A note on parameter estimation for discretely sampled SPDEs

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ABSTRACT: We consider a parameter estimation problem for one dimensional stochastic heat equations, when data is sampled discretely in time or spatial component. We establish some general results on derivation of consistent and asymptotically normal estimators based on computation of the p-variations of stochastic processes and their smooth perturbations. We apply these results to the considered SPDEs, by using some convenient representations of the solutions. For some equations such results were ready available, while for other classes of SPDEs we derived the needed representations along with their statistical asymptotical properties. We prove that the real valued parameter next to the Laplacian, and the constant parameter in front of the noise (the volatility) can be consistently estimated by observing the solution at a fixed time and on a discrete spatial grid, or at a fixed space point and at discrete time instances of a finite interval, assuming that the mesh-size goes to zero.

KEYWORDS: p-variation, statistics for SPDEs, discrete sampling, stochastic heat equation, inverse problems for SPDEs, Malliavin calculus.
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1 Introduction

Consider the following (parabolic) Stochastic Partial Differential Equations (SPDEs)

$$\mathrm{d}u(t) = (\theta\mathcal{A}_1 + \mathcal{A}_0)u(t)\,\mathrm{d}t + \sigma(\mathcal{M}u(t) + g(t))\,\mathrm{d}W(t)\,, \tag{1.1}$$

where $\mathcal{A}_0, \mathcal{A}_1, \mathcal{M}$ are some (linear or nonlinear) operators acting in suitable Hilbert spaces, g is an adapted vector-valued function, W is a cylindrical Brownian motion, and θ and σ are unknown parameters (to be estimated) belonging to a subset of real line. Implicitly we will assume that (1.1) is parabolic and admits a unique solution, although usually this has to be established on a case by case basis.

Major part of the existing literature on statistical inference for SPDEs (estimating θ and σ) lies within the spectral approach, where it is assumed that one path of the first N Fourier modes of the solution is observed continuously over a finite interval of time. In this case, the coefficient σ can be determine explicitly and exactly, similar to the case of finite dimensional diffusions, by employing quadratic variation type arguments, and due to the fact that a path is observed continuously in time. A general method of estimating θ is to construct Maximum Likelihood Estimators (MLEs) based on the information revealed by the first N Fourier modes, and prove that these estimators satisfy the desired statistical properties, such as consistency, asymptotic normality, and efficiency, as N increases. We refer the reader to the recent monograph [LR17, Chapter 6] for a comprehensive survey of this method applied to diagonalizable SPDEs. For MLE based estimators applied to nonlinear SPDEs see for instance [CGH11]. For other type of estimators, assuming the same observation scheme, see [CGH16]. Beyond spectral approach, the literature on parameter estimation for SPDEs is limited, and only few papers are devoted to discretely sampled SPDEs [PR97, Mar03, PvsT07]. Of course, one way to deal with discretely sampled data, is to discretize or approximate the MLEs using the available discrete data, and show that the statistical properties are preserved. On the other hand, if we assume that the solution itself is observed at some space-time grid points, one needs to approximate additionally the Fourier modes. To best of our knowledge, a rigourous asymptotic analysis of this idea is still to be done. Finally, it needs to be mentioned, that by its very nature, the Fourier decomposition has to be performed with the respect to the basis formed by the eigenfunctions of the operator $\mathcal{A}_1$. Usually, $\mathcal{A}_1$ is a differential operator, and thus essentially one has to deal with bounded domains.

The main goal of this notes is to study the parameter estimation problem for simple parabolic SPDEs, when data is sampled discretely. Namely, we consider the stochastic heat equation, one dimensional, driven by an additive or multiplicative space-time noise, either on bounded domain or whole space, and when the solution u is observed at some discrete space-time points. As such, we do not rely on spectral approach, but rather use some suitable representations of the solution to derive the corresponding estimators. The key idea of the proposed method relies on an intuitively simple observation: the p-variation of a stochastic process is invariant with respect to smooth perturbations. Hence, if the p-variation of a process X can be computed by an explicit formula, and the parameter of interest enters non-trivially into this formula, one can derive consistent estimators of this parameter (similar to estimating the volatility through quadratic variation). However, since the p-variation of the perturbed process $X+Y$ remains the same, given that Y is smooth enough, then the same estimator remains consistent assuming that $X+Y$ is observed. Analogous arguments remain valid for asymptotic normality property. See Section 2.1 for the formal result and some simple applications to parameter estimation problems. Thus, it remains to find suitable representations of the solution u as a sum of two processes. In Section 3, we start with the heat equation on the whole real line, and driven by an additive noise. It turns out that for any fixed instance of time $t>0$, the solution as a function of $x\in\mathbb{R}$ can be represented as a scaled two-sided Brownian motion plus a smooth process. Similarly, if we fix a spacial point, then the solution is a smoothly perturbed scaled fractional Brownian motion. We refer to [Kho14, Section 3] for details on these representations. With these at hand, using the p-variation idea described above, both θ and σ can be estimated in either time or space sampling regime. Hence, to construct a consistent, and asymptotically normal estimator for θ or σ it is enough to observe the solution at one time instant and discretely on a spacial grid of a finite interval, with mesh diameters going to zero. By the same token, it is sufficient to observe the solution just at one spacial point, and over a time-grid interval. We focus our study on these two sampling schemes. It should be mentioned that similar estimators, and same sampling schemes were studied in [PvsT07], where the authors considered the heat equation on $\mathbb{R}$ driven by a multiplicative noise. The methods of proof in [PvsT07] are different from ours. For the sake of completeness we present some relevant results in Section 5. In Section 4 we investigate the case of bounded domain and additive noise. There are no ready available results on the representations of the solution, and we first establish the corresponding result when time is fixed, which can be easily done via Karhunen–Loève expansion. The case of sampling the solution in time at a fixed spacial point for bounded domains is more delicate. We prove that the solution can be represented as a sum of a smooth process and a zero-mean Gaussian process with known finite fourth variation. Moreover, using some elements of Malliavin calculus, as well as a version of the cental limit theorem from [NOL08], we establish a central limit type theorem for the fourth variation of the solution. Consequently, we derive weakly consistent estimators for

θ and σ, and prove their asymptotic normality. The results on the representation of the solution are of independent interest, and could be used beyond statistical inference problems. It would be fair to note that a similar methodology of using Malliavin calculus technics to establish cental limit theorem can be found in [Cor12], albeit applied to similar processes but with a simpler covariance structure. To streamline the presentation, most of the proofs and some auxiliary technical results are moved to Appendix.

Finally, we want to mention that there are many natural questions that are left open, such as: considering more general sampling schemes, by sampling simultaneously in time and space; investigate equations in higher dimensions and of more complicated structure; equations on bounded domains and driven by multiplicative noise; nonlinear equations, etc. Some of these questions will be addressed by the authors in the future works.

2 Setup of the problem

Let $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \mathbb{P})$ be a stochastic basis satisfying the usual assumptions, and let G be either a bounded smooth domain in $\mathbb{R}$ or the whole real line $\mathbb{R}$. We consider the following stochastic partial differential equation on $H = L^2(G)$

$$\begin{aligned} \mathrm{d}u(t,x) &= \theta u_{xx}(t,x)\,\mathrm{d}t + \sigma g(u)\,\mathrm{d}W(t,x), \quad x\in G, \quad t>0, \\ u(0,x) &= u_0(x) \in L^2(G), \\ u(t,\cdot)|_{\partial D} &= 0, \quad t>0, \end{aligned} \tag{2.1}$$

where W is an H-valued (cylindrical) Brownian motion, $g:\mathbb{R}\to\mathbb{R}$, and θ,σ are some positive constants. Under some fairly general assumptions on the structure of the noise W, the function g and the parameters θ and σ, the solution to (2.1) exists and is unique [Cho07, LR17].

As usual, everywhere below, all equalities and inequalities between random variables, unless otherwise noted, will be understood in the $\mathbb{P}$-a.s. sense. The notations $\xrightarrow{\mathcal{D}}$ will be used for convergence in distribution, while $\xrightarrow{\mathbb{P}}$ or $\mathbb{P}-\lim$ will stand for convergence in probability.

We assume that $\theta\in\Theta\subset(0,+\infty)$ and $\sigma\in\mathbf{S}\subset(0,+\infty)$ are the (unknown) parameters of interest. In this work we focus on two sampling schemes[1]:

(A) *Fixed time and discrete space.* For a fixed instant of time $t>0$, and given interval $[a,b]\subset G$, the solution u is observed at points (t,x_j), $j=1,\ldots,m$, with $x_j = a+(b-a)j/m$, $j = 0,1,\ldots,m$.

(B) *Fixed space and discrete time.* For a fixed x from the interior of G, and given time interval $[c,d]\subset(0,+\infty)$, the solution u is observed at points $\{(t_i,x),\ i=1,\ldots,n\}$, where $t_i := c+(d-c)i/n,\ i=0,1,\ldots,n$.

The main goal of this paper is to derive consistent estimators for the parameters θ and σ under these sampling schemes, and to study the asymptotic properties of these estimators.

In what follows, we will use the notation $\Upsilon^m(a,b) = \{a_j \mid a_j = a+(b-a)j/m,\ j=0,1,\ldots,m\}$ for the uniform partition of size m of a given interval $[a,b]\subset\mathbb{R}$. For a given stochastic process X on some interval $[a,b]$, and $p\geq 1$, we will denote by $\mathsf{V}^p_m(X;[a,b])$ the sum

$$\mathsf{V}^p_m(X;[a,b]) := \sum_{j=1}^m |X(t_j)-X(t_{j-1})|^p,$$

[1]For simplicity of writing, we assume that the sampling points form a uniform grid. Generally speaking all the results hold true assuming only that the mesh size of the grid goes to zero.

where $t_j \in \Upsilon^m(a,b)$. Correspondingly,

$$\mathsf{V}^p(X;[a,b]) := \lim_{m\to\infty} \mathsf{V}^p_m(X;[a,b]), \quad \mathbb{P}-\text{a.s.},$$
$$\mathsf{V}^p_{\mathbb{P}}(X;[a,b]) := \mathbb{P}-\lim_{m\to\infty} \mathsf{V}^p_m(X;[a,b]),$$

will denote the p-variation of X on $[a,b]$, in $\mathbb{P}$-a.s. sense and respectively in probability. If no confusions arise, we will simply write $\mathsf{V}^p(X)$, and $\mathsf{V}^p_m(X)$ instead of $\mathsf{V}^p(X;[a,b])$ and $\mathsf{V}^p_m(X;[a,b])$; same applies to $\mathsf{V}^p_{\mathbb{P}}(X)$.

