‘rejection’.
where \( f : \mathbb{R} \to \mathbb{R} \) is a Borel measurable function, such that \( f(0) = 0 \). We note, that any such one step update rule \( \mu \) can be easily adapted to the case of random variables. Indeed, upon setting \( \tilde{\mu}_{t,t+1}(m) := \mu_{t,t+1}(m,0) \) we get a one step \( X \)-invariant update rule \( \tilde{\mu} \), which is suitable for random variables. Moreover, \( \tilde{\mu} \) will define the corresponding type of one step time consistency for random variables. Of course, this correspondence between update rule for processes and random variables is valid only for ‘one step’ setup.

Moreover, for update rules, which admit the so called nested composition property (cf. Ruszczyński (2010); Ruszczyński and Shapiro (2006) and references therein),

\[
\mu_{t,s}(m,V) = \mu_{t,t+1}(\mu_{t+1,t+2}(\ldots \mu_{s-2,s-1}(\mu_{s-1,s}(m,V),V)\ldots V),V),
\]

we have that \( \mu \)-acceptance (resp. \( \mu \)-rejection) time consistency is equivalent to one step \( \mu \)-acceptance (resp. \( \mu \)-rejection) time consistency.

### 3.1 Relation between update rule approach and the benchmark approach

As we will show in this section, there is a close relationship between our update rule approach to time consistency and the approach based on so called benchmark sets. The latter approach was initiated by Tutsch Tutsch (2008), where the author applied it in the context of dynamic risk measures. Essentially, a benchmark set is a collection of elements from \( X \) that satisfy some additional structural properties.

For simplicity, we shall assume here that \( X = L^p \), for \( p \in \{0, 1, \infty\} \). The definition of time consistency in terms of benchmark sets is as follows:

**Definition 3.8.** Let \( \varphi \) be a dynamic LM-measure and let \( \mathcal{Y} = \{\mathcal{Y}_t\}_{t \in \mathbb{T}} \) be a family of benchmark sets, that is, sets \( \mathcal{Y}_t \) such that \( \mathcal{Y}_t \subseteq X \), \( 0 \in \mathcal{Y}_t \) and \( \mathcal{Y}_t + \mathbb{R} = \mathcal{Y}_t \). We say that \( \varphi \) is acceptance (resp. rejection) time consistent with respect to \( \mathcal{Y} \), if

\[
\varphi_s(X) \geq \varphi_s(Y) \quad (\text{resp. } \leq) \quad \implies \quad \varphi_t(X) \geq \varphi_t(Y) \quad (\text{resp. } \leq),
\]

for all \( s \geq t \), \( X \in X \) and \( Y \in \mathcal{Y}_s \).

Informally, the “degree” of time consistency with respect to \( \mathcal{Y} \) is measured by the size of \( \mathcal{Y} \). Thus, the larger the sets \( \mathcal{Y}_s \) are, for each \( s \in \mathbb{T} \), the stronger is the degree of time consistency of \( \varphi \).

We now have the following important proposition,

**Proposition 3.9.** Let \( \varphi \) be a dynamic LM-measure and let \( \mathcal{Y} \) be a family of benchmark sets. Then, there exists an update rule \( \mu \) such that \( \varphi \) is acceptance (resp. rejection) time consistent with respect to \( \mathcal{Y} \) if and only if it is \( \mu \)-acceptance (resp. \( \mu \)-rejection) time consistent.

The update rule \( \mu \) is said to provide \( \varphi \) with the same type of time consistency as \( \mathcal{Y} \) does, and vice versa. Generally speaking, the converse implication does not hold true, i.e. given an LM-measure \( \varphi \) and an update rule \( \mu \) it may not be possible to construct \( \mathcal{Y} \) so that it provides the same type of time consistency as \( \mu \) does. In other words, the notion of time consistency given in terms of updates rule is more general.
4 Weak time consistency

In this section we will discuss examples of update rules, which relate to weak time consistency for random variables and for stochastic processes. This is meant to illustrate the framework developed earlier in this paper.

For a thorough presentation of application of our theory of update rules to other known types of time consistency, such as middle time consistency and strong time consistency, as well as for other related results and concepts, we refer to the survey paper Bielecki et al. (2015c).

The notion of weak time consistency was introduced in Tutsch (2008), and subsequently studied in Acciaio and Penner (2011); Artzner et al. (2007); Cheridito et al. (2006); Detlefsen and Scandolo (2005); Acciaio et al. (2012); Cheridito et al. (2006). The idea is that if ‘tomorrow’, say at time $s$, we accept $X \in \mathcal{X}$ at level $m_s \in \mathcal{F}_s$, then ‘today’, say at time $t$, we would accept $X$ at least at any level lower or equal to $m_s$, appropriately adjusted by the information $\mathcal{F}_t$ available at time $t$ (cf. (4.2)). Similarly, if tomorrow we reject $X$ at level higher or equal to $m_s \in \mathcal{F}_s$, then today, we should also reject $X$ at any level higher than $m_s$, adjusted to the flow of information $\mathcal{F}_t$. This suggests that the update rules should be taken as $\mathcal{F}_t$-conditional essential infimum and supremum, respectively. Towards this end, we first show that $\mathcal{F}_t$-conditional essential infimum and supremum are projective update rules.

**Proposition 4.1.** The family $\mu^{\inf} := \{\mu^{\inf}_t\}_{t \in \mathbb{T}}$ of maps $\mu^{\inf}_t : \bar{L}^0 \to \bar{L}_t^0$ given by

$$\mu^{\inf}_t(m) = \text{ess inf}_t m,$$

is a projective\(^7\) update rule. Similar result is true for family $\mu^{\sup} := \{\mu^{\sup}_t\}_{t \in \mathbb{T}}$ of maps $\mu^{\sup}_t : \bar{L}^0 \to \bar{L}_t^0$ given by $\mu^{\sup}_t(m) = \text{ess sup}_t m$.

4.1 Weak time consistency for random variables

Recall that the case of random variables corresponds to $\mathcal{X} = L^p$, for a fixed $p \in \{0, 1, \infty\}$. We proceed with the definition of weak acceptance and weak rejection time consistency (for random variables).

**Definition 4.2.** Let $\varphi$ be a dynamic LM-measure. Then $\varphi$ is said to be weakly acceptance (resp. weakly rejection) time consistent if it is $\mu^{\inf}$-acceptance (resp. $\mu^{\sup}$-rejection) time consistent.

Definition 4.2 of time consistency is equivalent to many forms of time consistency studied in the current literature. Usually, the weak time consistency is considered for dynamic monetary risk measures on $L^\infty$ (cf. Acciaio and Penner (2011) and references therein); we refer to this case as to the ‘classical weak time consistency.’ It was observed in Acciaio and Penner (2011) that in the classical weak time consistency framework, weak acceptance (resp. weak rejection) time consistency is equivalent to the statement that for any $X \in \mathcal{X}$ and $s > t$, we get

$$\varphi_s(X) \geq 0 \Rightarrow \varphi_t(X) \geq 0 \quad \text{(resp.} \leq).$$

---

\(^7\)See Remark 3.4 for the comment about notation.
This observation was the motivation for our definition of weak acceptance (resp. weak rejection) time consistency, and the next proposition explains why so.

**Proposition 4.3.** Let $\varphi$ be a dynamic LM-measure. The following conditions are equivalent

1) $\varphi$ is weakly acceptance time consistent, i.e. for any $X \in \mathcal{X}$, $t, s \in \mathbb{T}$, $s > t$, and $m_s \in \bar{L}_s^0$,

$$\varphi_s(X) \geq m_s \Rightarrow \varphi_t(X) \geq \text{ess inf}_t(m_s). \quad (4.2)$$

2) For any $X \in \mathcal{X}$, $s, t \in \mathbb{T}$, $s > t$, $\varphi_t(X) \geq \text{ess inf}_t \varphi_s(X)$.

3) For any $X \in \mathcal{X}$, $s, t \in \mathbb{T}$, $s > t$, and $m_t \in \bar{L}_t^0$,

$$\varphi_s(X) \geq m_t \Rightarrow \varphi_t(X) \geq m_t.$$ If additionally $\varphi$ is a dynamic monetary utility measure, then the above conditions are equivalent to

4) For any $X \in \mathcal{X}$ and $s, t \in \mathbb{T}$, $s > t$,

$$\varphi_s(X) \geq 0 \Rightarrow \varphi_t(X) \geq 0.$$ Similar result holds true for weak rejection time consistency.

