

DYNAMIC CONIC FINANCE: NO-ARBITRAGE PRICING AND
NO-GOOD-DEAL PRICING FOR DIVIDEND-PAYING SECURITIES IN
DISCRETE-TIME MARKETS WITH TRANSACTION COSTS

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ABSTRACT

This thesis studies *no-arbitrage pricing* and *dynamic conic finance* for dividend-paying securities in discrete-time markets with transaction costs.

The first part investigates no-arbitrage pricing for dividend-paying securities in discrete-time markets with transaction costs. We introduce the value process and the self-financing condition in our context. Then, we prove a version of First Fundamental Theorem of Asset Pricing. Specifically, we prove that the no-arbitrage condition under the efficient friction assumption is equivalent to the existence of a risk-neutral measure. We formulate an appropriate notion of a consistent pricing system in our set-up, and we prove that if there are no transaction costs on the dividends paid by the securities, then the no-arbitrage condition under the efficient friction assumption is equivalent to the existence of a consistent pricing system. We finish the chapter by deriving dual representations for the superhedging ask price and subhedging bid price of a derivative contract.

The second part studies dynamic conic finance in the set-up introduced in the first part. We formulate the no-good-deal condition in terms of a family of dynamic coherent risk measures, and then we prove a version of the Fundamental Theorem of No-Good-Deal Pricing. The Fundamental Theorem of No-Good-Deal Pricing provides a necessary and sufficient condition for the no-good-deal condition to hold. Next, we study the no-good-deal ask and bid prices of a derivative contract. We particularize our results to the dynamic Gain-Loss Ratio, and compute the no-good-deal prices of European-style Asian options in a market with transaction costs.

CHAPTER 1

INTRODUCTION

Three central themes in mathematical finance are no-arbitrage pricing, risk management, and performance measures. No-arbitrage pricing provides a framework for pricing derivative contracts, risk management identifies and assesses uncertainty related to financial positions, and performance measures allows for the attractiveness of different investments to be compared. There is a growing interest in establishing the general relationship between all the three themes—specifically the general relationship between no-arbitrage pricing, risk measures, and acceptability indices. Moreover, there is an increasing demand from practitioners for financial market models that include realistic features such as transaction costs. This thesis makes progress towards these efforts in several directions. In Chapter 2 we develop no-arbitrage pricing for dividend-paying securities in discrete-time markets with transaction costs¹. Then, in the set-up introduced in Chapter 2, we study dynamic conic finance in Chapter 3. Dynamic conic finance is a pricing framework that incorporates no-arbitrage pricing, dynamic coherent risk measures, and dynamic acceptability indices. To illustrate the theoretical results, we compute and compare the superhedging, subhedging, and no-good-deal prices of European-style Asian options in a market with transaction costs.

Let us first outline the contributions of Chapter 2. We begin the chapter by introducing the value process and the self-financing condition for trading strategies for markets with dividend-paying securities under transaction costs. The value process associated with a trading strategy is interpreted as the value of a portfolio of securities, and the self-financing condition requires no money to flow in or out of a

¹We define a transaction cost as the cost incurred in trading in a market in which securities' quoted prices have a bid-ask spread. We do not consider other costs such as broker's fees and taxes in the definition of a transaction cost.

portfolio. These two definitions are the building blocks of the both Chapter 2 and Chapter 3 because they allow us to formulate the no-arbitrage condition and the no-good-deal condition in our set-up. For frictionless markets, the value process and self-financing condition were originated by Harrison and Pliska [HP81]. The value process and the self-financing condition are rather straightforward to define for frictionless markets because securities can be bought and sold at the same price. On the contrary, there are several approaches to defining these concepts for markets with transaction costs (see for instance Jouini and Kallal [JK95]; Kabanov [Kab99]; Denis, Guasoni, and Rásonyi [DGR11]). However, these definitions are not suitable for dividend-paying securities because transaction costs associated to dividend-paying securities may *not* be proportional to the number of units of securities purchased or sold. For instance, transaction costs associated with interest rate swap contracts and credit default swap contracts *accrue* over time by merely holding the security—for a non-dividend paying security transaction costs are only charged whenever the security is bought or sold. Our consideration of dividends distinguishes the results in this chapter.