2.1 Statistical properties for smoothly perturbed stochastic processes

As already mentioned, the estimators proposed in this work are derived using the p-variation of some suitable processes. The next result shows that the 'quadratic variation type arguments' of estimating the diffusion coefficient are invariant with respect to smooth perturbations.

Proposition 2.1. *Let $X(t), Y(t)$, $t \in [a,b]$, be stochastic processes with continuous paths, and assume that the process Y has $C^1[a,b]$ sample paths, and there exists $p>1$, such that $0 < \mathsf{V}^p(X) < \infty$. Then,*

$$\mathsf{V}^p(X+Y;[a,b]) = \mathsf{V}^p(X;[a,b]). \tag{2.2}$$

Similarly, if $0 < \mathsf{V}^p_{\mathbb{P}}(X) < \infty$, then

$$\mathsf{V}^p_{\mathbb{P}}(X+Y;[a,b]) = \mathsf{V}^p_{\mathbb{P}}(X;[a,b]). \tag{2.3}$$

If in addition, there exist $\alpha, \sigma_0 > 0$ such that, $\alpha + 1/p < 1$,

$$n^\alpha \left(\mathsf{V}^p_n(X;[a,b]) - \mathsf{V}^p(X;[a,b])\right) \xrightarrow[n\to\infty]{\mathcal{D}} \mathcal{N}(0,\sigma_0^2), \tag{2.4}$$

then

$$n^\alpha \left(\mathsf{V}^p_n(X+Y;[a,b]) - \mathsf{V}^p(X;[a,b])\right) \xrightarrow[n\to\infty]{\mathcal{D}} \mathcal{N}(0,\sigma_0^2). \tag{2.5}$$

Moreover, if Y has $C^2[a,b]$ sample paths, and (2.4) holds for $p=2$ and $\alpha = 1/2$, then (2.5) holds true too, with $p=2, \alpha=1/2$.

The proof is deferred to Appendix B.

This result allows to construct directly consistent and asymptotically normal estimators for some parameter entering the true law of the perturbed process $X+Y$, given that the p-variation $\mathsf{V}^p(X;[a,b])$ of the unperturbed process X depends non-trivially on the parameter of interest, and this dependence can be computed explicitly.

For example, let B be a two-sided Brownian motion, and Y be a process with a $C^2(\mathbb{R})$ version, and consider the stochastic process

$$Z(x) = \sqrt{\beta} B(x) + Y(x), \quad x \in \mathbb{R},$$

where β is a positive, unknown parameter. Assume that Z is observed at grid points $\Upsilon^m(a,b)$, for some interval $[a,b] \subset \mathbb{R}$. In view of (2.2),

$$\mathsf{V}^2(Z;[a,b]) = \mathsf{V}^2(\sqrt{\beta}B;[a,b]) = \beta(b-a).$$

Consequently, the estimator

$$\widehat{\beta}_m = \frac{1}{b-a}\sum_{j=1}^m \left(Z(x_j) - Z(x_{j-1})\right)^2,$$

is a consistent estimator of β, namely $\lim_{m\to\infty}\widehat{\beta}_m = \beta$, $\mathbb{P}$-a.s.. Moreover, it is well known (cf. [Nou08, AES16]) that

$$\sqrt{m}(\mathsf{V}_m^2(B,[a,b]) - (b-a)) \xrightarrow[m\to\infty]{\mathcal{D}} \mathcal{N}(0, 2(b-a)^2),$$

and thus, by Proposition 2.1, the estimator $\widehat{\beta}_m$ is asymptotically normal, with the convergence

$$\sqrt{m}(\widehat{\beta}_m - \beta) \xrightarrow[m\to\infty]{\mathcal{D}} \mathcal{N}(0, 2\beta^2).$$

Similarly, let B^H be a fractional Brownian Motion (fBM) with Hurst index $H = \frac{1}{4}$, and Y be a process with continuously differentiable paths in $(0,+\infty)$. Assume that η is the parameter of interest, and suppose that the process

$$Z^H(t) = \eta^{1/4}B^H(t) + Y(t), \quad t > 0.$$

is sampled at grid points $t_i \in \Upsilon^n(c,d), i = 0,1,\ldots,n$, with $[c,d] \subset (0,\infty)$. Then,

$$\widehat{\eta}_n = \frac{1}{3(d-c)}\sum_{i=1}^{n}\left(Z^H(t_i) - Z^H(t_{i-1})\right)^4,$$

is a consistent estimator of η, since an fBM with Hurst index H has a finite, non-zero $p = 1/H$-variation. The asymptotic normality of $\mathsf{V}_n^4(B^H;[c,d])$ is established in Theorem A.1, and Corollary A.2, and hence, by (2.5), $\widehat{\eta}_n$ is also asymptotically normal, and satisfying

$$\sqrt{n}(\widehat{\eta}_n - \eta) \xrightarrow[n\to\infty]{\mathcal{D}} \mathcal{N}(0, \frac{1}{9}\breve{\sigma}^2\eta^2).$$

where $\breve{\sigma}^2$ is an explicit constant given in Corollary A.2.

3 Additive noise, whole space

In this section, we consider the SPDE (2.1) on the whole space $G = \mathbb{R}$, driven by an additive space-time white noise, and for simplicity we take zero initial data. Namely, we consider the following evolution equation

$$\begin{aligned} &\mathrm{d}u(t,x) = \theta u_{xx}(t,x)\,\mathrm{d}t + \sigma\,\mathrm{d}W(t,x), \quad x\in\mathbb{R}, \quad t>0, \\ &u(0,x) = 0, \quad x\in\mathbb{R}. \end{aligned} \tag{3.1}$$

The estimators for θ and σ are obtained by using the following representations (cf. [Kho14, Section 3]) of the solution u of (3.1):

(a) For every fixed $t > 0$, there exist a two-sided Brownian motion $B(x)$ and a Gaussian process $X(x)$ with a $C^\infty(\mathbb{R})$ version, such that

$$u(t,x) = \frac{\sigma}{\sqrt{2\theta}}B(x) + X(x), \quad x\in\mathbb{R}. \tag{3.2}$$

(b) For every fixed $x \in \mathbb{R}$, there exists a fractional Brownian motion $B^H(t)$ with Hurst index $H = 1/4$ and a Gaussian process $Y(t)$ that is continuous on $\mathbb{R}_+$ and infinitely differentiable on $(0,\infty)$, such that

$$u(t,x) = \frac{\sigma}{(\theta\pi)^{1/4}}B^H(t) + Y(t), \quad t>0. \tag{3.3}$$

It turns out that to estimate one of the parameters θ or σ, while the second one is known, it is enough to observe the solution u at one time instant, and at discrete space points with the mesh diameter going to zero. Similarly, it is enough to observe the solution at one fixed spatial point, and at discrete time points in a finite time interval (past initial time) with vanishing step size. These results are proved in the next subsections.

3.1 Space sampling at a fixed time instance

Assume that $t > 0$ is a fixed time instant, and consider the partition $\Upsilon^m(a,b)$ of the fixed interval $[a,b] \subset \mathbb{R}$. Suppose that the solution u of (3.1) is observed at the grid points $\{(t,x_j) \mid x_j \in \Upsilon^m(a,b), j = 1,\dots,m\}$. Consider the following estimators for θ and σ^2 respectively

$$\widehat{\theta}_{m,t} := \frac{(b-a)\sigma^2}{2\sum_{j=1}^m (u(t,x_j) - u(t,x_{j-1}))^2}, \tag{3.4}$$

$$\widehat{\sigma}^2_{m,t} := \frac{2\theta}{b-a}\sum_{j=1}^m (u(t,x_j) - u(t,x_{j-1}))^2. \tag{3.5}$$

Clearly, (3.4) assumes that σ is known, while (3.5) assumes that θ is known. The following results show that these estimators are consistent and asymptotically normal.

Theorem 3.1. *Assuming that σ is known, the estimator (3.4) of θ is:*

(i) consistent, that is $\lim_{m\to\infty} \widehat{\theta}_{m,t} = \theta, \ \mathbb{P} - a.s.$,

(ii) asymptotically normal,

$$\sqrt{m}(\widehat{\theta}_{m,t} - \theta) \xrightarrow[m\to\infty]{\mathcal{D}} \mathcal{N}(0, 2\theta^2). \tag{3.6}$$

Proof. Using the representation (3.2), and in view of Proposition 2.1, consistency of $\widehat{\theta}_{m,t}$ follows at once. In addition, we also have that

$$\sqrt{m}\left(\sum_{j=1}^m (u(t,x_j) - u(t,x_{j-1}))^2 - \frac{(b-a)\sigma^2}{2\theta}\right) \xrightarrow[m\to\infty]{\mathcal{D}} \mathcal{N}(0, \frac{(b-a)^2\sigma^4}{2\theta^2}).$$

Consequently, a direct application of Delta-Method yields (3.6), and this concludes the proof. □

Similarly, employing again Proposition 2.1, one has the following result.

Theorem 3.2. *Assuming that θ is known, the estimator (3.5) is a consistent and asymptotically normal estimator of σ^2, with*

$$\sqrt{m}(\widehat{\sigma}^2_{m,t} - \sigma^2) \xrightarrow[m\to\infty]{\mathcal{D}} \mathcal{N}(0, 2\sigma^4). \tag{3.7}$$

3.2 Time sampling at a fixed space point

In this section we assume that the solution u of (3.1) is observed at the grid points $\{(t_i,x) : i = 1,\dots,n\}$, where $x \in \mathbb{R}$ is a fixed spatial point, and $0 < c < d < \infty$. We consider the following estimators for θ, and σ^2 respectively,

$$\widehat{\theta}_{n,x} := \frac{3(d-c)\sigma^4}{\pi\sum_{i=1}^n (u(t_i,x)-u(t_{i-1},x))^4}, \tag{3.8}$$

$$\widehat{\sigma}^2_{n,x} := \sqrt{\frac{\theta\pi}{3(d-c)}\sum_{i=1}^n (u(t_i,x)-u(t_{i-1},x))^4}. \tag{3.9}$$

Similar the previous section, the following results about asymptotic properties of these estimators hold.