Property 3) in Proposition 4.3 was also suggested as the notion of (weak) acceptance and (weak) rejection time consistency in the context of scale invariant measures, called acceptability indices (cf. Biagini and Bion-Nadal (December 2014); Bielecki et al. (2014b)).

In many papers studying risk measurement theory (cf. Detlefsen and Scandolo (2005) and references therein), the weak form of time consistency theory is defined using dual approach to the measurement of risk. Rather than directly updating the level of preferences $m$, as in our approach, in the dual approach the level of preference is updated indirectly by manipulating probabilistic scenarios and explaining the update procedure by using so called pasting property (see e.g. (Detlefsen and Scandolo, 2005, Def. 9)). As shown in the next result, our update rule related to weak form of time consistency admits dual representation, allowing us to link our definition with the dual approach.

**Proposition 4.4.** For any $m \in \bar{L}^0$ and $t \in \mathbb{T}$ we get

$$\mu_t^{\text{inf}}(m) = \text{ess inf}_{Z \in P_t} E[Zm|\mathcal{F}_t]. \quad (4.3)$$

where $P_t := \{Z \in L^0 \mid Z \geq 0, E[Z|\mathcal{F}_t] = 1\}$. Similar result is true for $\text{ess sup}_t m$.

---

\[ ^8 \text{i.e } \varphi_t(0) = 0 \text{ and } \varphi_t(X + c_t) = \varphi_t(X) + c_t \text{ for any } t \in \mathbb{T}, X \in \bar{L}^0 \text{ and } c_t \in L_t^\infty. \]
In (4.3), the random variables \( Z \in P_t \) may be treated as Radon-Nikodym derivatives w.r.t. \( P \) of some probability measures \( Q \) such that \( Q \ll P \) and \( Q|_{\mathcal{F}_t} = P|_{\mathcal{F}_t} \). The family \( P_t \) may thus be thought of as the family of all possible \( \mathcal{F}_t \)-conditional probabilistic scenarios. Accordingly, \( \mu_t^{\inf}(m) \) represents the \( \mathcal{F}_t \)-conditional worst-case preference update with respect to all such scenarios. Note that combining Propositions 3.6 and 4.4, we obtain that weak acceptance time consistency of \( \varphi \) is equivalent to the condition

\[
\varphi_t(X) \geq \operatorname{ess} \inf_{Z \in P_t} E[Z \varphi_s(X)|\mathcal{F}_t],
\]

which in fact is a starting point for almost all robust definitions of weak time consistency, for \( \varphi \)'s admitting dual representation Detlefsen and Scandolo (2005).

As next result shows, the weak time consistency is indeed one of the weakest forms of time consistency, being implied by any other concept of time consistency generated by a projective rule.

**Proposition 4.5.** Let \( \varphi \) be a dynamic LM-measure and let \( \mu \) be a projective update rule. If \( \varphi \) is \( \mu \)-acceptance (resp. \( \mu \)-rejection) time consistent, then \( \varphi \) is weakly acceptance (resp. weakly rejection) time consistent.

In particular, recall that time consistency is preserved under monotone transformations, Remark 3.7. Thus, for any strictly monotone function \( g : \mathbb{R} \to \mathbb{R} \), if \( \varphi \) is weakly acceptance (resp. weakly rejection) time consistent, then \( \{g \circ \varphi_t\}_{t \in T} \) also is weakly acceptance (resp. weakly rejection) time consistent.

### 4.2 Weak and Semi-weak time consistency for stochastic processes

In this subsection we introduce and discuss the concept of semi-weak time consistency for stochastic processes. Thus, we take \( \mathcal{X} = \mathbb{V}^p \), for a fixed \( p \in \{0, 1, \infty\} \). As it will turn out, in the case of random variables semi-weak time consistency coincides with the property of weak time consistency; that is why we omitted discussion of semi-weak consistency in the previous section.

To provide a better perspective for the concept of semi-weak time consistency, we start with the definition of weak time consistency for stochastic processes, which transfers directly from the case of random variables using (3.5).

**Definition 4.6.** Let \( \varphi \) be a dynamic LM-measure. We say that \( \varphi \) is weakly acceptance (resp. weakly rejection) time consistent for stochastic processes if it is one step \( \mu \)-acceptance (resp. one step \( \mu^* \)-rejection) time consistent, where the update rule is given by

\[
\mu_t^{\inf}(m, V) = \mu_t^{\inf}(m) + V_t \quad (\text{resp. } \mu_t^{\sup}(m, V) = \mu_t^{\sup}(m) + V_t).
\]

As mentioned earlier, the update rule, and consequently weak time consistency for stochastic processes, depends also on the value of the process (the dividend paid) at time \( t \). If tomorrow, at time \( t+1 \), we accept \( X \in \mathcal{X} \) at level greater than \( m_{t+1} \in \mathcal{F}_{t+1} \), then today at time \( t \), we will accept \( X \) at least at level \( \operatorname{ess} \inf_t m_{t+1} \) (i.e. the worst level of \( m_{t+1} \) adapted to the information \( \mathcal{F}_t \)) plus the dividend \( V_t \) received today.
For counterparts of Propositions 4.3 and 4.5 for the case of stochastic processes, see the survey paper Bielecki et al. (2015c).

As it was shown in Bielecki et al. (2014b), none of the existing, at that time, forms of time consistency were suitable for scale-invariant maps, such as acceptability indices. In fact, even the weak acceptance and the weak rejection time consistency for stochastic processes are too strong in case of acceptability indices. Because of that we need even a weaker notion of time consistency, which we will refer to as semi-weak acceptance and semi-weak rejection time consistency. The notion of semi-weak time consistency for stochastic processes, introduced next, is suited precisely for acceptability indices, and we refer the reader to Bielecki et al. (2014b) for a detailed discussion on time consistency for acceptability indices and their dual representations.

**Definition 4.7.** Let $\varphi$ be a dynamic LM-measure (for processes). Then $\varphi$ is said to be:

- **Semi-weakly acceptance time consistent** if it is one step $\mu$-acceptance time consistent, where the update rule is given by
  \[
  \mu_{t,t+1}(m, V) = 1_{\{V_t \geq 0\}} \mu_{t}^{\inf}(m) + 1_{\{V_t < 0\}} (-\infty).
  \]

- **Semi-weakly rejection time consistent** if it is one step $\mu'$-rejection time consistent, where the update rule is given by
  \[
  \mu'_{t,t+1}(m, V) = 1_{\{V_t \leq 0\}} \mu_{t}^{\sup}(m) + 1_{\{V_t > 0\}} (+\infty).
  \]

It is straightforward to check that weak acceptance/rejection time consistency for stochastic processes always implies semi-weak acceptance/rejection time consistency.

Next, we will show that the definition of semi-weak time consistency is indeed equivalent to time consistency introduced in Bielecki et al. (2014b), that was later studied in Biagini and Bion-Nadal (December 2014); Bielecki et al. (2014a).

**Proposition 4.8.** Let $\varphi$ be a dynamic LM-measure on $\mathbb{V}^p$. The following conditions are equivalent

1) $\varphi$ is semi-weakly acceptance time consistent, i.e. for all $V \in \mathcal{X}$, $t \in T$, $t < T$, and $m_t \in \bar{L}_t^0$,
   \[
   \varphi_{t+1}(V) \geq m_{t+1} \Rightarrow \varphi_t(V) \geq 1_{\{V_t \geq 0\}} \text{ess inf}_{t}(m_{t+1}) + 1_{\{V_t < 0\}} (-\infty).
   \]

2) For all $V \in \mathcal{X}$ and $t \in T$, $t < T$, $\varphi_t(V) \geq 1_{\{V_t \geq 0\}} \text{ess inf}_t(\varphi_{t+1}(V)) + 1_{\{V_t < 0\}} (-\infty)$.

3) For all $V \in \mathcal{X}$, $t \in T$, $t < T$, and $m_t \in \bar{L}_t^0$, such that $V_t \geq 0$ and $\varphi_{t+1}(V) \geq m_t$, then $\varphi_t(V) \geq m_t$.

Similar result is true for semi-weak rejection time consistency.