The main topic of Chapter 1 is *the no-arbitrage condition* (**NA**). Loosely speaking², **NA** is satisfied if “it is impossible to make something out of nothing.” Understanding **NA** is of great practical importance because market participants usually require for the financial market models that they use to satisfy this condition. The key to understanding **NA** is the First Fundamental Theorem of Asset Pricing (FFTAP), which in our context asserts that **NA** under *the efficient friction assumption* (**EF**) (See (2.16)) is satisfied if and only if there exists a risk-neutral measure³. Proving a version of the FFTAP in a discrete-time setting for a general state space is

²For the definition of the no-arbitrage condition, see Definition 2.3.1.

³For the definition of a risk-neutral measure, see Definition 2.4.1.

the main contribution of Chapter 2. The pivotal (and most challenging) step in the proof is showing that the set of values that can be superhedged at zero cost is closed with respect to convergence in probability. Once this topological property is proved, the Kreps-Yan Theorem (Kreps [Kre81]; Yan [Yan80]), which is a general result from convex analysis, essentially implies the FFTAP.

In the theory of no-arbitrage pricing in markets with transaction costs, *consistent pricing systems*⁴ (CPSs), which are interpreted as corresponding frictionless markets, play a central role. We prove that if there exists a CPS, then **NA** is satisfied. This result is crucial from the modeling point of view because it provides a straightforward condition that can be used to verify whether a financial market model satisfies **NA**. We also prove that whenever there are no transaction costs on the dividends paid by the securities (there may still be transaction costs on the ex-dividend prices of the securities), **NA** under **EF** holds if and only if there exists a CPS.

Finally, we derive dual representations for the superhedging ask price and subhedging bid price of a derivative contract. These prices are important because they provide two unique no-arbitrage prices that have meaningful financial interpretations in the context of hedging.

The FFTAP has been proved in varying levels of generality for frictionless markets. In a discrete-time setting for a finite state space, the theorem was first proved in Harrison and Pliska [HP81]. Almost a decade later, Dalang, Morton, and Willinger [DMW90] proved the FFTAP for the more technically challenging setting in which the state space is general. Their approach requires the use of advanced, measurable selection arguments, which motivated several authors to provide alternative proofs using more accessible techniques (see Schachermayer [Sch92]; Kabanov

⁴For the definition of a consistent pricing system, see Definition 2.5.1.

and Kramkov [KK94]; Rogers [Rog94]; Jacod and Shiryaev [JS98]; Kabanov and Stricker [KS01b]). Using advanced concepts from functional and stochastic analysis, the FFTAP was first proved in a general continuous-time set-up in the celebrated paper by Delbaen and Schachermayer [DS94]. A comprehensive review of the literature pertaining to no-arbitrage pricing in frictionless markets can be found in Delbaen and Schachermayer [DS06].

The first rigorous treatment of the FFTAP for markets with transaction costs in a discrete-time setting was carried out by Kabanov and Stricker [KS01a]. Under the assumption that the state space is finite, it was proved that **NA** is equivalent to the existence of a consistent pricing system. However, their results did not extend to the case of a general state space. As in the frictionless case, the transition from a finite state space to a general state space is nontrivial due to measure-theoretic and topological related difficulties. These difficulties were overcome in Kabanov, Rásonyi, and Stricker [KRS02], where a version of the FFTAP was proven under **EF**. It was proved that *the strict no-arbitrage condition*, a condition which is stronger than **NA**, is equivalent to the existence of a strictly consistent pricing system. In that paper, it was asked whether **EF** can be discarded. Schachermayer [Sch04] answered this question negatively by showing that neither **NA** nor the strict no-arbitrage condition alone is sufficiently strong to yield a version of the FFTAP. More importantly, Schachermayer [Sch04] proved a new version of the FFTAP that does not require **EF**. Namely, that *the robust no-arbitrage condition*, which is stronger than the strict no-arbitrage condition, is equivalent to the existence of a strictly consistent pricing system. Subsequent studies that treat the robust no-arbitrage condition are Bouchard [Bou06], Vallière, Kabanov, and Stricker [DVK07], Jacka, Berkaoui, and Warren [JBW08]. Recently, Pennanen [Pen11d, Pen11a, Pen11b, Pen11c] studied no-arbitrage pricing in a general context in which markets can have constraints, and transaction costs may depend nonlinearly on traded amounts. Therein, the problem of superhedging

a claims process (e.g. swaps) is also investigated. An excellent survey of the literature pertaining to no-arbitrage pricing in markets with transaction costs can be found in Kabanov and Safarian [KS09]. Let us mention that the results in the papers above cannot be applied to markets with dividend-paying securities in which transaction costs accrue over time. There is literature that considers arbitrary transaction costs (cf. Jouini and Kallal [JK95]; Cherny [Che07b]; Denis, Guasoni, and Rásonyi [DGR11]). However, they consider a continuous-time setting and stronger versions of **NA**. The results in this chapter can also be found in Bielecki, Cialenco, and Rodriguez [BCR12].