Theorem 3.3. *Given that σ is known, we have that*

$$\lim_{n\to\infty}\widehat{\theta}_{n,x} = \theta, \quad \mathbb{P}-a.s.$$
$$\sqrt{n}(\widehat{\theta}_{n,x}-\theta) \xrightarrow[n\to\infty]{\mathcal{D}} \mathcal{N}(0, \frac{1}{9}\theta^2\check{\sigma}^2).$$

Assuming that θ is known, we have that

$$\lim_{n\to\infty}\widehat{\sigma}^2_{n,x} = \sigma^2, \quad \mathbb{P}-a.s.$$
$$\sqrt{n}(\widehat{\sigma}^2_{n,x}-\sigma^2) \xrightarrow[n\to\infty]{\mathcal{D}} \mathcal{N}(0, \frac{1}{36}\sigma^4\check{\sigma}^2).$$

where $\check{\sigma}^2$ is the constant given in (A.2).

The proof is analogous to the proofs of Theorems 3.1 and 3.2 and is omitted here.

4 Additive noise, bounded domain

In this section we consider the stochastic evolution equation (2.1) on bounded domain $G=[0,\pi]$, with zero initial data, zero boundary conditions, and driven by a space-time white noise:

$$\begin{aligned} \mathrm{d}u(t,x) &= \theta u_{xx}(t,x)\,\mathrm{d}t + \sigma\,\mathrm{d}W(t,x), \quad x\in(0,\pi), \quad t>0,\\ u(0,x) &= 0, \quad x\in(0,\pi),\\ u(t,0) &= u(t,\pi)=0, \quad t>0, \end{aligned} \tag{4.1}$$

In this case, the Laplace operator $\Delta=\partial_{xx}$ has only discrete spectrum, with eigenvalues $\lambda_k = -k^2$, $k\in\mathbb{N}$, and corresponding eigenfunctions $h_k(x)=\sqrt{2/\pi}\sin(kx)$, $k\in\mathbb{N}$. Moreover, the functions $\{h_k,\ k\in\mathbb{N}\}$ form a complete orthonormal system in $L^2(G)$, and the noise term can be conveniently written as

$$W(t,x)=\sum_{k\geq 1} w_k(t)h_k(x),$$

where $w_k, k\in\mathbb{N}$, are independent standard Brownian motions. The solution of this equation admits a Fourier series decomposition,

$$u(t,x)=\sum_{k\geq 1} u_k(t)h_k(x), \quad t>0, \quad x\in(0,\pi),$$

where each Fourier mode $u_k(t)$ is an Ornstein–Uhlenbeck process of the form

$$\begin{aligned}\mathrm{d}u_k(t) &= -\theta k^2 u_k(t)\,\mathrm{d}t + \sigma\,\mathrm{d}w_k(t), \quad t>0,\\ u_k(0) &= 0.\end{aligned}$$

Equivalently, we have that

$$u_k(t) = \sigma\int_0^t e^{-\theta k^2(t-s)}\,\mathrm{d}w_k(s). \tag{4.2}$$

Clearly, $u_k(t)\sim\mathcal{N}(0,\frac{(1-e^{-2\theta k^2 t})\sigma^2}{2\theta k^2})$, and u_k, $k\in\mathbb{N}$, are independent random variable.

4.1 Space sampling at a fixed time instance

First we will establish the counterpart of the representation (3.2).

Theorem 4.1. *For every fixed $t>0$, there is a Brownian motion $B(x)$ on $[0,\pi]$, and a Gaussian process $R(x)$, $x\in[0,\pi]$ with a $C^\infty(0,\pi)$ version, such that*

$$u(t,x) = \frac{\sigma}{\sqrt{2\theta}}B(x) + R(x), \quad x\in[0,\pi].$$

Proof. A similar result, left as an exercise, can be found in [Wal86, Exercise 3.10]. For the sake of completeness, we sketch the proof here. It is enough to note that the solution u can be represented, for any $t>0$, as

$$u(t,x) = \frac{\sigma}{\sqrt{2\theta}}B(x) + R(x),$$

where

$$\begin{aligned}B(x) &= \xi_0 + \sum_{k\geq 1}\frac{1}{k}\xi_k h_k(x), & R(x) &= -\frac{\sigma x}{\sqrt{2\theta\pi}}\xi_0 + \frac{\sigma}{\sqrt{2\theta}}\sum_{k\geq 1}\frac{a_k-1}{k}\xi_k h_k(x),\\ \xi_k &= \sqrt{\frac{2\theta k^2}{(1-e^{-2\theta k^2 t})\sigma^2}}u_k(t), & a_k &= \sqrt{1-e^{-2\theta k^2 t}}.\end{aligned}$$

Note that ξ_k are i.i.d. standard Gaussian random variables. It is easy to check that B is a standard Brownian motion on $[0,\pi]$, for example by noting that v is the Karhunen–Loève expansion for the Brownian motion, up to some change of variables. It is also straightforward to show that R is smooth. This completes the proof. □

With this at hand, similar to Theorem 3.1, we have the following result.

Theorem 4.2. *Let u be the solution to (4.1), and assume that u is sampled at discrete points $\{(t,x_j)\mid x_j\in\Upsilon^m(a,b)\}$, for some fixed $t>0$ and $a,b\in(0,\pi)$. Then, assuming σ is known, $\widehat{\theta}_{m,t}$ given by (3.4) is a consistent and asymptotically normal estimator for θ, satisfying (3.6). Respectively, if θ is known, then $\widehat{\sigma}^2_{m,t}$ in (3.5) is a consistent and asymptotically normal estimator of σ^2, satisfying (3.7).*

4.2 Time sampling at a fixed space point

The case of sampling the solution in time at a fixed spatial point for bounded domains is more delicate, primarily since there is no ready available representation similar to (3.3). In [Wal81] the author proved that for a similar SPDE at $x = 0$ the 4-variation (in time) of the solution converges to a constant. We start by proving that the 4−variation converges to a constant at any fixed space point x. In addition, we also establish the asymptotic normality property of the 4-variation.

Proposition 4.3. *Let $x \in (0, \pi)$ be a fixed space point. Then, the solution $u(t, x)$ of the equation (4.1) admits the following decomposition*

$$u(t, x) = \frac{\sigma}{(\pi\theta)^{1/4}} v(t) + S(t), \quad t > 0, \tag{4.3}$$

where v and S are zero-mean Gaussian processes such that:

(a) $S(t)$ is continuous on $[0, +\infty)$, and infinitely differentiable on $(0, \infty)$;

(b) $v(t)$ has finite 4−variation (with convergence in probability)

$$\mathbb{P} - \lim_{n\to\infty} \mathsf{V}_n^4(v; [c, d]) = 3(d - c). \tag{4.4}$$

(c) the 4-variation admits the asymptotic normality property

$$\sqrt{n}\left(\frac{\mathsf{V}_n^4(v; [c, d])}{n\sigma_n^4} - 3\right) \xrightarrow[n\to\infty]{\mathcal{D}} \mathcal{N}(0, \bar{\sigma}_2^2 + \bar{\sigma}_4^2), \tag{4.5}$$

where

$$\sigma_n^2 = \frac{2}{\sqrt{\pi\theta}} \sum_{k\geq 1} \frac{\sin^2(kx)}{k^2}(1 - e^{-(d-c)\theta k^2/n}),$$

$$\bar{\sigma}_2^2 = 72 + 144 \lim_{n\to\infty} \sum_{j=1}^{n-1}(1 - \frac{j}{n}) \mid \frac{F(j)}{\sigma_n^2} \mid^2, \quad \bar{\sigma}_4^2 = 24 + 48 \lim_{n\to\infty} \sum_{j=1}^{n-1}(1 - \frac{j}{n}) \mid \frac{F(j)}{\sigma_n^2} \mid^4,$$

and

$$F(j) = \frac{1}{\sqrt{\pi\theta}} \sum_{k\geq 1} \frac{\sin^2(kx)}{k^2} \left(2e^{-j(d-c)\theta k^2/n} - e^{-(j+1)(d-c)\theta k^2/n} - e^{-(j-1)(d-c)\theta k^2/n}\right).$$

Moreover,

$$\sqrt{n}\left(\frac{\pi\theta \mathsf{V}_n^4\left(u(\cdot, x); [c, d]\right)}{n\sigma_n^4\sigma^4} - 3\right) \xrightarrow[n\to\infty]{\mathcal{D}} \mathcal{N}(0, \bar{\sigma}_2^2 + \bar{\sigma}_4^2). \tag{4.6}$$

where $\sigma_n^2, \bar{\sigma}_2^2$ and $\bar{\sigma}_4^2$ are given above.

The proof is deferred to the Appendix B. To prove (4.5), we use some techniques from Malliavin calculus (cf. [NOL08]). The general idea of the proof is in line with the proof of the central limit theorem in [Cor12] established for a similar but simpler covariance structure.

Next, we present the main results of this subsection on consistency and asymptotic normality of the estimators (3.8) and (3.9).

Theorem 4.4. *Let u be the solution to (4.1), and assume that u is sampled at discrete points $\{(t_i,x) \mid t_i \in \Upsilon^n(c,d)\}$, for some fixed $x\in(0,\pi)$, and $0<c<d<\infty$. Then, assuming σ is known, $\widehat{\theta}_{n,x}$ given by (3.8) is a weakly consistent estimator for θ, that is*

$$\mathbb{P}-\lim_{n\to\infty}\widehat{\theta}_{n,x}=\theta. \tag{4.7}$$

Respectively, if θ is known, then $\widehat{\sigma}^2_{n,x}$ in (3.9) is a weakly consistent estimator of σ^2. Moreover, $\widehat{\theta}_{n,x}$ and $\widehat{\sigma}^2_{n,x}$ satisfy the following central limit type convergence

$$\sqrt{n}\left(\widehat{\theta}_{n,x}-\frac{(d-c)\theta}{n\sigma_n^4}\right)\xrightarrow[n\to\infty]{\mathcal{D}}\mathcal{N}(0,\theta^2\left(\bar{\sigma}_2^2+\bar{\sigma}_4^2\right)), \tag{4.8}$$

$$\sqrt{n}\left(\widehat{\sigma}^2_{n,x}-\frac{\sqrt{n}\sigma_n^2}{\sqrt{d-c}}\sigma^2\right)\xrightarrow[n\to\infty]{\mathcal{D}}\mathcal{N}(0,\frac{1}{36}\sigma^4\left(\bar{\sigma}_2^2+\bar{\sigma}_4^2\right)). \tag{4.9}$$

Proof. Consistency is a direct consequence of Proposition 4.3.(a)-(b) and (2.3) from Proposition 2.1.