---

9In Bielecki et al. (2014b) the authors combined both semi-weak acceptance and rejection time consistency into one single definition and call it time consistency.
Property 3) in Proposition 4.8 illustrates best the financial meaning of semi-weak acceptance time consistency: if tomorrow we accept the dividend stream $V \in \mathcal{X}$ at level $m_t$, and if we get a positive dividend $V_t$ paid today at time $t$, then today we accept the cash-flow $V$ at least at level $m_t$ as well. Similar interpretation is valid for semi-weak rejection time consistency.

The next two results give an important (dual) connection between cash additive risk measures and acceptability indices. In particular, these results shed light on the relation between time-consistency property of dynamic acceptability indices, represented by the family $\{\alpha_t\}_{t \in \mathbb{T}}$ below, and time consistency of the corresponding family $\{\varphi^x\}_{x \in \mathbb{R}^+}$, where $\varphi^x = \{\varphi^x_t\}_{t \in \mathbb{T}}$ is a dynamic risk measure (for any $x \in \mathbb{R}^+$).

**Proposition 4.9.** Let $\{\varphi^x\}_{x \in \mathbb{R}^+}$, be a decreasing family of dynamic LM-measure\(^{10}\). Assume that for each $x \in \mathbb{R}^+$, $\{\varphi^x_t\}_{t \in \mathbb{T}}$ is weakly acceptance (resp. weakly rejection) time consistent. Then, the family $\{\alpha_t\}_{t \in \mathbb{T}}$ of maps $\alpha_t : \mathcal{X} \to L^0_{\mathbb{T}}$ defined by\(^{11}\)

$$\alpha_t(V) := \text{ess sup}_{x \in \mathbb{R}^+} \{x \mathbf{1}_{\varphi^x_t(V) \geq 0}\},$$

is a semi-weakly acceptance (resp. semi-weakly rejection) time consistent dynamic LM-measure.

Observe that

$$\alpha_t(V)(\omega) = \sup \{x \in \mathbb{R}^+ : \varphi^x_t(V)(\omega) \geq 0\}.$$  \hspace{1cm} (4.6)

As the representation (4.6) is more convenient than (4.5), it will be used in the proofs given in the Appendix.

**Proposition 4.10.** Let $\{\alpha_t\}_{t \in \mathbb{T}}$ be a dynamic LM-measure, which is independent of the past and translation invariant\(^{12}\). Assume that $\{\alpha_t\}_{t \in \mathbb{T}}$ is semi-weakly acceptance (resp. semi-weakly rejection) time consistent. Then, for any $x \in \mathbb{R}^+$, the family $\{\varphi^x_t\}_{t \in \mathbb{T}}$ defined by

$$\varphi^x_t(V) = \text{ess inf}_{c \in \mathbb{R}} \{c \mathbf{1}_{\alpha_t(V - c 1_{\{t\}}) \leq x}\},$$

is a weakly acceptance (resp. weakly rejection) time consistent dynamic LM-measure.

In the proofs given in the Appendix, we will use representation

$$\varphi^x_t(V)(\omega) = \inf \{c \in \mathbb{R} : \alpha_t(V - c 1_{\{t\}})(\omega) \leq x\},$$

rather than (4.7), as it is more convenient.

This type of dual representation, i.e. (4.5) and (4.7), first appeared in Cherny and Madan (2009) where the authors studied static (one period of time) scale invariant measures. Subsequently, in

\(^{10}\)A family, indexed by $x \in \mathbb{R}^+$, of maps $\{\varphi^x_t\}_{t \in \mathbb{T}}$, will be called decreasing, if $\varphi^x_t(X) \leq \varphi^y_t(X)$ for all $X \in \mathcal{X}$, $t \in \mathbb{T}$ and $x, y \in \mathbb{R}^+$, such that $x \geq y$.

\(^{11}\)Note that the map defined in (4.5) is $\mathcal{F}_t$-measurable as the essential supremum over an uncountable family of $\mathcal{F}_t$-measurable random variables. See Appendix A.1.

\(^{12}\)We say that $\alpha$ is translation invariant if $\alpha_t(V + m 1_{\{t\}}) = \alpha_t(V + m 1_{\{s\}})$, for any $m \in L^\infty_{\mathbb{T}}$ and $V \in \mathcal{X}$, where $1_{\{t\}}$ corresponds to process equal to 1 a time $t$ and 0 elsewhere; We say that $\alpha$ is independent of the past if $\alpha_t(V) = \alpha_t(V - 0 \cdot 1_{\{t\}} V)$, for any $V \in \mathcal{X}$. 

Bielecki et al. (2014b), the authors extended these results to the case of stochastic processes with special emphasis on time consistency property. In contrast to Bielecki et al. (2014b), we consider an arbitrary probability space, not just a finite one.

We conclude this section by presenting two examples that illustrate the concept of semi-weak time consistency and show the connection between maps introduced in Propositions 4.9 and 4.10. For more examples we refer to the survey paper Bielecki et al. (2015c).

**Example 4.11** (Dynamic Gain Loss Ratio). Dynamic Gain Loss Ratio (dGLR) is a popular measure of performance, which essentially improves on some drawbacks of Sharpe Ratio (such as penalizing for positive returns), and it is equal to the ratio of expected return over expected losses. Formally, for $\mathcal{X} = \mathbb{Y}^1$, dGLR is defined as

$$
\varphi_t(V) := \begin{cases} 
\frac{E[\sum_{i=t}^{T} V_i|\mathcal{F}_t]}{E[\sum_{i=t}^{T} V_i - |\mathcal{F}_t|]}, & \text{if } E[\sum_{i=t}^{T} V_i|\mathcal{F}_t] > 0, \\
0, & \text{otherwise.}
\end{cases}
$$

(4.9)

For various properties and dual representations of dGLR see for instance Bielecki et al. (2014b, 2015a). In Bielecki et al. (2014b), the authors showed that dGLR is both semi-weak acceptance and semi-weakly rejection time consistent, although assuming that $\Omega$ is finite. For sake of completeness we will show here that dGLR is semi-weakly acceptance time consistency; semi-weakly rejection time consistent, although assuming that $\Omega$ is finite. For sake of completeness we will show here that dGLR is semi-weakly acceptance time consistency; semi-weakly rejection time consistency is left to an interested reader as an exercise.

Assume that $t \in \mathbb{T} \setminus \{T\}$, and $V \in \mathcal{X}$. In view of Proposition 3.6, it is enough to show that

$$
\varphi_t(V) \geq \mathbb{1}_{\{V_i \geq 0\}} \text{ess inf}_t(\varphi_{t+1}(V)) + \mathbb{1}_{\{V_i < 0\}}(-\infty).
$$

(4.10)

On the set $\{V_i < 0\}$ the inequality (4.10) is trivial. Since $\varphi_t$ is non-negative and local, without loss of generality, we may assume that $\text{ess inf}_t(\varphi_{t+1}(V)) > 0$. Moreover, $\varphi_{t+1}(V) \geq \text{ess inf}_t(\varphi_{t+1}(V))$, which implies

$$
E[\sum_{i=t+1}^{T} V_i|\mathcal{F}_{t+1}] \geq \text{ess inf}_t(\varphi_{t+1}(V)) \cdot E[\sum_{i=t+1}^{T} V_i - |\mathcal{F}_{t+1}|].
$$

(4.11)

Using (4.11) we obtain

$$
\mathbb{1}_{\{V_i \geq 0\}} E[\sum_{i=t}^{T} V_i|\mathcal{F}_t] \geq \mathbb{1}_{\{V_i \geq 0\}} E[\sum_{i=t+1}^{T} V_i|\mathcal{F}_{t+1}|] E[\sum_{i=t+1}^{T} V_i - |\mathcal{F}_{t+1}|] \\
\geq \mathbb{1}_{\{V_i \geq 0\}} \text{ess inf}_t(\varphi_{t+1}(V)) \cdot E[\sum_{i=t+1}^{T} V_i - |\mathcal{F}_{t+1}|] \\
\geq \mathbb{1}_{\{V_i \geq 0\}} \text{ess inf}_t(\varphi_{t+1}(V)) \cdot E[\sum_{i=t}^{T} V_i - |\mathcal{F}_t|].
$$

(4.12)

Note that $\text{ess inf}_t(\varphi_{t+1}(V)) > 0$ implies that $\varphi_{t+1}(V) > 0$, and thus $E[\sum_{i=t+1}^{T} V_i|\mathcal{F}_{t+1}] > 0$. Hence, on set $\{V_i \geq 0\}$, we have

$$
\mathbb{1}_{\{V_i \geq 1\}} E[\sum_{i=t}^{T} V_i|\mathcal{F}_t] \geq \mathbb{1}_{\{V_i \geq 1\}} E[\sum_{i=t+1}^{T} V_i|\mathcal{F}_{t+1}|] > 0.
$$
Combining this and (4.12), we conclude the proof.