Next, we outline the contributions in Chapter 3. As we mentioned in the opening paragraph, this chapter develops *dynamic conic finance*, which is a pricing framework that incorporates no-arbitrage pricing, dynamic coherent risk measures (DCRMs), and dynamic coherent acceptability indices (DCAIs). Let us first give a brief overview of coherent risk measures and coherent acceptability indices. *Static coherent risk measures* (SCRM) were introduced in the seminal paper by Artzner et al. [ADEH99]. A function ρ from the set of all risks to the real numbers is a SCRM if it satisfies the following four properties: monotonicity, positive homogeneity, translation invariance, and subadditivity. These four properties are considered to be desirable for a measure of risk to have⁵. In Cherny and Madan [CM09], the notion of a *static coherent acceptability index* (SCAI) was introduced. A SCAI is a function from the set of all risks to the (extended) nonnegative real numbers that satisfies the following four properties: monotonicity, scale invariance, quasi-concavity, and the Fatou property. SCAIs generalize the concept of performance measures such as the Sharpe ratio [Sha66] and the Gain-Loss ratio [BL00]. It was found in Cherny

⁵There is also the notion of a convex risk measure. In the definition of a convex risk measure, the subadditivity and positive homogeneity property are replaced by a convexity property (see Föllmer and Schied [FS04]).

and Madan [CM09] that there is a duality between SCRM and SCAI. Specifically, that every SCAI can be characterized in terms of an increasing family of SCRM and vice versa. Recently, Bielecki, Cialenco, and Zhang [BCZ11] extended the notions of SCRM and SCAI to a multi-period setting by introducing dynamic coherent risk measures (DCRM) and dynamic coherent acceptability indices (DCAI). It was shown therein that DCRM and DCAI enjoy a duality property, among other properties. We direct the reader to Appendix B for definitions and results regarding DCRM and DCAI. DCRM and DCAI are central to developing dynamic conic finance because they allow us to formulate a multi-period version of the no-good-deal condition.

The main topic of Chapter 3 is *the no-good-deal condition (NGD)* for dividend-paying securities in discrete-time markets with transaction costs. We remark that this chapter adopts the financial market model with transaction costs that is introduced in Chapter 2. Loosely speaking⁶ **NGD** holds if “there does not exist a zero-cost cash flow with negative risk.” In view of Theorem B.0.2, under an additional continuity assumption, we may alternatively say that **NGD** is satisfied if “there does not exist an acceptable zero-cost cash flow”. At first glance, it appears that **NGD** is not related to **NA**. However, the main result of this chapter, which is the Fundamental Theorem of No-Good-Deal Pricing (FTNGDP), asserts that **NGD** and **NA** are deeply connected. An immediate consequence of the FTNGDP is that **NA** holds whenever **NGD** is satisfied. This is a desirable consequence from the modeling point of view because it assures that a financial market model satisfying **NGD** also satisfies **NA**.

In the second part of Chapter 3, we study the no-good-deal pricing framework introduced in Bielecki et al. [BCIR12] for narrowing the theoretical spread between

⁶For the definition of **NGD**, see Definition 3.1.2.

ask price and bid price of a derivative security using a DCAI. In this study, we particularize the framework to the dynamic Gain-Loss ratio, which is a DCAI introduced in Bielecki et al. [BCZ11]. We illustrate the theoretical results by computing the no-good-deal ask and bid prices of European-style Asian options.