Combining (3.8) and (4.6), we have

$$\sqrt{n}\left(\frac{3(d-c)\theta}{\widehat{\theta}_{n,x}n\sigma_n^4}-3\right)\xrightarrow[n\to\infty]{\mathcal{D}}\mathcal{N}(0,\bar{\sigma}_2^2+\bar{\sigma}_4^2).$$

Due to (4.7), and by Slutsky's theorem, (4.8) follows at once. Relationship (4.9) is proved similarly.

This completes the proof. □

5 Multiplicative noise, whole space

Let us consider the SPDE (2.1), on $G=\mathbb{R}$, and driven by a multiplicative noise:

$$\begin{aligned} \mathrm{d}u(t,x)&=\theta u_{xx}(t,x)\,\mathrm{d}t+\sigma u(t,x)\,\mathrm{d}W(t,x), \quad x\in\mathbb{R}, \quad t>0, \\ u(0,x)&=u_0, \quad x\in\mathbb{R}. \end{aligned} \tag{5.1}$$

The problem of estimating θ and σ for this equation, assuming sampling scheme (A) or (B), have been essentially studied in [PvsT07]. The estimators are similar to those derived above for the additive noise, and for sake of completeness, we present them here too.

Assume that the solution u of (5.1) is observed according to sampling scheme (A). Then, given that σ is known, the estimator

$$\widehat{\theta}_{m,t}:=\frac{(b-a)\sigma^2\sum_{j=1}^m u^2(t,x_j)}{2m\sum_{j=1}^m(u(t,x_j)-u(t,x_{j-1}))^2}.$$

is an weakly consistent estimator of θ. Respectively, if θ is known, then

$$\widehat{\sigma}^2_{m,t}=\frac{2m\theta\sum_{j=1}^m(u(t,x_j)-u(t,x_{j-1}))^2}{(b-a)\sum_{j=1}^m u^2(t,x_j)}.$$

is an weakly consistent estimator of σ^2.

Analogously, let u being observed by sampling scheme (B), and let

$$\begin{aligned} \widehat{\theta}_{n,x}&:=\frac{3(d-c)\sigma^4\sum_{i=1}^n u^4(t_i,x)}{n\pi\sum_{i=1}^n(u(t_i,x)-u(t_{i-1},x))^4}, \\ \widehat{\sigma}^2_{n,x}&:=\sqrt{\frac{n\theta\pi\sum_{i=1}^n(u(t_i,x)-u(t_{i-1},x))^4}{3(d-c)\sum_{i=1}^n u^4(t_i,x)}}. \end{aligned}$$

Then, assuming that σ is known (resp. θ is known), then estimator $\widehat{\theta}_{n,x}$ (resp. $\widehat{\theta}_{n,x}$) is an weakly consistent estimator of θ (resp. σ^2).

The asymptotic normality of the estimators in this section remains an open problem.

A Appendix

A.1 Auxiliary technical results

In this section we will provide some technical results used in the paper.

Theorem A.1. *Let $\{X_t, t \geq 0\}$ be a Gaussian process with the following properties*

(i) $X_0 = 0$, *and* $\mathbb{E}X_t = 0, \quad t \geq 0$.

(ii) $X_{t+s} - X_t \sim \mathcal{N}(0, \sigma^2(s))$, *where $\sigma(s)$ is a deterministic function of s.*

(iii) There exists a constant $\gamma > 0$ such that $(X_{\alpha t}, t \geq 0) \overset{law}{=} \alpha^\gamma (X_t, t \geq 0)$, for any $\alpha > 0$.

(iv) For any $t \geq 0, \Delta t > 0$, the sequence $X_{t+n\Delta t} - X_{t+(n-1)\Delta t}$, $n \in \mathbb{N}$ is stationary. In particular, $Y_n = \frac{X_n - X_{n-1}}{\sigma(1)}, \quad n \in \mathbb{N}$, is a zero mean and stationary Gaussian sequence with unit variance.

(v) Let r be the covariance function of Y, $r(n) = \mathbb{E}Y_m Y_{m+n}$, and assume that for some positive integer k, $\sum_{n\geq 1} r^k(n) < \infty$.

Then,

$$\frac{1}{\sqrt{n}} \sum_{j=1}^{n} H\left(\frac{n^\gamma}{\sigma(1)} \left(X_{j/n} - X_{(j-1)/n} \right); k \right) \xrightarrow[n\to\infty]{\mathcal{D}} \check{\sigma}\mathcal{N}(0,1), \tag{A.1}$$

where

$$\check{\sigma}^2 = \sum_{l=k}^{\infty} c_l^2 l! \check{\sigma}_l^2, \qquad \check{\sigma}_l^2 = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{n} r^l(|i-j|).$$

Proof. By [BM83, Theorem 1], applied to the sequence Y, we immediately get

$$\frac{1}{\sqrt{n}} \sum_{j=1}^{n} H(Y_j; k) \xrightarrow[n\to\infty]{\mathcal{D}} \check{\sigma}\mathcal{N}(0,1),$$

where

$$\check{\sigma}^2 = \sum_{l=k}^{\infty} c_l^2 l! \check{\sigma}_l^2, \qquad \check{\sigma}_l^2 = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{n} r^l(|i-j|).$$

Since

$$(X_{j/n} - X_{(j-1)/n}, j = 1, 2, \ldots, n) \overset{\text{law}}{=} \frac{1}{n^\gamma}(X_j - X_{j-1}, j = 1, 2, \ldots, n),$$

we conclude that (A.1) holds. □

The following result is an immediate consequence of Theorem A.1.

Corollary A.2. *Let B^H be a fractional Brownian motion with Hurst parameter $H = 1/4$. Then,*

$$\sqrt{n}\left(\mathsf{V}_n^4(B^H;[a,b]) - 3(b-a)\right) \xrightarrow[n\to\infty]{\mathcal{D}} (b-a)\breve{\sigma}\mathcal{N}(0,1),$$

where

$$\breve{\sigma}^2 = 72\breve{\sigma}_2^2 + 24\breve{\sigma}_4^2, \qquad \breve{\sigma}_l^2 = \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^n\sum_{j=1}^n r^l(|i-j|). \tag{A.2}$$

For reader's convenience we also present here a result from [NOL08], used in the proof of Proposition 4.3. For most of this part, we will use the standard notations from [Nua06] and [NOL08]. We will denote by $H(x;k)$ a polynomial with Hermite rank k, that is, H can be expanded in the form

$$H(x;k) = \sum_{j=k}^{\infty} c_j H_j(x),$$

where H_j is the jth Hermite polynomial (with leading coefficient 1), and $c_k \neq 0$. Let H be a separable Hilbert space. For every $n \geq 1$, the notation $H^{\otimes n}$ will stand for the nth tensor product of H, and $H^{\odot n}$ will denote the nth symmetric tensor product of H, endowed with the modified norm $\sqrt{n!}\|\cdot\|_{H^{\otimes n}}$. Suppose that $X = \{X(h), h \in H\}$ is an isonormal Gaussian process on H, on some fixed probability space, say $(\Omega, \mathscr{F}, \mathbb{P})$, and assume that $\mathscr{F}$ is generated by X.

For every $n \geq 1$, let $\mathcal{H}_n$ be the nth Wiener chaos of X, that is, the closed linear subspace of $L^2(\Omega, \mathcal{F}, \mathbb{P})$ generated by the random variables $\{H_n(X(h)), h \in H, \|h\|_H = 1\}$, where H_n is the nth Hermite polynomial. We denote by $\mathcal{H}_0$ the space of constant random variables. The mapping $I_n(h^{\otimes n}) = H_n(X(h))$, for $n \geq 1$, provides a linear isometry between $H^{\odot n}$ and $\mathcal{H}_n$. For $n = 0$, we have that $\mathcal{H}_0 = \mathbb{R}$, and take I_0 to be the identity map. It is well known that any square intergrable random variable $F \in L^2(\Omega, \mathscr{F}, \mathbb{P})$ admits the following expansion

$$F = \sum_{n=0}^{\infty} I_n(f_n),$$

where $f_0 = \mathbb{E}F$, and the $f_n \in H^{\odot n}$ are uniquely determined by F.

Let $\{e_k, k \geq 1\}$ be a complete orthonormal system in H. Given $f \in H^{\odot n}$ and $g \in H^{\odot m}$, for $\ell = 0, \ldots, n \wedge m$, the contraction of f and g of order ℓ is the element of $H^{\otimes(n+m-2\ell)}$ defined by

$$f \otimes_\ell g = \sum_{i_1,\ldots,i_\ell} \langle f, e_{i_1} \otimes \cdots \otimes e_{i_\ell}\rangle_{H^{\otimes l}} \otimes \langle g, e_{i_1} \otimes \cdots \otimes e_{i_\ell}\rangle_{H^{\otimes l}}$$

Theorem A.3 ([NOL08])**.** *For $d \geq 2$, fix d natural numbers $1 \leq n_1 \leq \cdots \leq n_d$. Let $\{F_k\}_{k\in\mathbb{N}}$ be a sequence of random vectors of the form*

$$F_k = (F_k^1, \ldots, F_k^d) = (I_{n_1}(f_k^1), \ldots, I_{n_d}(f_k^d)),$$

where $f_k^i \in H^{\odot n_i}$ and I_{n_i} is the Wiener integral of order n_i, such that, for every $1 \leq i, j \leq d$,

$$\lim_{k\to\infty} \mathbb{E}\left[F_k^i F_k^j\right] = \delta_{ij}. \tag{A.3}$$

The following two[2] statements are equivalent.

[2] The original result [NOL08, Theorem 7] contains six equivalent conditions; we list only those two that we use in this paper.