**Example 4.12** (Dynamic RAROC for processes). Risk Adjusted Return On Capital (RAROC) is a popular scale invariant measure of performance; we refer the reader to Cherny and Madan (2009) for study of static RAROC, and to Bielecki et al. (2014b) for its extension to dynamic setup. We consider the space $\mathcal{X} = \mathbb{V}^1$ and we fix $\alpha \in (0,1)$. Dynamic RAROC, at level $\alpha$, is the family $\{\varphi_t\}_{t \in \mathbb{T}}$, with $\varphi_t$ given by

$$
\varphi_t(V) := \begin{cases} 
\frac{E[\sum_{i=t}^T V_i | F_t]}{-\rho^\alpha_t(V)} & \text{if } E[\sum_{i=t}^T V_i | F_t] > 0, \\
0 & \text{otherwise},
\end{cases}
$$

(4.13)

where $\rho^\alpha_t(V) = \operatorname{ess \ inf}_{Z \in D^\alpha_t} E[Z \sum_{i=t}^T V_i | F_t]$, and where the family of sets $\{D^\alpha_t\}_{t \in \mathbb{T}}$ is defined by\(^{13}\)

$$
D^\alpha_t := \{Z \in L^1 : 0 \leq Z \leq \alpha^{-1}, E[Z|F_t] = 1\}.
$$

(4.14)

We use the convention $\varphi_t(V) = +\infty$, if $\rho_t(V) \geq 0$. In Bielecki et al. (2014b) it was shown that dynamic RAROC is a dynamic acceptability index for processes. Moreover, it admits the following dual representation (cf. (4.6)): for any fixed $t \in \mathbb{T}$,

$$
\varphi_t(V) = \sup\{x \in \mathbb{R}_+ : \phi^x_t(V) \geq 0\},
$$

where $\phi^x_t(V) = \operatorname{ess \ inf}_{Z \in B^x_t} E[Z(\sum_{i=t}^T V_i)|F_t]$ with

$$
B^x_t = \{Z \in L^1 : Z = \frac{1}{1+x} + \frac{x}{1+x}Z_1, \text{ for some } Z_1 \in D^\alpha_t\}.
$$

It is easy to check, that the family $\{\varphi^x_t\}_{t \in \mathbb{T}}$ is a dynamic coherent risk measure for processes, see Bielecki et al. (2014b) for details. Since $1 \in D^\alpha_t$, we also get that $\{\phi^x_t\}_{t \in \mathbb{T}}$ is increasing with $x \in \mathbb{R}_+$.

Moreover, it is known that $\{\phi^x_t\}_{t \in \mathbb{T}}$ is weakly acceptance time consistent but not weakly rejection time consistent, for any fixed $x \in \mathbb{R}_+$ (see Bielecki et al., 2015c, Example 1). Thus, using Propositions 4.9 and 4.10 we immediately conclude that $\{\varphi^x_t\}_{t \in \mathbb{T}}$ is semi-weakly acceptance time consistent and not semi-weakly rejection time consistent.

\[\text{A Appendix}\]

\section{Conditional expectation and essential supremum/infimum on } \hat{L}^0

First, we will present some elementary properties of the generalized conditional expectation.

**Proposition A.1.** For any $X, Y \in \hat{L}^0$ and $s, t \in \mathbb{T}$, $s > t$ we get

1) $E[\lambda X | F_t] \leq \lambda E[X | F_t]$ for $\lambda \in L^0_t$, and $E[\lambda X | F_t] = \lambda E[X | F_t]$ for $\lambda \in L^0_t, \lambda \geq 0$;

---

\(^{13}\)The family $\{D^\alpha_t\}_{t \in \mathbb{T}}$ represents risk scenarios, which define the dynamic version of the conditional value at risk at level $\alpha$ (cf. Cherny (2010)).
2) \(E[X|\mathcal{F}_t] \leq E[E[X|\mathcal{F}_s]|\mathcal{F}_t]\), and \(E[X|\mathcal{F}_t] = E[E[X|\mathcal{F}_s]|\mathcal{F}_t]\) for \(X \geq 0\);

3) \(E[X|\mathcal{F}_t] + E[Y|\mathcal{F}_t] \leq E[X + Y|\mathcal{F}_t]\), and \(E[X|\mathcal{F}_t] + E[Y|\mathcal{F}_t] = E[X + Y|\mathcal{F}_t]\) if \(X, Y \geq 0\);

**Remark A.2.** All inequalities in Proposition A.1 can be strict. Assume that \(t = 0\) and \(k,s \in \mathbb{T}\), \(k > s > 0\), and let \(\xi \in L^0_k\) be such that \(\xi = \pm 1\), \(\xi\) is independent of \(\mathcal{F}_s\), and \(P(\xi = 1) = P(\xi = -1) = 1/2\). We consider \(Z \in L^0_s\) such that \(Z \geq 0\), and \(E[Z] = \infty\). By taking \(\lambda = -1\), \(X = \xi Z\) and \(Y = -X\), we get strict inequalities in 1), 2) and 3).

Next, we will discuss some important features of conditional essential infimum and conditional essential supremum, in the context of \(\bar{L}^0\).

Before that, we will recall the definition of conditional essential infimum for bounded random variables. For \(X \in L^\infty\) and \(t \in \mathbb{T}\), we will denote by \(\essinf_{t} X\) the unique (up to a set of probability zero), \(\mathcal{F}_t\)-measurable random variable, such that for any \(A \in \mathcal{F}_t\), the following equality holds true

\[
\essinf_{\omega \in A} X = \essinf_{\omega \in A} (\essinf_{t} X).
\]

We will call this random variable the \(\mathcal{F}_t\)-**conditional essential infimum of \(X**. We refer the reader to Barron et al. (2003) for a detailed proof of the existence and uniqueness of the conditional essential infimum. We will call \(\esssup_{t} (X) := -\essinf_{t} (-X)\) the \(\mathcal{F}_t\)-**conditional essential supremum of \(X \in L^\infty\).

As stated in the preliminaries we extend these two notions to the space \(\bar{L}^0\). For any \(t \in \mathbb{T}\) and \(X \in \bar{L}^0\), we define the \(\mathcal{F}_t\)-conditional essential infimum by

\[
\essinf_{t} X := \lim_{n \to \infty} \left[ \essinf_{t} (X^+ \wedge n) \right] - \lim_{n \to \infty} \left[ \esssup_{t} (X^- \wedge n) \right], \tag{A.2}
\]

and respectively we put \(\esssup_{t} (X) := -\essinf_{t} (-X)\).

**Remark A.3.** Extending the function \arctan to \([-\infty, \infty]\) by continuity, and observing that \(\arctan X \in L^\infty\) for any \(X \in \bar{L}^0\), one can naturally extend conditional essential infimum to \(\bar{L}^0\) by setting

\[
\essinf_{t} X = \arctan^{-1}(\essinf_{t}(\arctan X)).
\]

We proceed with the following result:

**Proposition A.4.** For any \(X, Y \in \bar{L}^0\), \(s, t \in \mathbb{T}\), \(s \geq t\), and \(A \in \mathcal{F}_t\) we have

1) \(\essinf_{\omega \in A} X = \essinf_{\omega \in A} (\essinf_{t} X)\);

2) If \(\essinf_{\omega \in A} X = \essinf_{\omega \in A} U\) for some \(U \in \bar{L}^0_t\), then \(U = \essinf_{t} X\);

3) \(X \geq \essinf_{t} X\);

4) If \(Z \in \bar{L}^0_t\), is such that \(X \geq Z\), then \(\essinf_{t} X \geq Z\);

5) If \(X \geq Y\), then \(\essinf_{t} X \geq \essinf_{t} Y\);

6) \(1_A \essinf_{t} X = 1_A \essinf_{t} (1_A X)\);
7) \( \text{ess inf}_s X \geq \text{ess inf}_t X \);

The analogous results are true for \( \{\text{ess sup}_t\}_{t \in \mathcal{T}} \).