The FTNGDP has been studied in several contexts. In Carr, Geman, and Madan [CGM01], a version of FTNGDP was formulated and proved in terms of the no-strictly-acceptable-opportunities condition for frictionless markets, and subsequently Pinar, Salih, and Camci [PSC10] proved a version of the FTNGDP in the context of the Gain-Loss ratio in markets with proportional transaction costs. The FTNGDP has been obtained for markets with transaction costs in the context of static coherent risk measures, and for frictionless markets using discrete-time coherent risk measures by Cherny [Che07c] and [Che07d], respectively. Let us remark that Chapter 3 was mainly motivated by results obtained in Cherny and Madan [CM10], where a version of the FTNGDP that is formulated in terms of a family of SCRMs is proved.

Our treatment of the FTNGDP differs from the papers mentioned above. First, our framework allows for (hedging) cash flows to pay dividends and be subjected to transaction costs. In particular, our no-good-deal pricing approach can be applied to interest rate swaps and credit default swaps. Second, we prove a version of the FTNGDP formulated in terms of a no-good-deal-condition that is dynamically consistent in time, in the sense that it is defined in terms of a family of DCRMs.

The thesis is organized as follows. Chapter 2 studies no-arbitrage pricing for dividend-paying securities in discrete-time markets with transaction costs. In Section 2.1 we introduce our financial market model, and in Section 2.2 we study the value process and self-financing condition. Then, in Section 2.3, we formulate the no-arbitrage condition under the efficient friction assumption, and prove a key closedness property of the set of values that can be superhedged at zero cost. Next, in

Section 2.4, we prove a version of the First Fundamental Theorem of Asset Pricing. In Section 2.5, we investigate the relationship between the no-arbitrage condition and consistent pricing systems. We conclude Chapter 2 with Section 2.6, where we introduce the superhedging ask price and subhedging bid price of a derivative contract, and derive dual representations for these prices. In Chapter 3, we study dynamic conic finance. We begin the chapter by studying the no-good-deal condition in Section 3.1. Subsequently, in Section 3.2, we prove the Fundamental Theorem of No-Good-Deal Pricing. In Section 3.3, the no-good-deal prices are studied, and then we particularize them to the dynamic Gain-Loss ratio in Section 3.4. We finish Chapter 3 with Section 3.5, where we compute the no-good-deal prices of European-style Asian options in a market with transaction costs.

CHAPTER 2

NO-ARBITRAGE PRICING IN MARKETS WITH TRANSACTION COSTS

In this chapter, we develop no-arbitrage pricing for dividend-paying securities in discrete-time markets with transaction costs.

2.1 Financial Market Model

Let T be a fixed time horizon, and define $\mathcal{T} := \{0, 1, \dots, T\}$. Next, let $(\Omega, \mathcal{F}_T, \mathbb{F} = (\mathcal{F}_t)_{t \in \mathcal{T}}, \mathbb{P})$ be the underlying filtered probability space.

On this probability space, we consider a market consisting of a savings account B and of N traded securities satisfying the following properties:

- The savings account can be purchased and sold according to the price process $B := \left(\left(\prod_{s=0}^t (1 + r_s) \right) \right)_{t=0}^T$, where $(r_t)_{t=0}^T$ is a nonnegative process specifying the risk-free rate.
- The N securities can be purchased according to the ex-dividend price process $P^{ask} := \left((P_t^{ask,1}, \dots, P_t^{ask,N}) \right)_{t=0}^T$; the associated (cumulative) dividend process is denoted by $A^{ask} := \left((A_t^{ask,1}, \dots, A_t^{ask,N}) \right)_{t=1}^T$.
- The N securities can be sold according to the ex-dividend price process $P^{bid} := \left((P_t^{bid,1}, \dots, P_t^{bid,N}) \right)_{t=0}^T$; the associated (cumulative) dividend process is denoted by $A^{bid} := \left((A_t^{bid,1}, \dots, A_t^{bid,N}) \right)_{t=1}^T$.

We assume that the processes introduced above are adapted. In what follows, we will denote by Δ the backward difference operator: $\Delta X_t := X_t - X_{t-1}$, and we take the convention that $A_0^{ask} = A_0^{bid} = 0$. It is straightforward to verify that Δ satisfies

following product rule:

$$\begin{aligned}
\Delta(X_t Y_t) &= X_{t-1} \Delta Y_t + Y_t \Delta X_t \\
&= X_t \Delta Y_t + Y_{t-1} \Delta X_t \\
&= X_t \Delta Y_t + Y_t \Delta X_t - \Delta X_t \Delta Y_t,
\end{aligned}$$

for any processes X and Y .