(N1) For all $1 \leq i \leq d, 1 \leq \ell \leq n_i - 1$, $\|f_k^{(i)} \otimes_\ell f_k^{(i)}\|^2_{H^{2\otimes(n_i-\ell)}} \to 0$, *as* $k \to \infty$.

(N2) The sequence $\{F_k\}_{k\in\mathbb{N}}$, *as* $k \to \infty$, *converges in distribution to a d-dimensional standard Gaussian vector* $\mathcal{N}_d(0, I_d)$.

We conclude this section with a result used to obtain the exact rates of convergence of some estimators from Section 4.2.

Lemma A.4. *For any* $x \in (0, \pi)$ *and* $\theta > 0$, *the following holds true*

$$\lim_{n\to\infty} \sqrt{n} \sum_{k\geq 1} \frac{\sin^2(kx)}{k^2} \left(1 - e^{-\theta k^2/n}\right) = \frac{\sqrt{\pi\theta}}{2}. \tag{A.4}$$

Proof. Note that

$$\sin^2(kx) = \frac{1}{2} - \frac{\sin((2k+1)x) - \sin((2k-1)x)}{4\sin x},$$

and therefore,

$$\begin{aligned}
&\sqrt{n} \sum_{k\geq 1} \frac{\sin^2(kx)}{k^2} \left(1 - e^{-\theta k^2/n}\right) \\
&\quad \sqrt{n} \sum_{k\geq 1} \frac{1}{2k^2} \left(1 - e^{-\theta k^2/n}\right) - \sqrt{n} \sum_{k\geq 1} \frac{\sin((2k+1)x) - \sin((2k-1)x)}{4k^2 \sin x} \left(1 - e^{-\theta k^2/n}\right) \\
&\quad =: L_n^1 - L_n^2.
\end{aligned}$$

To prove (A.4), we will show that $L_n^1 \to \sqrt{\pi\theta}/2$, and $L_n^2 \to 0$.

It is straightforward to check that for any $\varepsilon > 0$, the function $(1 - e^{-\epsilon x})/x$, $x > 0$, is decreasing. It is also easy to show that

$$\int_0^\infty \frac{1 - e^{-z^2}}{z^2} dz = \sqrt{\pi}.$$

Using these, we obtain

$$\begin{aligned}
L_n^1 &= \sqrt{n} \sum_{k\geq 1} \int_{k-1}^k \frac{1}{2k^2} \left(1 - e^{-\theta k^2/n}\right) \mathrm{d}z \leq \sqrt{n} \sum_{k\geq 1} \int_{k-1}^k \frac{1}{2z^2} \left(1 - e^{-\theta z^2/n}\right) \mathrm{d}z \\
&= \frac{\sqrt{n}}{2} \int_0^\infty \frac{1}{z^2} \left(1 - e^{-\theta z^2/n}\right) \mathrm{d}z = \frac{\sqrt{n}}{2} \int_0^\infty \frac{1}{y^2 n/\theta} \left(1 - e^{-y^2}\right) \mathrm{d}y \sqrt{n/\theta} \\
&= \frac{\sqrt{\theta}}{2} \int_0^\infty \frac{1}{y^2} \left(1 - e^{-y^2}\right) \mathrm{d}y = \frac{\sqrt{\pi\theta}}{2}.
\end{aligned} \tag{A.5}$$

On the other hand,

$$\begin{aligned}
L_n^1 &= \sqrt{n} \sum_{k\geq 1} \int_k^{k+1} \frac{1}{2k^2} \left(1 - e^{-\theta k^2/n}\right) \mathrm{d}z \geq \sqrt{n} \sum_{k\geq 1} \int_k^{k+1} \frac{1}{2z^2} \left(1 - e^{-\theta z^2/n}\right) \mathrm{d}z \\
&= \frac{\sqrt{n}}{2} \int_1^\infty \frac{1}{z^2} \left(1 - e^{-\theta z^2/n}\right) \mathrm{d}z = \frac{\sqrt{n}}{2} \int_{\sqrt{\theta/n}}^\infty \frac{1}{y^2 n/\theta} \left(1 - e^{-y^2}\right) \mathrm{d}y \sqrt{n/\theta} \\
&= \frac{\sqrt{\theta}}{2} \int_{\sqrt{\theta/n}}^\infty \frac{1}{y^2} \left(1 - e^{-y^2}\right) \mathrm{d}y \xrightarrow[n\to\infty]{} \frac{\sqrt{\pi\theta}}{2}.
\end{aligned} \tag{A.6}$$

Combing (A.5) and (A.6), we conclude that $L_n^1 \to \sqrt{\pi\theta}/2$.

Denote by

$$f_k := \frac{1-e^{-\theta k^2/n}}{k^2}, \quad k \geq 1,$$

and as above, one can show that $\{f_k, k \in \mathbb{N}\}$ is a decreasing sequence. By simple rearrangement of terms, we get

$$L_2^n = \sqrt{n}\sum_{k\geq 2} \sin((2k-1)x)\,(f_{k-1}-f_k) - \sqrt{n}\sin x f_1.$$

Thus,

$$\begin{aligned}
|L_n^2| &\leq \sqrt{n}\sum_{k\geq 2}\Big|\sin((2k-1)x)\Big|\,(f_{k-1}-f_k) + \sqrt{n}\sin x f_1 \\
&\leq \sqrt{n}\sum_{k\geq 2}(f_{k-1}-f_k) + \sqrt{n}f_1 \leq 2\sqrt{n}f_1 = 2\sqrt{n}\left(1-e^{-\theta/n}\right) \\
&\leq 2\sqrt{n}\frac{\theta}{n} = 2\frac{\theta}{\sqrt{n}} \xrightarrow[n\to\infty]{} 0.
\end{aligned}$$

The proof is complete. □

B Proofs

Proof of Proposition 2.1

First we prove (2.2). It should be noted that a similar result is proved in [CNW06, Corollary 2]. For completeness, we outline out proof too. All 'p-variations' below are on the fixed interval $[a,b]$, and as agreed above, we will omit writing their dependence on $[a,b]$. By Minkowski's inequality, we have that

$$\left|\ (\mathsf{V}_n^p(X))^{1/p} - (\mathsf{V}_n^p(Y))^{1/p}\ \right| \leq (\mathsf{V}_n^p(X+Y))^{1/p} \leq (\mathsf{V}_n^p(X))^{1/p} + (\mathsf{V}_n^p(Y))^{1/p}. \tag{B.1}$$

Since Y has $C^1[a,b]$ sample paths, we have $\lim_{n\to\infty}\mathsf{V}_n^p(Y) = 0$. Hence, passing to the limit in (B.1), the identity (2.2) follows. As far as (2.3), note that in view of (B.1), for any $\epsilon > 0$,

$$\begin{aligned}
&\left\{\left|(\mathsf{V}_n^p(X+Y))^{1/p} - (\mathsf{V}_{\mathbb{P}}^p(X))^{1/p}\right| \geq \epsilon\right\} \\
&\quad = \left\{(\mathsf{V}_n^p(X+Y))^{1/p} \geq \left(\mathsf{V}_{\mathbb{P}}^p(X)\right)^{1/p} + \epsilon\right\} \cup \left\{(\mathsf{V}_n^p(X+Y))^{1/p} \leq \left(\mathsf{V}_{\mathbb{P}}^p(X)\right)^{1/p} - \epsilon\right\} \\
&\quad \subset \left\{(\mathsf{V}_n^p(X))^{1/p} + (\mathsf{V}_n^p(Y))^{1/p} \geq \left(\mathsf{V}_{\mathbb{P}}^p(X)\right)^{1/p} + \epsilon\right\} \\
&\qquad\qquad \cup \left\{\left|(\mathsf{V}_n^p(X))^{1/p} - (\mathsf{V}_n^p(Y))^{1/p}\right| \leq \left(\mathsf{V}_{\mathbb{P}}^p(X)\right)^{1/p} - \epsilon\right\} \\
&\quad \subset \left\{\left|(\mathsf{V}_n^p(X))^{1/p} + (\mathsf{V}_n^p(Y))^{1/p} - \left(\mathsf{V}_{\mathbb{P}}^p(X)\right)^{1/p}\right| \geq \epsilon\right\} \\
&\qquad\qquad \cup \left\{\left|(\mathsf{V}_n^p(X))^{1/p} - (\mathsf{V}_n^p(Y))^{1/p} - \left(\mathsf{V}_{\mathbb{P}}^p(X)\right)^{1/p}\right| \geq \epsilon\right\} \\
&\quad = \left\{\left|(\mathsf{V}_n^p(X))^{1/p} - \left(\mathsf{V}_{\mathbb{P}}^p(X)\right)^{1/p}\right| \geq \epsilon/2\right\} \cup \left\{(\mathsf{V}_n^p(Y))^{1/p} \geq \epsilon/2\right\}
\end{aligned} \tag{B.2}$$

Due to continuity of $x^{1/p}$, based on our initial assumptions, we have that $\mathbb{P}-\lim_{n\to\infty}\left(\mathsf{V}_n^p(X)\right)^{1/p} = \left(\mathsf{V}_{\mathbb{P}}^p(X)\right)^{1/p}$, and $\mathbb{P}-\lim_{n\to\infty}\left(\mathsf{V}_n^p(Y)\right)^{1/p}=0$. Thus, by (B.2), we get at once that

$$\mathbb{P}-\lim_{n\to\infty}\left(\mathsf{V}_n^p(X+Y)\right)^{1/p} = \left(\mathsf{V}_{\mathbb{P}}^p(X)\right)^{1/p},$$

which consequently implies (2.3).