The proof for the case \( X, Y \in L^\infty \) can be found in Barron et al. (2003). Since for any \( n \in \mathbb{N} \) and \( X, Y \in \tilde{L}^0 \) we get \( X^+ \wedge n \in L^\infty, X^- \wedge n \in L^\infty \) and \( X^+ \wedge X^- = 0 \), the extension of the proof to the case \( X, Y \in \tilde{L}^0 \) is straightforward, and we omit it here.

Remark A.5. Similarly to Barron et al. (2003), the conditional essential infimum \( \text{ess inf}_t(X) \) can be alternatively defined as the largest \( \mathcal{F}_t \)-measurable random variable, which is smaller than \( X \), i.e. properties 3) and 4) from Proposition A.4 are characteristic properties for conditional essential infimum.

Next, we define the generalized versions of ess inf and ess sup of a (possibly uncountable) family of random variables: For \( \{X_i\}_{i \in I} \), where \( X_i \in \tilde{L}^0 \), we let

\[
\text{ess inf}_{i \in I} X_i := \lim_{n \to \infty} \left[ \text{ess inf}_{i \in I} (X_i^+ \wedge n) \right] - \lim_{n \to \infty} \left[ \text{ess sup}_{i \in I} (X_i^- \wedge n) \right]. \tag{A.3}
\]

Note that, in view of (Karatzas and Shreve, 1998, Appendix A), \( \text{ess inf}_{i \in I} X_i \wedge n \) and \( \text{ess sup}_{i \in I} X_i \wedge n \) are well defined, so that \( \text{ess inf}_{i \in I} X_i \) is well defined. It needs to be observed that the operations of the right hand side of (A.3) preserve measurability. In particular, if \( X_i \in \mathcal{F}_t \) for all \( i \in I \), then \( \text{ess inf}_{i \in I} X_i \in \mathcal{F}_t \).

Furthermore, if for any \( i, j \in I \), there exists \( k \in I \), such that \( X_k \leq X_i \wedge X_j \), then there exists a sequence \( i_n \in I, n \in \mathbb{N} \), such that \( \{X_{i_n}\}_{n \in \mathbb{N}} \) is nonincreasing and \( \text{ess inf}_{i \in I} X_i = \text{inf}_{n \in \mathbb{N}} X_{i_n} = \lim_{n \to \infty} X_{i_n} \). Analogous results hold true for \( \text{ess sup}_{i \in I} X_i \).

A.2 Proofs

Proof of Proposition 3.6.

**Proof.** Let \( \mu \) be an update rule.

1) The implication \((\Rightarrow)\) follows immediately, by taking in the definition of acceptance time consistency \( m_s = \varphi_s(X) \).

\((\Leftarrow)\) Assume that \( \varphi_t(X) \geq \mu_{t,s}(\varphi_s(X), X) \), for any \( s, t \in \mathcal{T}, s > t, \) and \( X \in \mathcal{X} \). Let \( m_s \in \tilde{L}_s^0 \) be such that \( \varphi_s(X) \geq m_s \). Using monotonicity of \( \mu \), we get \( \varphi_t(X) \geq \mu_{t,s}(\varphi_s(X), X) \geq \mu_{t,s}(m_s, X) \).

2) The proof is similar to 1).

\[ \square \]

Proof of Proposition 3.9.

**Proof.** We do the proof only for acceptance time consistency. The proof for rejection time consistency is analogous.

**Step 1.** We will show that \( \varphi \) is acceptance time consistent with respect to \( \mathcal{Y} \), if and only if

\[
1_A \varphi_s(X) \geq 1_A \varphi_s(Y) \implies 1_A \varphi_t(X) \geq 1_A \varphi_t(Y), \tag{A.4}
\]
for all \( s \geq t, X \in \mathcal{X}, Y \in \mathcal{Y}_s \) and \( A \in \mathcal{F}_t \). For sufficiency it is enough to take \( A = \Omega \). For necessity let us assume that
\[
\mathbf{1}_A \varphi_s(X) \geq \mathbf{1}_A \varphi_s(Y). \tag{A.5}
\]
Using locality of \( \varphi \), we get that (A.5) is equivalent to
\[
\mathbf{1}_A \varphi_s(\mathbf{1}_A X + \mathbf{1}_{A^c} Y) + \mathbf{1}_{A^c} \varphi_s(\mathbf{1}_A X + \mathbf{1}_{A^c} Y) \geq \mathbf{1}_A \varphi_s(Y) + \mathbf{1}_{A^c} \varphi_s(Y)
\]
and consequently to \( \varphi_s(\mathbf{1}_A X + \mathbf{1}_{A^c} Y) \geq \varphi_s(Y) \). Thus, using (3.7), we get
\[
\varphi_s(\mathbf{1}_A X + \mathbf{1}_{A^c} Y) \geq \varphi_s(Y) \implies \varphi_t(\mathbf{1}_A X + \mathbf{1}_{A^c} Y) \geq \varphi_t(Y).
\]
By the same arguments we get that \( \varphi_t(\mathbf{1}_A X + \mathbf{1}_{A^c} Y) \geq \varphi_t(Y) \) is equivalent to \( \mathbf{1}_A \varphi_t(X) \geq \mathbf{1}_A \varphi_t(Y) \), which concludes this part of the proof.

Step 2. Now we demonstrate that \( \varphi \) is acceptance time consistent with respect to \( \mathcal{Y} \) if and only if \( \varphi \) is acceptance time consistent with respect to the family \( \mathcal{Y}_t = \{ \hat{Y}_t \}_{t \in \mathbb{T}} \) of benchmark sets given by
\[
\hat{Y}_t := \{ \mathbf{1}_A Y_1 + \mathbf{1}_{A^c} Y_2 : Y_1, Y_2 \in \mathcal{Y}_t, A \in \mathcal{F}_t \}. \tag{A.6}
\]
Noting that for any \( t \in \mathbb{T} \) we have \( \mathcal{Y}_t \subseteq \hat{Y}_t \), we get the sufficiency part. For necessity let us assume that
\[
\varphi_s(X) \geq \varphi_s(Y) \tag{A.7}
\]
for some \( Y \in \hat{Y}_t \). From (A.6) we know that there exists \( A \in \mathcal{F}_t \) and \( Y_1, Y_2 \in \mathcal{Y}_s \), such that \( Y = \mathbf{1}_A Y_1 + \mathbf{1}_{A^c} Y_2 \). Consequently, using locality of \( \varphi \), and the fact that (A.7) is equivalent to
\[
\mathbf{1}_A \varphi_s(X) + \mathbf{1}_{A^c} \varphi_s(X) \geq \mathbf{1}_A \varphi_s(\mathbf{1}_A Y_1 + \mathbf{1}_{A^c} Y_2) + \mathbf{1}_{A^c} \varphi_s(\mathbf{1}_A Y_1 + \mathbf{1}_{A^c} Y_2),
\]
we conclude that (A.7) is equivalent to
\[
\mathbf{1}_A \varphi_s(X) + \mathbf{1}_{A^c} \varphi_s(X) \geq \mathbf{1}_A \varphi_s(Y_1) + \mathbf{1}_{A^c} \varphi_s(Y_2)
\]
As the sets \( A \) and \( A^c \) are disjoint, using (A.4) twice, we get
\[
\mathbf{1}_A \varphi_t(X) + \mathbf{1}_{A^c} \varphi_t(X) \geq \mathbf{1}_A \varphi_t(Y_1) + \mathbf{1}_{A^c} \varphi_t(Y_2).
\]
Using similar arguments as before, we get that the above inequality is equivalent to \( \varphi_t(X) \geq \varphi_t(Y) \), which in fact concludes this part of the proof.

Step 3. For any \( m_s \in \tilde{L}_s^0 \) we set,
\[
\mu_{t,s}(m_s) := \text{ess sup}_{A \in \mathcal{F}_t} \left[ \mathbf{1}_A \text{ess sup}_{Y \in \mathcal{Y}^\neg}_{A,s} \varphi_t(Y) + \mathbf{1}_{A^c}(-\infty) \right],
\]
where \( \mathcal{Y}^\neg_{A,s}(m_s) := \{ Y \in \hat{Y}_s : \mathbf{1}_A m_s \geq \mathbf{1}_A \varphi_s(Y) \} \), and show that the corresponding family of maps \( \mu \) is an projective update rule.
Adaptiveness. For any $m_s \in \bar{L}^{0}_t$, ess sup of the set of $\mathcal{F}_t$-measurable random variables $\{\varphi_t(Y)\}_{Y \in \mathcal{Y}^{-}_{A,s}(m_s)}$ is $\mathcal{F}_t$-measurable (see Karatzas and Shreve (1998), Appendix A), which implies that $\mu_{t,s}(m_s) \in \bar{L}^{0}_t$.