Remark 2.1.1. For any $t = 1, 2, \dots, T$ and $j = 1, 2, \dots, N$, the random variable $\Delta A_t^{\text{ask},j}$ is interpreted as amount of dividend associated with holding a long position in security j from time $t - 1$ to time t , and the random variable $\Delta A_t^{\text{bid},j}$ is interpreted as amount of dividend associated with holding a short position in security j from time $t - 1$ to time t .

We illustrate the processes introduced above in the context of a vanilla Credit Default Swap (CDS) contract.

Example 2.1.1. A CDS contract is a contract between two parties, a protection buyer and a protection seller, in which the protection buyer pays periodic fees to the protection seller in exchange for some payment made by the protection seller to the protection buyer if a pre-specified credit event of a reference entity occurs. Let τ be the nonnegative random variable specifying the time of the credit event of the reference entity. Suppose the CDS contract admits the following specifications: initiation date $t = 0$, expiration date $t = T$, and nominal value \$1. For simplicity, we assume that the loss-given-default is a nonnegative scalar δ and is paid at default. Typically, CDS contracts are traded on over-the-counter markets in which dealers quote CDS spreads to investors. Suppose that the CDS spread quoted by the dealer to sell a CDS contract with above specifications is κ^{bid} (to be received every unit of time), and the CDS spread quoted by the dealer to buy a CDS contract with above specifications is κ^{ask} (to be paid every unit of time). We remark that the CDS spreads κ^{ask} and κ^{bid} are specified in

the CDS contract, so the CDS contract to sell protection is a different contract than the CDS contract to buy protection.

The cumulative dividend processes A^{ask} and A^{bid} associated with buying and selling the CDS with specifications above, respectively, are defined as

$$A_t^{ask} := 1_{\{\tau \leq t\}}\delta - \kappa^{ask} \sum_{u=1}^t 1_{\{u < \tau\}}, \quad A_t^{bid} := 1_{\{\tau \leq t\}}\delta - \kappa^{bid} \sum_{u=1}^t 1_{\{u < \tau\}}$$

for all $t \in \mathcal{T}^*$. In this case, the ex-dividend ask and bid price processes P^{bid} and P^{ask} specify the mark-to-market values of the CDS.

From now on, we make the following standing assumption.

Assumption (A): $P^{ask} \geq P^{bid}$ and $\Delta A^{ask} \leq \Delta A^{bid}$.

Note that if this assumption is violated, then market exhibits arbitrage by simultaneously buying and selling the corresponding security.

For convenience, we define $\mathcal{T}^* := \{1, 2, \dots, T\}$, $\mathcal{J} := \{0, 1, \dots, N\}$, and $\mathcal{J}^* := \{1, 2, \dots, N\}$. Unless stated otherwise, all inequalities and equalities between processes and random variables are understood \mathbb{P} -a.s. and coordinate-wise.

2.2 The Value Processes and the Self-Financing Condition

A *trading strategy* is a predictable process $\phi := ((\phi_t^0, \phi_t^1, \dots, \phi_t^N))_{t=1}^T$, where ϕ_t^j is interpreted as the number of units of security j held from time $t - 1$ to time t . Processes ϕ^1, \dots, ϕ^N correspond to the holdings in the N securities, and process ϕ^0 corresponds to the holdings in the savings account B . We take the convention $\phi_0 = (0, \dots, 0)$.

Definition 2.2.1. The value process $(V_t(\phi))_{t=0}^T$ associated with a trading strategy ϕ

is defined as

$$V_t(\phi) = \begin{cases} \phi_1^0 + \sum_{j=1}^N \phi_1^j (1_{\{\phi_1^j \geq 0\}} P_0^{ask,j} + 1_{\{\phi_1^j < 0\}} P_0^{bid,j}), & \text{if } t = 0, \\ \phi_t^0 B_t + \sum_{j=1}^N \phi_t^j (1_{\{\phi_t^j \geq 0\}} P_t^{bid,j} + 1_{\{\phi_t^j < 0\}} P_t^{ask,j}) \\ \quad + \sum_{j=1}^N \phi_t^j (1_{\{\phi_t^j < 0\}} \Delta A_t^{bid,j} + 1_{\{\phi_t^j \geq 0\}} \Delta A_t^{ask,j}), & \text{if } 1 \leq t \leq T. \end{cases}$$

Remark 2.2.1. (i) Note the difference in the use of bid and ask prices, in the above definition, between the time $t = 0$ and the time $t = 1, \dots, T$. At time $t = 0$, $V_0(\phi)$ is interpreted as the cost of setting up the portfolio associated with ϕ . For $t = 1, \dots, T$, the value process $V_t(\phi)$ equals the sum of the liquidation value of the portfolio associated with trading strategy ϕ before any time t transactions and the dividends associated with ϕ from time $t - 1$ to t .