In view of Slutsky's Theorem, to prove (2.5), it is enough to show that

$$\lim_{n\to\infty} n^{\alpha}\left(\mathsf{V}_n^p(X+Y)-\mathsf{V}_n^p(X)\right)=0.$$

By (B.1) and by mean-value theorem, we have

$$\begin{aligned}\mathsf{V}_n^p(X+Y) &\leq \left(\left(\mathsf{V}_n^p(X)\right)^{1/p}+\left(\mathsf{V}_n^p(Y)\right)^{1/p}\right)^p \\ &= \mathsf{V}_n^p(X)+p\left(\left(\mathsf{V}_n^p(X)\right)^{1/p}+\eta_{1,n}\left(\mathsf{V}_n^p(Y)\right)^{1/p}\right)^{p-1}\left(\mathsf{V}_n^p(Y)\right)^{1/p},\end{aligned} \tag{B.3}$$

for some $\eta_{1,n}\in[0,1]$. Since Y has $C^1[a,b]$ sample paths, denoting $M=\sup_{a\leq t\leq b}\mid Y'(t)\mid$, and again by mean-value theorem, we get

$$\mathsf{V}_n^p(Y)=\sum_{j=1}^n|Y(t_j)-Y(t_{j-1})|^p=\sum_{j=1}^n|(t_j-t_{j-1})Y'(\zeta_j)|^p\leq n(M/n)^p. \tag{B.4}$$

Therefore, by (B.3), and since $\alpha+1/p<1$, we conclude that

$$n^{\alpha}\left(\mathsf{V}_n^p(X+Y)-\mathsf{V}_n^p(X)\right)\leq p\left(\left(\mathsf{V}_n^p(X)\right)^{1/p}+\eta_1\left(\mathsf{V}_n^p(Y)\right)^{1/p}\right)^{p-1} n^{\alpha+1/p-1}M \underset{n\to\infty}{\longrightarrow} 0.$$

Similarly, we have that

$$n^{\alpha}\left(\mathsf{V}_n^p(X+Y)-\mathsf{V}_n^p(X)\right)\geq -p\left(\left(\mathsf{V}_n^p(X)\right)^{1/p}-\eta_2\left(\mathsf{V}_n^p(Y)\right)^{1/p}\right)^{p-1} n^{\alpha+1/p-1}M \underset{n\to\infty}{\longrightarrow} 0,$$

and therefore, (2.5) is proved.

Now suppose that Y has $C^2[a,b]$ sample paths, and assume that (2.4) holds true for $p=2,\alpha=1/2$. To show that (2.5) also holds true, it is enough to prove that

$$\lim_{n\to\infty} n^{1/2}\left(\mathsf{V}_n^2(X+Y)-\mathsf{V}_n^2(X)\right)=0. \tag{B.5}$$

Note that,

$$\mathsf{V}_n^2(X+Y)-\mathsf{V}_n^2(X)=2\sum_{j=1}^n\left(X(t_j)-X(t_{j-1})\right)\left(Y(t_j)-Y(t_{j-1})\right)+\mathsf{V}_n^2(Y).$$

Using (B.4), we have $n^{1/2}\mathsf{V}_n^2(Y)\leq n^{3/2}(M/n)^2\to 0$.

By mean value theorem,

$$\begin{aligned}n^{1/2}\sum_{j=1}^n\left(X(t_j)-X(t_{j-1})\right)\left(Y(t_j)-Y(t_{j-1})\right) &= n^{-1/2}(b-a)\sum_{i=1}^n\left(X(t_j)-X(t_{j-1})\right)\left(Y'(\zeta_j)-Y'(t_{j-1})\right) \\ &\quad + n^{-1/2}(b-a)\sum_{i=1}^n\left(X(t_j)-X(t_{j-1})\right)Y'(t_{j-1}) \\ &=: K_1+K_2.\end{aligned}$$

Applying Cauchy-Schwartz inequality, we get

$$\begin{aligned} |K_1| &\leq n^{-3/2}(b-a)^2 \sum_{i=1}^{n} \left| \ (X(t_j) - X(t_{j-1})) \max_{a\leq t\leq b} \ | \, Y''(t) \, | \ \right| \\ &\leq n^{-1}(b-a)^2 \max_{a\leq t\leq b} \ | \, Y''(t) \, | \, \sqrt{\mathsf{V}_n^2(X)} \underset{n\to\infty}{\longrightarrow} 0. \end{aligned}$$

We rewrite K_2 as

$$K_2 = n^{-1/2}(b-a)\left(X(b)Y'(b) - X(a)Y'(a) - \sum_{j=1}^{n} X(t_j)\left(Y'(t_j) - Y'(t_{j-1})\right)\right).$$

Since, $\lim_{n\to\infty}\sum_{j=1}^{n} X(t_j)\left(Y'(t_j) - Y'(t_{j-1})\right) = \int_a^b X(t)dY'(t) = \int_a^b X(t)Y''(t)dt$, we have at once that

$$\lim_{n\to\infty} K_2 = \lim_{n\to\infty} n^{-1/2}(b-a)\left(X(b)Y'(b) - X(a)Y'(a) - \int_a^b X(t)Y''(t)dt\right) = 0.$$

Combining the above, (B.5) is proved.

This concludes the proof.

Proof of Proposition 4.3

Assume that $x \in (0,\pi)$ is fixed. We start by constructing the Gaussian processes S, v. Let $\{\eta_k, k \in \mathbb{N}\}$ be a sequence of i.i.d. standard normal random variables, independent of $\{u_k, k \in \mathbb{N}\}$, and let

$$\begin{aligned} S_k(t) &:= \frac{\sigma}{\sqrt{2\theta}k} e^{-\theta k^2 t}\eta_k, \qquad k \in \mathbb{N},\ t \geq 0, \\ S(t) &:= \sum_{k=1}^{\infty} S_k(t)h_k(x), \qquad t \geq 0. \end{aligned}$$

Consequently, we put

$$\begin{aligned} v_k(t) &:= \frac{(\theta\pi)^{1/4}}{\sigma}\left(u_k(t) - S_k(t)\right), \qquad k \in \mathbb{N},\ t \geq 0, \\ v(t) &:= \sum_{k\geq 1} v_k(t)h_k(x), \quad t \geq 0, \quad x \in (0,\pi). \end{aligned}$$

Clearly, S and v are zero-mean Gaussian processes that satisfying (4.3).

(a) It is straightforward to check that S is continuous on $[0,+\infty)]$ and infinitely differentiable on $(0,\infty)$. Moreover,

$$\mathbb{E}\left|S_k(t+\epsilon) - S_k(t)\right|^2 = \frac{\sigma^2}{2\theta k^2} e^{-2\theta k^2 t}\left(1 - e^{-\theta k^2 \epsilon}\right)^2, \qquad k \in \mathbb{N},\ t \geq 0. \tag{B.6}$$

(b) By direct computations, using (4.2), one can show that

$$\mathbb{E}\left|u_k(t+\epsilon) - u_k(t)\right|^2 = \frac{\sigma^2}{2\theta k^2}(1 - e^{-\theta k^2 \epsilon})\left(2 - (1 - e^{-\theta k^2 \epsilon})e^{-2\theta k^2 t}\right), \tag{B.7}$$

for $t \geq 0,\ \varepsilon > 0,\ k \in \mathbb{N}$. Combining (B.6), (B.7) and the independence between S_k and u_k, we deduce that

$$\mathbb{E}\,|v_k(t+\epsilon) - v_k(t)|^2 = \frac{\sqrt{\pi}}{\sqrt{\theta}k^2}(1 - e^{-\theta k^2 \epsilon}), \qquad k \in \mathbb{N},\ t \geq 0.$$

Consequently, we have that

$$\mathbb{E}\,|v(t+\epsilon) - v(t)|^2 = \sum_{k\geq 1} \mathbb{E}\,|v_k(t+\epsilon) - v_k(t)|^2\, h_k^2(x) = \frac{2}{\sqrt{\pi\theta}} \sum_{k\geq 1} \frac{\sin^2(kx)}{k^2}(1 - e^{-\theta k^2 \epsilon}).$$

We will prove (4.4) by showing that

$$\lim_{n\to\infty} \mathbb{E}\left(\mathsf{V}_n^4(v;[c,d])\right) = 3(d-c), \tag{B.8}$$

$$\lim_{n\to\infty} \operatorname{Var}\left(\mathsf{V}_n^4(v;[c,d])\right) = 0. \tag{B.9}$$

Denote by

$$\sigma_n^2 := \mathbb{E}\,|v(t_j) - v(t_{j-1})|^2 = \frac{2}{\sqrt{\pi\theta}} \sum_{k\geq 1} \frac{\sin^2(kx)}{k^2}(1 - e^{-(d-c)\theta k^2/n}), \quad n \in \mathbb{N}.$$

In view of Lemma A.4,

$$\lim_{n\to\infty} \sqrt{n}\sigma_n^2 = \sqrt{d-c}. \tag{B.10}$$

Since v is a zero-mean Gaussian process, we have

$$\mathbb{E}\,|v(t_j) - v(t_{j-1})|^4 = 3\left(\mathbb{E}\,|v(t_j) - v(t_{j-1})|^2\right)^2 = 3\sigma_n^4,$$

therefore

$$\lim_{n\to\infty} \mathbb{E}\left(\mathsf{V}_n^4(v;[c,d])\right) = \lim_{n\to\infty} \sum_{j=1}^n \mathbb{E}\,|v(t_j) - v(t_{j-1})|^4 = \lim_{n\to\infty} 3n\sigma_n^4 = 3(d-c),$$

and hence (B.8) is proved. Next, note that

$$\begin{aligned}
\operatorname{Var}\left(\mathsf{V}_n^4(v;[c,d])\right) &= \mathbb{E}\left(\mathsf{V}_n^4(v;[c,d]) - \mathbb{E}\left(\mathsf{V}_n^4(v;[c,d])\right)\right)^2 \\
&= \sum_{j=1}^n \mathbb{E}\left(|v(t_j,x) - v(t_{j-1},x)|^4 - 3\sigma_n^4\right)^2 \\
&\qquad + 2\sum_{i<j} \mathbb{E}\left(|v(t_i,x) - v(t_{i-1},x)|^4 - 3\sigma_n^4\right)\left(|v(t_j,x) - v(t_{j-1},x)|^4 - 3\sigma_n^4\right) \\
&=: J_1 + J_2.
\end{aligned}$$

According to (B.10), we deduce that

$$J_1 = \sum_{j=1}^n \mathbb{E}\left(|v(t_j,x) - v(t_{j-1},x)|^8\right) - 9n\sigma_n^8 = 96n\sigma_n^8 \underset{n\to\infty}{\longrightarrow} 0. \tag{B.11}$$