Monotonicity. If $m_s \geq m'_s$ then for any $A \in \mathcal{F}_t$ we get $\mathcal{Y}^{-}_{A,s}(m_s) \supseteq \mathcal{Y}^{-}_{A,s}(m'_s)$, which implies $\mu_{t,s}(m_s) \geq \mu_{t,s}(m'_s)$.

Locality. Let $B \in \mathcal{F}_t$ and $m_s \in \bar{L}^{0}_t$. It is enough to consider $A \in \mathcal{F}_t$, such that $\mathcal{Y}^{-}_{A,s}(m_s) \neq \emptyset$, as otherwise we get

$$\left[1_A \text{ ess sup } \varphi_t(Y) + 1_A^c(-\infty) \right] \equiv -\infty.$$ 

For any such $A \in \mathcal{F}_t$, we get

$$1_{A \cap B} \text{ ess sup } \varphi_t(Y) = 1_{A \cap B} \text{ ess sup } \varphi_t(Y). \quad (A.8)$$

Indeed, since $\mathcal{Y}^{-}_{A,s}(m_s) \subseteq \mathcal{Y}^{-}_{A \cap B,s}(m_s)$, we have

$$1_{A \cap B} \text{ ess sup } \varphi_t(Y) \leq 1_{A \cap B} \text{ ess sup } \varphi_t(Y).$$

On the other hand, for any $Y \in \mathcal{Y}^{-}_{A \cap B,s}(m_s)$ and for a fixed $Z \in \mathcal{Y}^{-}_{A,s}(m_s)$ we get, in view of (A.6), that

$$1_B Y + 1_B^c Z \in \mathcal{Y}^{-}_{A,s}(m_s).$$

Thus, using locality of $\varphi_t$, we deduce

$$1_{A \cap B} \text{ ess sup } \varphi_t(Y) = 1_{A \cap B} \text{ ess sup } 1_B \varphi_t(1_B Y + 1_B^c Z) \leq 1_{A \cap B} \text{ ess sup } \varphi_t(Y),$$

which proves (A.8). Now, note that $\mathcal{Y}^{-}_{A \cap B,s}(m_s) = \mathcal{Y}^{-}_{A \cap B,s}(1_B m_s)$, and thus

$$1_A \text{ ess sup } \varphi_t(Y) = 1_A \text{ ess sup } \varphi_t(Y). \quad (A.9)$$

Combining (A.8), (A.9), and the fact that $\mathcal{Y}^{-}_{A,s}(m_s) \neq \emptyset$ implies $\mathcal{Y}^{-}_{A,s}(1_B m_s) \neq \emptyset$, we obtain the following chain of equalities

$$1_B \mu_{t,s}(m_s) = 1_B \text{ ess sup } \left[1_A \text{ ess sup } \varphi_t(Y) + 1_A^c(-\infty) \right]$$

$$= 1_B \text{ ess sup } \left[1_{A \cap B} \text{ ess sup } \varphi_t(Y) + 1_{A \cap B}^c(-\infty) \right]$$

$$= 1_B \text{ ess sup } \left[1_{A \cap B} \text{ ess sup } \varphi_t(Y) + 1_{A \cap B}^c(-\infty) \right]$$

$$= 1_B \text{ ess sup } \left[1_{A \cap B} \text{ ess sup } \varphi_t(Y) + 1_{A \cap B}^c(-\infty) \right]$$

$$= 1_B \text{ ess sup } \left[1_A \text{ ess sup } \varphi_t(Y) + 1_A^c(-\infty) \right]$$

$$= 1_B \mu_{t,s}(1_B m_s).$$
Thus, \( \mu \) is an \( X \)-invariant update rule.

**Step 4.** By locality of \( \varphi \) and (A.4), we note that acceptance time consistency with respect to \( Y \) is equivalent to

\[
\varphi_t(X) \geq \operatorname{ess sup}_{A \in \mathcal{F}_t} \left[ 1_A \operatorname{ess sup}_{Y \in Y_A,\operatorname{s}} \varphi_t(Y) + 1_A(-\infty) \right].
\]  

(A.10)

Thus, using (3.4), we deduce that \( \varphi \) satisfies (3.7) if and only if \( \varphi \) is time consistent with respect to the update rule \( \mu \). Since (3.4) is equivalent to (A.10), we conclude the proof.

---

**Proof of Proposition 4.1.**

**Proof.** Monotonicity and locality of \( \mu^\operatorname{inf} \) is a straightforward implication of Proposition A.4. Thus, \( \mu^\operatorname{inf} \) is \( sX \)-invariant update rule. The projectivity comes straight from the definition (see Remark A.5).

---

**Proof of Proposition 4.3.**

**Proof.** We will only show the proof for acceptance consistency. The proof for rejection consistency is similar. Let \( \{ \varphi_t \}_{t \in \mathcal{T}} \) be a dynamic LM-measure.

1) \( \Leftrightarrow \) 2). This is a direct application of Proposition 3.6.

1) \( \Rightarrow \) 3). Assume that \( \varphi \) is weakly acceptance consistent, and let \( m_t \in \tilde{L}^0_t \) be such that \( \varphi_s(X) \geq m_t \). Then, using Proposition 3.6, we get \( \varphi_t(X) \geq \operatorname{ess inf}_t(\varphi_s(X)) \geq \operatorname{ess inf}_t(m_t) = m_t \), and hence 3) is proved.

3) \( \Rightarrow \) 1). By the definition of conditional essential infimum, \( \operatorname{ess inf}_t(\varphi_s(X)) \in \tilde{L}^0_t \), for any \( X \in \mathcal{X} \), and \( t, s \in \mathcal{T} \). Moreover, by Proposition A.4.(3), we have that \( \varphi_s(X) \geq \operatorname{ess inf}_t(\varphi_s(X)) \). Using assumption 3) with \( m_t = \operatorname{ess inf}_t(\varphi_s(X)) \), we immediately obtain \( \varphi_t(X) \geq \operatorname{ess inf}_t(\varphi_s(X)) \). Due to Proposition 3.6 this concludes the proof.

3) \( \Leftrightarrow \) 4). Clearly 3) \( \Rightarrow \) 4). Thus, it remains to show the converse implication. Since \( \varphi \) is a monetary utility measure, then invoking locality of \( \varphi \), we conclude that for any \( m_t \in \tilde{L}^0_t \), such that \( \varphi_s(X) \geq m_t \), and for any \( n \in \mathbb{N} \), we have

\[ \varphi_s(1_{\{m_t \in (-n,n)\}}(X - m_t)) \geq 0. \]

Now, in view of 4), we get that \( \varphi_t(1_{\{m_t \in (-n,n)\}}(X - m_t)) \geq 0 \), and consequently

\[ 1_{\{m_t \in (-n,n)\}}\varphi_t(X) \geq 1_{\{m_t \in (-n,n)\}}m_t. \]

Thus, 3) is proved on the \( \mathcal{F}_t \)-measurable set \( \{ m_t \in (-\infty,\infty) \} = \bigcup_{n \in \mathbb{N}} \{ m_t \in (-n,n) \} \). On the set \( \{ m_t = -\infty \} \) inequality \( \varphi_t(X) \geq m_t \) is trivial. Finally, on the set \( \{ m_t = \infty \} \), in view of the monotonicity of \( \varphi \), we have that \( \varphi_s(X) = \varphi_t(X) = \infty \), which implies 3). This concludes the proof.
A unified approach to time consistency

Proof of Proposition 4.4.

Proof. Let a family $\mu = \{\mu_t\}_{t \in T}$ of maps $\mu_t : \bar{L}^0 \rightarrow \bar{L}^0$ be given by

$$
\mu_t(m) = \ess inf_{Z \in P_t} E[Zm|\mathcal{F}_t]
$$

(A.11)

Before proving (4.3), we will need to prove some facts about $\mu$.