(ii) Due to the presence of transaction costs, the value process V may not be linear in its argument, i.e. $V_t(\phi) + V_t(\psi) \neq V_t(\phi + \psi)$, and $V_t(\alpha\phi) \neq \alpha V_t(\phi)$ for some $\alpha < 0$, and some trading strategies ϕ, ψ , some time $t \in \mathcal{T}$. This is the major difference from the frictionless setting.

Next, we introduce the self-financing condition, which is appropriate in the context of this paper.

Definition 2.2.2. A trading strategy ϕ is self-financing if

$$\begin{aligned} B_t \Delta \phi_{t+1}^0 + \sum_{j=1}^N \Delta \phi_{t+1}^j (1_{\{\Delta \phi_{t+1}^j \geq 0\}} P_t^{ask,j} + 1_{\{\Delta \phi_{t+1}^j < 0\}} P_t^{bid,j}) \\ = \sum_{j=1}^N \phi_t^j (1_{\{\phi_t^j \geq 0\}} \Delta A_t^{ask,j} + 1_{\{\phi_t^j < 0\}} \Delta A_t^{bid,j}) \end{aligned} \quad (2.1)$$

for $t = 1, 2, \dots, T - 1$.

The self-financing condition imposes the restriction that no money can flow in or out of the portfolio. We note that if $P^{ask} = P^{bid}$ and $A^{ask} = A^{bid}$, then the self-financing condition in the frictionless case is recovered.

Remark 2.2.2. *The self-financing condition not only takes into account transaction costs due to purchases and sales of securities (left hand side of (2.1)), but also transaction costs accrued through the dividends (right hand side of (2.1)).*

The next result gives a useful characterization of the self-financing condition in terms of the value process.

Proposition 2.2.1. *A trading strategy ϕ is self-financing if and only if the value process $V(\phi)$ satisfies*

$$\begin{aligned} V_t(\phi) = & V_0(\phi) + \sum_{u=1}^t \phi_u^0 \Delta B_u + \sum_{j=1}^N \phi_t^j \left(1_{\{\phi_t^j \geq 0\}} P_t^{bid,j} + 1_{\{\phi_t^j < 0\}} P_t^{ask,j} \right) \\ & - \sum_{j=1}^N \sum_{u=1}^t \Delta \phi_u^j \left(1_{\{\Delta \phi_u^j \geq 0\}} P_{u-1}^{ask,j} + 1_{\{\Delta \phi_u^j < 0\}} P_{u-1}^{bid,j} \right) \\ & + \sum_{j=1}^N \sum_{u=1}^t \phi_u^j \left(1_{\{\phi_u^j \geq 0\}} \Delta A_u^{ask,j} + 1_{\{\phi_u^j < 0\}} \Delta A_u^{bid,j} \right) \end{aligned}$$

for all $t \in \mathcal{T}^*$.

Proof. By the definition of $V(\phi)$,

$$\begin{aligned} V_t(\phi) = & \sum_{u=2}^t \Delta(\phi_u^0 B_u) + \phi_1^0 B_1 + \sum_{j=1}^N \phi_t^j \left(1_{\{\phi_t^j \geq 0\}} P_t^{bid,j} + 1_{\{\phi_t^j < 0\}} P_t^{ask,j} \right) \\ & + \sum_{j=1}^N \left(\sum_{u=2}^t \Delta \left(1_{\{\phi_u^j \geq 0\}} \phi_u^j \Delta A_u^{ask,j} \right) + 1_{\{\phi_1^j \geq 0\}} \phi_1^j \Delta A_1^{ask,j} \right) \\ & + \sum_{j=1}^N \left(\sum_{u=2}^t \Delta \left(1_{\{\phi_u^j < 0\}} \phi_u^j \Delta A_u^{bid,j} \right) + 1_{\{\phi_1^j < 0\}} \phi_1^j \Delta A_1^{bid,j} \right) \end{aligned}$$