As far as J_2, for $j \geq 1$, we put

$$
\begin{aligned}
F(j) &:= \mathbb{E}\left(v(t_i,x)-v(t_{i-1},x)\right)\left(v(t_{i+j},x)-v(t_{i+j-1},x)\right) \\
&= \frac{1}{\sqrt{\pi\theta}}\sum_{k\geq 1}\frac{\sin^2(kx)}{k^2}\left(2e^{-j(d-c)\theta k^2/n}-e^{-(j+1)(d-c)\theta k^2/n}-e^{-(j-1)(d-c)\theta k^2/n}\right) \\
&= G_j - G_{j-1},
\end{aligned}
$$

where

$$
G_j := \frac{1}{\sqrt{\pi\theta}}\sum_{k\geq 1}\frac{\sin^2(kx)}{k^2}\left(e^{-j(d-c)\theta k^2/n}-e^{-(j+1)(d-c)\theta k^2/n}\right), \quad j\geq 0,
$$

and also put $F(0) := \sigma_n^2$. Since $F(j)<0$, we have that $G_j < G_{j-1}$. Using the property of joint normal distributions, we continue

$$
\begin{aligned}
J_2 &= 2\sum_{i<j}\mathbb{E}\left(|v(t_i,x)-v(t_{i-1},x)|^4-3\sigma_n^4\right)\left(|v(t_j,x)-v(t_{j-1},x)|^4-3\sigma_n^4\right) \\
&= 2\sum_{i<j}\left(24F^4(j-i)+72F^2(j-i)\sigma_n^4\right).
\end{aligned}
$$

From here, since $|F(j-i)|\leq\sigma_n^2$, we deduce that

$$
\begin{aligned}
J_2 &\leq 2\sum_{i<j}\left(24|F(j-i)|\sigma_n^6+72|F(j-i)|\sigma_n^6\right) = 192\sum_{i<j}|F(j-i)|\sigma_n^6 \\
&= 192\sigma_n^6\sum_{j=1}^{n-1}(n-j)\left(G_{j-1}-G_j\right).
\end{aligned}
$$

Note that $\sum_{j=1}^{n-1}(n-j)\left(G_{j-1}-G_j\right) = nG_0 - \sum_{j=0}^{n-1}G_j$, and since

$$
\begin{aligned}
\sum_{j=0}^{n-1}G_j &= \sum_{j=0}^{n-1}\frac{1}{\sqrt{\pi\theta}}\sum_{k\geq 1}\frac{\sin^2(kx)}{k^2}\left(e^{-j(d-c)\theta k^2/n}-e^{-(j+1)(d-c)\theta k^2/n}\right) \\
&= \frac{1}{\sqrt{\pi\theta}}\sum_{k\geq 1}\frac{\sin^2(kx)}{k^2}\left(1-e^{-(d-c)\theta k^2}\right) = \frac{1}{2}\sigma_1^2,
\end{aligned}
$$

and $G_0 = \frac{1}{2}\sigma_n^2$, we conclude that

$$
\begin{aligned}
J_2 &\leq 192\sigma_n^6\left(n\frac{1}{\sqrt{\pi\theta}}\sum_{k\geq 1}\frac{\sin^2(kx)}{k^2}\left(1-e^{-(d-c)\theta k^2/n}\right)-\frac{1}{\sqrt{\pi\theta}}\sum_{k\geq 1}\frac{\sin^2(kx)}{k^2}\left(1-e^{-(d-c)\theta k^2}\right)\right) \\
&= 192\sigma_n^6\left(\frac{n}{2}\sigma_n^2-\frac{1}{2}\sigma_1^2\right)\overset{n\to\infty}{\longrightarrow} 0.
\end{aligned}
\tag{B.12}
$$

according to (B.10). Combining (B.11) and (B.12), (B.9) is proved. Consequently, by (B.8) and (B.9), we also have that $\mathsf{V}_n^4(v;[c,d])$ converges to $3(d-c)$, both in L^2 and in probability.

(c) We will apply Theorem A.3, by showing that (A.3) and along with condition (N1) are satisfied. We begin by establishing the following estimates

$$
\sum_{j=-l}^{r}|F(|j|)|^m \leq 2\sigma_n^{2m}, \tag{B.13}
$$

for any $m \geq 1$, $\ell, r \in \mathbb{N}$. Since $m \geq 1$,

$$\begin{aligned}\sum_{j=1}^{r} |F(j)|^m &= \sum_{j=1}^{r} |F(j)|^{m-1} |F(j)| \leq \sum_{j=1}^{r} \sigma_n^{2(m-1)} |F(j)| \\ &= \sum_{j=1}^{r} \sigma_n^{2(m-1)} (G_{j-1} - G_j) = \sigma_n^{2(m-1)} (G_0 - G_{r-1}) \\ &\leq \sigma_n^{2(m-1)} G_0 = \frac{1}{2} \sigma_n^{2m},\end{aligned}$$

where we used the fact that $G_j \geq 0$ and $G_0 = \frac{1}{2}\sigma_n^2$. Therefore,

$$\begin{aligned}\sum_{j=-l}^{r} |F(|j|)|^m &= (\sigma_n^2)^m + \sum_{j=1}^{r} |F(j)|^m + \sum_{j=1}^{l} |F(j)|^m \\ &\leq \sigma_n^{2m} + \frac{1}{2}\sigma_n^{2m} + \frac{1}{2}\sigma_n^{2m} = 2\sigma_n^{2m}.\end{aligned}$$

With slight abuse of notations, just in this proof, we denote by $\Delta v_j^n := v(t_j, x) - v(t_{j-1}, x)$. Let $\mathcal{H}$ be the closed subspace of $L^2(\Omega, \mathcal{F}, \mathbb{P})$ generated by the random variables $\frac{\Delta v_j^n}{\sigma_n}$, $1 \leq j \leq n$; $j, n \in \mathbb{N}$. Then,

$$\begin{aligned}\left|\frac{\Delta v_j^n}{\sigma_n}\right|^4 - 3 &= \left(\left|\frac{\Delta v_j^n}{\sigma_n}\right|^4 - 6\left|\frac{\Delta v_j^n}{\sigma_n}\right|^2 + 3\right) + 6\left(\left|\frac{\Delta v_j^n}{\sigma_n}\right|^2 - 1\right) \\ &= H_4\left(\frac{\Delta v_j^n}{\sigma_n}\right) + 6H_2\left(\frac{\Delta v_j^n}{\sigma_n}\right) = I_4\left[\left(\frac{\Delta v_j^n}{\sigma_n}\right)^{\otimes 4}\right] + 6I_2\left[\left(\frac{\Delta v_j^n}{\sigma_n}\right)^{\otimes 2}\right].\end{aligned}$$

Therefore,

$$\sqrt{n}\left(\frac{\mathsf{V}_n^4(v; [c,d])}{n\sigma_n^4} - 3\right) = I_4\left[\frac{1}{\sqrt{n}} \sum_{j=1}^{n} \left(\frac{\Delta v_j^n}{\sigma_n}\right)^{\otimes 4}\right] + I_2\left[\frac{6}{\sqrt{n}} \sum_{j=1}^{n} \left(\frac{\Delta v_j^n}{\sigma_n}\right)^{\otimes 2}\right] \tag{B.14}$$

Let

$$f_n^{(2)} := \frac{6}{\sqrt{n}} \sum_{j=1}^{n} \left(\frac{\Delta v_j^n}{\sigma_n}\right)^{\otimes 2}, \qquad f_n^{(4)} := \frac{1}{\sqrt{n}} \sum_{j=1}^{n} \left(\frac{\Delta v_j^n}{\sigma_n}\right)^{\otimes 4}, \tag{B.15}$$

and consider the sequence of two dimensional random vectors $F_n := \left(I_2(f_n^{(2)}), I_4(f_n^{(4)})\right), n \in \mathbb{N}$, to which we will apply Theorem A.3. Using the properties of Wiener integral, we obtain that

$$\lim_{n\to\infty} \mathbb{E}\left(I_2(f_n^{(2)}) I_4(f_n^{(4)})\right) = 0,$$

and hence (A.3) is satisfied.

Next, we move to verification of condition (N1), which in this case becomes

$$\lim_{n\to\infty} \|f_n^{(m)} \otimes_r f_n^{(m)}\|_{H^{2\otimes(m-r)}}^2 = 0. \tag{B.16}$$

for $m = 2, 4$, and $1 \leq r \leq m - 1$.

Using the linearity of the inner products and the properties of the tensor products of Hilbert spaces, we obtain

$$\begin{aligned}
\mathbb{E}\left(I_2(f_n^{(2)})\right)^2 &= 2\langle f_n^{(2)}, f_n^{(2)}\rangle_{\mathcal{H}^{\otimes 2}} = \frac{72}{n}\Big\langle \sum_{j=1}^n \left(\frac{\Delta v_j^n}{\sigma_n}\right)^{\otimes 2}, \sum_{j=1}^n \left(\frac{\Delta v_j^n}{\sigma_n}\right)^{\otimes 2}\Big\rangle_{\mathcal{H}^{\otimes 2}} \\
&= \frac{72}{n}\sum_{i,j=1}^n \Big\langle \left(\frac{\Delta v_i^n}{\sigma_n}\right)^{\otimes 2}, \left(\frac{\Delta v_j^n}{\sigma_n}\right)^{\otimes 2}\Big\rangle_{\mathcal{H}^{\otimes 2}} = \frac{72}{n}\sum_{i,j=1}^n \Big\langle \frac{\Delta v_i^n}{\sigma_n}, \frac{\Delta v_j^n}{\sigma_n}\Big\rangle_{\mathcal{H}}^2 \\
&= \frac{72}{n}\sum_{i,j=1}^n \left[\mathbb{E}\left(\frac{\Delta v_i^n}{\sigma_n}\cdot\frac{\Delta v_j^n}{\sigma_n}\right)\right]^2 = \frac{72}{n}\sum_{i,j=1}^n \frac{|F(|j-i|)|^2}{\sigma_n^4} \\
&= \frac{72}{n\sigma_n^4}\left(\sum_{j=1}^n |F(0)|^2 + 2\sum_{i<j}|F(j-i)|^2\right) = \frac{72}{n\sigma_n^4}\left(n\sigma_n^4 + 2\sum_{j=1}^{n-1}(n-j)|F(j)|^2\right) \\
&= 72 + \frac{144}{\sigma_n^4}\sum_{j=1}^{n-1}(1-\frac{j}{n})|F(j)|^2 = 72 + 144\sum_{j=1}^{n-1}(1-\frac{j}{n})\left|\frac{F(j)}{\sigma_n^2}\right|^2.
\end{aligned}$$