First, let us show that

$$
\forall t \in T. \text{ Monotonicity is straightforward. Indeed, let } m, m' \in \bar{L}^0 \text{ be such that } m \geq m'. \text{ For any } Z \in P_t, \text{ using the fact that } Z \geq 0, \text{ we get } Zm \geq Zm'. \text{ Thus, } E[Zm|\mathcal{F}_t] \geq E[Zm'|\mathcal{F}_t] \text{ and consequently } \ess inf_{Z \in P_t} E[Zm|\mathcal{F}_t] \geq \ess inf_{Z \in P_t} E[Zm'|\mathcal{F}_t]. \text{ Locality follows from the fact, for any } A \in \mathcal{F}_t \text{ and } m \in \bar{L}^0, \text{ using Proposition A.1, convention } 0 \cdot \pm \infty = 0, \text{ and the fact that for any } Z_1, Z_2 \in P_t \text{ we have } 1_A Z_1 + 1_A' Z_2 \in P_t, \text{ we get}
$$

$$
1_A \mu_t(m) = 1_A \ess inf_{Z \in P_t} E[Zm|\mathcal{F}_t]
$$

$$
= 1_A \ess inf_{Z \in P_t} (E[(1_A Z)m|\mathcal{F}_t] + E[(1_A'|Z)m|\mathcal{F}_t])
$$

$$
= 1_A \ess inf_{Z \in P_t} (E[(1_A Z)m|\mathcal{F}_t] + 1_A \ess inf_{Z \in P_t} E[(1_A'|Z)m|\mathcal{F}_t])
$$

$$
= 1_A \ess inf_{Z \in P_t} E[Z(1_A m)|\mathcal{F}_t] + 1_A \ess inf_{Z \in P_t} 1_A E[Zm|\mathcal{F}_t]
$$

$$
= 1_A \mu_t(1_A m).
$$

Secondly, let us prove that we get

$$
m \geq \mu_t(m),
$$

(A.12)

for any $m \in \bar{L}^0$. Let $m \in L^0$. For $\alpha \in (0,1)$ let

$$
Z_\alpha := 1_{\{m \leq q_\alpha^+(m)\}} E[1_{\{m \leq q_\alpha^+(m)\}}|\mathcal{F}_t]^{-1}.
$$

(A.13)

where $q_\alpha^+(\alpha)$ is $\mathcal{F}_t$-conditional (upper) $\alpha$ quantile of $m$, defined as

$$
q_\alpha^+(\alpha) := \ess sup\{Y \in L^0_t \mid E[1_{\{m \leq Y\}}|\mathcal{F}_t] \leq \alpha\}.
$$

For $\alpha \in (0,1)$, noticing that $Z_\alpha < \infty$, due to convention $0 \cdot \infty = 0$ and the fact that

$$
\{E[1_{\{m \leq q_\alpha^+(m)\}}|\mathcal{F}_t] = 0\} \subseteq \{1_{\{m \leq q_\alpha^+(m)\}} = 0\} \cup B,
$$

for some $B$, such that $P[B] = 0$, we conclude that $Z_\alpha \in P_t$.

Moreover, by the definition of $q_\alpha^+(\alpha)$, there exists a sequence $Y_n \in L^0_t$, such that $Y_n \nearrow q_\alpha^+(\alpha)$, and

$$
E[1_{\{m < Y_n\}} | \mathcal{F}_t] \leq \alpha.
$$

Consequently, by monotone convergence theorem, we have

$$
E[1_{\{m < q_\alpha^+(\alpha)\}} | \mathcal{F}_t] \leq \alpha.
$$

\footnote{In the risk measure framework, it might be seen as the risk minimizing scenario for conditional $CV@R_\alpha$.}
Hence, we deduce
\[ P[m < q_t^+(\alpha)] = E[1_{\{m < q_t^+(\alpha)\}}] \leq E[E[1_{\{m < q_t^+(\alpha)\}}|\mathcal{F}_t]] \leq E[\alpha] = \alpha, \]
which implies that
\[ P[m \geq q_t^+(\alpha)] \geq (1 - \alpha). \] (A.14)

On the other hand
\[ 1_{\{m \geq q_t^+(\alpha)\}}m \geq 1_{\{m \geq q_t^+(\alpha)\}}q_t^+(\alpha) = 1_{\{m \geq q_t^+(\alpha)\}}q_t^+(\alpha)E[Z_\alpha|\mathcal{F}_t] \]
\[ \geq 1_{\{m \geq q_t^+(\alpha)\}}E[Z_\alpha q_t^+(\alpha)|\mathcal{F}_t] \geq 1_{\{m \geq q_t^+(\alpha)\}}E[Z_\alpha m|\mathcal{F}_t], \]
which combined with (A.14), implies that
\[ P[m \geq E[Z_\alpha m|\mathcal{F}_t]] \geq 1 - \alpha. \] (A.15)

Hence, using (A.15), and the fact that
\[ E[Z_\alpha m|\mathcal{F}_t] \geq \mu_t(m), \quad \alpha \in (0, 1), \]
we get that
\[ P[m \geq \mu_t(m)] \geq 1 - \alpha. \]

Letting \( \alpha \to 0 \), we conclude that (A.12) holds true for \( m \in \bar{L}^0 \).

Now, assume that \( m \in \bar{L}^0 \), and let \( A := \{E[1_{\{m = -\infty\}}|\mathcal{F}_t] = 0\} \). Similar to the arguments above, we get
\[ 1_A m \geq \mu_t(1_A m). \]

Since \( \mu_t(0) = 0 \), and due to locality of \( \mu_t \), we deduce
\[ 1_A m \geq \mu_t(1_A m) = 1_A \mu_t(1_A m) = 1_A \mu_t(m). \] (A.16)

Moreover, taking \( Z = 1 \) in (A.11), we get
\[ 1_A \cdot \mu_t(1_A \cdot (-\infty)) = 1_A \cdot E[m|\mathcal{F}_t] \geq 1_A \cdot \mu_t(m). \] (A.17)

Combining (A.16) and (A.17), we concludes the proof of (A.12) for all \( m \in \bar{L}^0 \).

Finally, we will show that \( \mu \) defined as in (A.11) satisfies property 1) from Proposition A.4, which will consequently imply equality (4.3). Let \( m \in \bar{L}^0 \) and \( A \in \mathcal{F}_t \). From the fact that \( m \geq \mu_t(m) \) we get
\[ \text{ess inf}_{\omega \in A} \ m \geq \text{ess inf}_{\omega \in A} \ \mu_t(m). \]

On the other hand we know that \( 1_A \text{ ess inf}_{\omega \in A} \ m \leq 1_A m \) and \( 1_A \text{ ess inf}_{\omega \in A} \ m \in \bar{L}_t^0 \), so
\[ \text{ess inf}_{\omega \in A} \ m = \text{ess inf}_{\omega \in A} (1_A \text{ ess inf}_{\omega \in A} \ m) = \text{ess inf}_{\omega \in A} (1_A \mu_t(1_A \text{ ess inf}_{\omega \in A} \ m)) \leq \text{ess inf}_{\omega \in A} (1_A \mu_t(1_A m)) = \text{ess inf}_{\omega \in A} (1_A \mu_t(m)) = \text{ess inf}_{\omega \in A} \mu_t(m) \]
which proves the equality. The proof for ess sup \( \mu \) is similar and we omit it here. This concludes the proof.
Proof of Proposition 4.5.

Proof. Then, using Proposition A.4, for any \( t, s \in \mathbb{T}, s > t \), and any \( X \in \mathcal{X} \), we get
\[
\varphi_t(X) \geq \mu_t(\varphi_s(X)) \geq \mu_t(\text{ess inf}_s(\varphi_s(X))) \geq \mu_t(\text{ess inf}_t(\varphi_s(X))) = \text{ess inf}_t(\varphi_s(X)).
\]
The proof for rejection time consistency is similar. \(\square\)

Proof of Proposition 4.8.

Proof. We will only show the proof for acceptance consistency. The proof for rejection consistency is similar. Let \( \varphi \) be a dynamic LM-measure.

1) \(\Leftrightarrow\) 2). This is a direct implication of Proposition 3.6.

2) \(\Rightarrow\) 3). Assume that \( \varphi \) is semi-weakly acceptance consistent. Let \( V \in \mathcal{X} \) and \( m_t \in \mathcal{L}_t^0 \) be such that \( \varphi_{t+1}(V) \geq m_t \) and \( V_t \geq 0 \). Then, by monotonicity of \( \mu_t^{\text{inf}} \), we have
\[
\varphi_t(V) \geq \mathbb{1}_{\{V_t \geq 0\}} \mu_t^{\text{inf}}(\varphi_{t+1}(V)) \geq \mu_t^{\text{inf}}(m_t) = \text{ess inf}_t(m_t) = m_t,
\]
and hence 3) is proved.

3) \(\Rightarrow\) 2). Let \( V \in \mathcal{X} \). We need to show that
\[
\varphi_t(V) = \mathbb{1}_{\{V_t \geq 0\}} \mu_t^{\text{inf}}(\varphi_{t+1}(V)) + \mathbb{1}_{\{V_t < 0\}}(-\infty).
\] (A.18)

On the set \( \{V_t < 0\} \) inequality (A.18) is trivial. We know that
\[
(\mathbb{1}_{\{V_t \geq 0\}} \cdot V)_t \geq 0 \quad \text{and} \quad \varphi_{t+1}(\mathbb{1}_{\{V_t \geq 0\}} \cdot V) \geq \text{ess inf}_t \varphi_{t+1}(\mathbb{1}_{\{V_t \geq 0\}} \cdot V).
\]

Thus, for \( m_t = \text{ess inf}_t \varphi_{t+1}(\mathbb{1}_{\{V_t \geq 0\}} \cdot V) \), using locality of \( \varphi \) and \( \mu_t^{\text{inf}} \) as well as 3), we get
\[
\mathbb{1}_{\{V_t \geq 0\}} \varphi_t(V) = \mathbb{1}_{\{V_t \geq 0\}} \varphi_t(\mathbb{1}_{\{V_t \geq 0\}} \cdot V) \geq \mathbb{1}_{\{V_t \geq 0\}} m_t = \mathbb{1}_{\{V_t \geq 0\}} \mu_t^{\text{inf}}(\varphi_{t+1}(V)).
\]

and hence (A.18) is proved on the set \( \{V_t \geq 0\} \). This conclude the proof of 2). \(\square\)

Proof of Proposition 4.9

Proof. The proof of locality and monotonicity of (4.5) is straightforward (see Bielecki et al. (2014b) for details). Let us assume that \( \{\varphi_t^x\}_{t \in \mathbb{T}} \) is weakly acceptance time consistent. Using counterpart of Proposition 4.3 for stochastic processes (see Bielecki et al. (2015c)) we get
\[
\mathbb{1}_{\{V_t \geq 0\}} \alpha_t(V) = \mathbb{1}_{\{V_t \geq 0\}} \left( \sup \{ x \in \mathbb{R}_+ : \mathbb{1}_{\{V_t \geq 0\}} \varphi_t^x(V) \geq 0 \} \right) \\
\geq \mathbb{1}_{\{V_t \geq 0\}} \left( \sup \{ x \in \mathbb{R}_+ : \mathbb{1}_{\{V_t \geq 0\}} \text{ess inf}_t \varphi_{t+1}^x(V) + V_t \geq 0 \} \right) \\
\geq \mathbb{1}_{\{V_t \geq 0\}} \left( \sup \{ x \in \mathbb{R}_+ : \mathbb{1}_{\{V_t \geq 0\}} \text{ess inf}_t \varphi_{t+1}^x(V) \geq 0 \} \right) \\
= \mathbb{1}_{\{V_t \geq 0\}} \text{ess inf}_t \left( \sup \{ x \in \mathbb{R}_+ : \mathbb{1}_{\{V_t \geq 0\}} \varphi_{t+1}^x(V) \geq 0 \} \right) \\
= \mathbb{1}_{\{V_t \geq 0\}} \text{ess inf}_t \alpha_{t+1}(V).
\]
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This leads to inequality

\[ \alpha_t(V) \geq 1_{\{V_t \geq 0\}} \text{ess inf}_t \alpha_{t+1}(V) + 1_{\{V_t < 0\}}(-\infty), \]

which, by Proposition 4.8, is equivalent to semi-weak rejection time consistency. The proof of weak acceptance time consistency is similar.

Proof of Proposition 4.10

Proof. The proof of locality and monotonicity of (4.7) is straightforward (see Bielecki et al. (2014b) for details). Let us prove weak acceptance time consistency. Let us assume that \{\alpha_t\}_{t \in \mathbb{T}} is semi-weakly acceptance time consistent. Using Proposition 3.6 we get

\[ \varphi^T_t(V) = \inf\{c \in \mathbb{R} : \alpha_t(V - c1_{\{t\}}) \leq x\} \]
\[ = \inf\{c \in \mathbb{R} : \alpha_t(V - c1_{\{t+1\}}) \leq x\} \]
\[ = \inf\{c \in \mathbb{R} : \alpha_t(V - c1_{\{t+1\}} - V_t1_{\{t\}}) \leq x\} + V_t \]
\[ \geq \inf\{c \in \mathbb{R} : 1_{\{0 \geq 0\}} \text{ess inf}_t \alpha_{t+1}(V - c1_{\{t+1\}} - V_t1_{\{t\}}) + 1_{\{0 < 0\}}(-\infty) \leq x\} + V_t \]
\[ = \inf\{c \in \mathbb{R} : \text{ess inf}_t \alpha_{t+1}(V - c1_{\{t+1\}}) \leq x\} + V_t \]
\[ = \text{ess inf}_t (\inf\{c \in \mathbb{R} : \alpha_{t+1}(V - c1_{\{t+1\}}) \leq x\}) + V_t \]

which, is equivalent to weak acceptance time consistency of \varphi. The proof of rejection time consistency is similar.

Proof of Proposition A.1.

Proof. First note that for any \(X,Y \in \bar{L}^0, \lambda \in L_0^0\) such that \(X,Y,\lambda \geq 0\), and for any \(s,t \in \mathbb{T}, s > t\), by Monotone Convergence Theorem, and using the convention \(0 \cdot \pm \infty = 0\) we get

\[ E[\lambda X | \mathcal{F}_t] = \lambda E[X | \mathcal{F}_t]; \quad (A.19) \]
\[ E[X | \mathcal{F}_t] = E[E[X | \mathcal{F}_s] | \mathcal{F}_t]; \quad (A.20) \]
\[ E[X | \mathcal{F}_t] + E[Y | \mathcal{F}_t] = E[X + Y | \mathcal{F}_t]. \quad (A.21) \]

Moreover, for \(X \in L^0\), we also have

\[ E[-X | \mathcal{F}_t] \leq -E[X | \mathcal{F}_t]. \quad (A.22) \]

For the last inequality we used the convention \(\infty - \infty = -\infty\).

Next, using (A.19)-(A.22), we will prove the announced results. Assume that \(X,Y \in \bar{L}^0\).

1) If \(\lambda \in L_0^0\), and \(\lambda \geq 0\), then, by (A.19) we get

\[ E[\lambda X | \mathcal{F}_t] = E[(\lambda X)^+ | \mathcal{F}_t] - E[(\lambda X)^- | \mathcal{F}_t] = E[\lambda X^+ | \mathcal{F}_t] - E[\lambda X^- | \mathcal{F}_t] = \lambda E[X^+ | \mathcal{F}_t] - \lambda E[X^- | \mathcal{F}_t] = \lambda E[X | \mathcal{F}_t]. \]
From here, and using (A.22), for a general $\lambda \in L_t^0$, we deduce

$$E[\lambda X|F_t] = E[1_{\{\lambda \geq 0\}} \lambda X + 1_{\{\lambda < 0\}} \lambda X|F_t] = 1_{\{\lambda \geq 0\}} \lambda E[X|F_t] + 1_{\{\lambda < 0\}} (-\lambda) E[-X|F_t] \leq 1_{\{\lambda \geq 0\}} \lambda E[X|F_t] + 1_{\{\lambda < 0\}} \lambda E[X|F_t] = \lambda E[X|F_t].$$

2) The proof of 2) follows from (A.20) and (A.22); for $X \in L^0$ see also the proof in (Cherny, 2010, Lemma 3.4).

3) On the set $\{E[X|F_t] = -\infty\} \cup \{E[Y|F_t] = -\infty\}$ the inequality is trivial due to the convention $\infty - \infty = -\infty$. On the other hand the set $\{E[X|F_t] > -\infty\} \cap \{E[Y|F_t] > -\infty\}$ could be represented as the union of the sets $\{E[X|F_t] > n\} \cap \{E[Y|F_t] > n\}$ for $n \in \mathbb{Z}$ on which the inequality becomes the equality, due to (A.21).

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References


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