In view of (B.13), we have that

$$\sum_{j=1}^{n-1}(1-\frac{j}{n})\left|\frac{F(j)}{\sigma_n^2}\right|^2 \leq \sum_{j=1}^{\infty}\left|\frac{F(j)}{\sigma_n^2}\right|^2 < \infty,$$

and thus

$$\bar{\sigma}_2^2 := \lim_{n\to\infty}\mathbb{E}\left(I_2(f_n^{(2)})\right)^2 = 72 + 144\lim_{n\to\infty}\sum_{j=1}^{n-1}(1-\frac{j}{n})\left|\frac{F(j)}{\sigma_n^2}\right|^2 < \infty.$$

Similarly,

$$\begin{aligned}
\mathbb{E}\left(I_4(f_n^{(4)})\right)^2 &= 24\Big\langle f_n^{(4)}, f_n^{(4)}\Big\rangle_{\mathcal{H}^{\otimes 4}} = \frac{24}{n}\Big\langle \sum_{j=1}^n \left(\frac{\Delta v_j^n}{\sigma_n}\right)^{\otimes 4}, \sum_{j=1}^n \left(\frac{\Delta v_j^n}{\sigma_n}\right)^{\otimes 4}\Big\rangle_{\mathcal{H}^{\otimes 4}} \\
&= \frac{24}{n}\sum_{i,j=1}^n \Big\langle \left(\frac{\Delta v_i^n}{\sigma_n}\right)^{\otimes 4}, \left(\frac{\Delta v_j^n}{\sigma_n}\right)^{\otimes 4}\Big\rangle_{\mathcal{H}^{\otimes 4}} = \frac{24}{n}\sum_{i,j=1}^n \Big\langle \frac{\Delta v_i^n}{\sigma_n}, \frac{\Delta v_j^n}{\sigma_n}\Big\rangle_{\mathcal{H}}^4 \\
&= \frac{24}{n}\sum_{i,j=1}^n \left[\mathbb{E}\left(\frac{\Delta v_i^n}{\sigma_n}\cdot\frac{\Delta v_j^n}{\sigma_n}\right)\right]^4 = \frac{24}{n}\sum_{i,j=1}^n \frac{|F(|j-i|)|^4}{\sigma_n^8} \\
&\leq 24 + 48\sum_{j=1}^{n-1}\left|\frac{F(j)}{\sigma_n^2}\right|^4 \leq 24 + 48\sum_{j=1}^{\infty}\left|\frac{F(j)}{\sigma_n^2}\right|^4 < \ \infty,
\end{aligned}$$

and consequently,

$$\bar{\sigma}_4^2 := \lim_{n\to\infty}\mathbb{E}\left(I_4(f_n^{(4)})\right)^2 = 24 + 48\lim_{n\to\infty}\sum_{j=1}^{n-1}(1-\frac{j}{n})\left|\frac{F(j)}{\sigma_n^2}\right|^4 < \infty.$$

Let $a_2 = 6, a_4 = 1$. Then,

$$
\begin{aligned}
\|f_n^{(m)} \otimes_r f_n^{(m)}\|^2_{H^{2\otimes(m-r)}} &= \| \frac{a_m}{\sqrt{n}} \sum_{j=1}^n \left(\frac{\Delta v_j^n}{\sigma_n}\right)^{\otimes m} \otimes_r \frac{a_m}{\sqrt{n}} \sum_{j=1}^n \left(\frac{\Delta v_j^n}{\sigma_n}\right)^{\otimes m} \|^2_{H^{\otimes 2(m-r)}} \\
&= \| \frac{a_m^2}{n} \sum_{i,j=1}^n \left(\frac{\Delta v_i^n}{\sigma_n}\right)^{\otimes m} \otimes_r \left(\frac{\Delta v_j^n}{\sigma_n}\right)^{\otimes m} \|^2_{H^{\otimes 2(m-r)}} \\
&= \| \frac{a_m^2}{n} \sum_{i,j=1}^n \langle \frac{\Delta v_i^n}{\sigma_n}, \frac{\Delta v_j^n}{\sigma_n} \rangle_H^r \left(\frac{\Delta v_i^n}{\sigma_n}\right)^{\otimes(m-r)} \otimes \left(\frac{\Delta v_j^n}{\sigma_n}\right)^{\otimes(m-r)} \|^2_{H^{\otimes 2(m-r)}} \\
&= \| \frac{a_m^2}{n} \sum_{i,j=1}^n \frac{|F(|j-i|)|^r}{\sigma_n^{2r}} \left(\frac{\Delta v_i^n}{\sigma_n}\right)^{\otimes(m-r)} \otimes \left(\frac{\Delta v_j^n}{\sigma_n}\right)^{\otimes(m-r)} \|^2_{H^{\otimes 2(m-r)}} \\
&= \frac{a_m^4}{n^2 \sigma_n^{4m}} \sum_{i,j,i',j'=1}^n |F(|j-i|)|^r |F(|j'-i'|)|^r |F(|i'-i|)|^{m-r} |F(|j'-j|)|^{m-r} \\
&\le \frac{a_m^4}{n^2 \sigma_n^{4m}} \sum_{i,j,i',j'=1}^n \left| F(|j-i|) F(|j'-i'|) F(|i'-i|) F(|j'-j|) \right| \sigma_n^{4m-8} \\
&= \frac{a_m^4}{n^2 \sigma_n^{8}} \sum_{i,j,i',j'=1}^n \left| F(|j-i|) F(|j'-i'|) F(|i'-i|) F(|j'-j|) \right| \\
&= O_1 + 2O_2,
\end{aligned}
$$

where

$$
\begin{aligned}
O_1 &:= \frac{a_m^4}{n^2 \sigma_n^8} \sum_{i',j'=1}^n \sum_{i=1}^n \left| \, F(0) F(|j'-i'|) F(|i'-i|) F(|j'-i|) \, \right|, \\
O_2 &:= \frac{a_m^4}{n^2 \sigma_n^8} \sum_{i',j'=1}^n \sum_{i<j} \left| \, F(|j-i|) F(|j'-i'|) F(|i'-i|) F(|j'-j|) \, \right|.
\end{aligned}
$$

First note that, by direct computations and using (B.13), we have

$$
\begin{aligned}
O_1 &= \frac{a_m^4}{n^2 \sigma_n^6} \sum_{i',j'=1}^n \sum_{i=1}^n \left| \, F(|j'-i'|) F(|i'-i|) F(|j'-i|) \, \right| \\
&\le \frac{a_m^4}{n^2 \sigma_n^6} \sum_{i',j'=1}^n \sum_{i=1}^n \mid F(|j'-i'|) \mid \frac{F(|i'-i|)^2 + F(|j'-i|)^2}{2} \\
&\le \frac{a_m^4}{n^2 \sigma_n^6} \sum_{i',j'=1}^n \mid F(|j'-i'|) \mid \frac{2\sigma_n^4 + 2\sigma_n^4}{2} \le \frac{2a_m^4}{n^2 \sigma_n^2} \sum_{i',j'=1}^n \mid F(|j'-i'|) \mid \\
&\le \frac{2a_m^4}{n^2 \sigma_n^2} \left(\sum_{j=1}^n |F(0)| + 2 \sum_{i<j} \mid F(j-i) \mid \right) \\
&\le \frac{2a_m^4}{n} + \frac{4a_m^4}{n^2 \sigma_n^2} \sum_{j=1}^{n-1} (n-j) \mid F(j) \mid \; = \frac{2a_m^4}{n} + \frac{4a_m^4}{n} \sum_{j=1}^{n-1} (1 - \frac{j}{n}) \mid \frac{F(j)}{\sigma_n^2} \mid \\
&\underset{n\to\infty}{\longrightarrow} 0.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
O_2 &= \frac{a_m^4}{n^2\sigma_n^8}\sum_{i',j'=1}^{n}\sum_{i=1}^{n-1}\sum_{k=1}^{n-i}\Big|F(|i+k-i|)F(|j'-i'|)F(|i'-i|)F(|j'-i-k|)\Big| \\
&= \frac{a_m^4}{n^2\sigma_n^8}\sum_{i',j'=1}^{n}\sum_{i=1}^{n-1}\sum_{k=1}^{n-i}\Big|F(k)F(|j'-i'|)F(|i'-i|)F(|j'-i-k|)\Big| \\
&\le \frac{a_m^4}{n^2\sigma_n^8}\sum_{i',j'=1}^{n}\sum_{i=1}^{n-1}\sum_{k=1}^{n-i}\Big|F(|j'-i'|)F(|i'-i|)\Big|\frac{F(k)^2+F(|j'-i-k|)^2}{2} \\
&\le \frac{2a_m^4}{n^2\sigma_n^4}\sum_{i',j'=1}^{n}\sum_{i=1}^{n-1}\Big|F(|j'-i'|)F(|i'-i|)\Big| \le \frac{4a_m^4}{n^2\sigma_n^2}\sum_{i',j'=1}^{n}\Big|F(|j'-i'|)\Big| \\
&\xrightarrow[n\to\infty]{} 0.
\end{aligned}
$$

Thus, (B.16) holds true. Therefore, (N2) from Theorem A.3 holds true, namely, we have that

$$
F_n \xrightarrow[n\to\infty]{\mathcal{D}} \mathcal{N}\left(0, \begin{pmatrix} \bar\sigma_2^2 & 0 \\ 0 & \bar\sigma_4^2 \end{pmatrix}\right). \tag{B.17}
$$

Consequently, (4.5) follows from (B.14),(B.15) and (B.17). Finally, (4.5) implies (4.6), by using that

$$
\sqrt{n}\left(\frac{\pi\theta \mathsf{V}_n^4\left(u(\cdot,x);[c,d]\right)}{n\sigma_n^4\sigma^4} - \frac{\mathsf{V}_n^4(v;[c,d])}{n\sigma_n^4}\right) \to 0, \quad \text{in } L^2 \text{ and in probability.} \tag{B.18}
$$

The proof of (B.18) follows by similar arguments as in proof of Proposition 2.1 and we omit it here.
The proof is complete.

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