for any trading strategy ϕ , for all $t \in \mathcal{T}^*$. Using the product rule for the backwards

difference operator Δ ,

$$\begin{aligned}
V_t(\phi) &= \sum_{u=2}^t \left(\phi_u^0 \Delta B_u + B_{u-1} \Delta \phi_u^0 \right) + \phi_1^0 B_1 + \sum_{j=1}^N \phi_t^j \left(1_{\{\phi_t^j \geq 0\}} P_t^{bid,j} + 1_{\{\phi_t^j < 0\}} P_t^{ask,j} \right) \\
&+ \sum_{j=1}^N \sum_{u=2}^t \left(1_{\{\phi_u^j \geq 0\}} \phi_u^j \Delta (\Delta A_u^{ask,j}) + \Delta A_{u-1}^{ask,j} \Delta (1_{\{\phi_u^j \geq 0\}} \phi_u^j) + 1_{\{\phi_1^j \geq 0\}} \phi_1^j \Delta A_1^{ask,j} \right) \\
&+ \sum_{j=1}^N \sum_{u=2}^t \left(1_{\{\phi_u^j < 0\}} \phi_u^j \Delta (\Delta A_u^{bid,j}) + \Delta A_{u-1}^{bid,j} \Delta (1_{\{\phi_u^j < 0\}} \phi_u^j) + 1_{\{\phi_1^j < 0\}} \phi_1^j \Delta A_1^{bid,j} \right).
\end{aligned}$$

Rearranging terms, we see that

$$\begin{aligned}
V_t(\phi) &= \sum_{u=1}^t \phi_u^0 \Delta B_u + \sum_{u=2}^t B_{u-1} \Delta \phi_u^0 + \phi_1^0 + \sum_{j=1}^N \phi_t^j \left(1_{\{\phi_t^j \geq 0\}} P_t^{bid,j} + 1_{\{\phi_t^j < 0\}} P_t^{ask,j} \right) \\
&+ \sum_{j=1}^N \sum_{u=1}^t 1_{\{\phi_u^j \geq 0\}} \phi_u^j \Delta A_u^{ask,j} - \sum_{j=1}^N \sum_{u=2}^t 1_{\{\phi_{u-1}^j \geq 0\}} \phi_{u-1}^j \Delta A_{u-1}^{ask,j} \\
&+ \sum_{j=1}^N \sum_{u=1}^t 1_{\{\phi_u^j < 0\}} \phi_u^j \Delta A_u^{bid,j} - \sum_{j=1}^N \sum_{u=2}^t 1_{\{\phi_{u-1}^j < 0\}} \phi_{u-1}^j \Delta A_{u-1}^{bid,j}.
\end{aligned}$$

From the definition of $V_0(\phi)$, we have that

$$\phi_1^0 = V_0(\phi) - \sum_{j=1}^N \phi_1^j \left(1_{\{\phi_1^j \geq 0\}} P_0^{ask,j} + 1_{\{\phi_1^j < 0\}} P_0^{bid,j} \right).$$

Therefore,

$$\begin{aligned}
V_t(\phi) &= V_0(\phi) + \sum_{u=1}^t \phi_u^0 \Delta B_u + \sum_{u=2}^t B_{u-1} \Delta \phi_u^0 \tag{2.2} \\
&- \sum_{j=1}^N \phi_1^j \left(1_{\{\phi_1^j < 0\}} P_0^{bid,j} + 1_{\{\phi_1^j \geq 0\}} P_0^{ask,j} \right) + \sum_{j=1}^N \phi_t^j \left(1_{\{\phi_t^j \geq 0\}} P_t^{bid,j} + 1_{\{\phi_t^j < 0\}} P_t^{ask,j} \right) \\
&+ \sum_{j=1}^N \sum_{u=1}^t \phi_u^j \left(1_{\{\phi_u^j \geq 0\}} \Delta A_u^{ask,j} + 1_{\{\phi_u^j < 0\}} \Delta A_u^{bid,j} \right) \\
&- \sum_{j=1}^N \sum_{u=2}^t \phi_{u-1}^j \left(1_{\{\phi_{u-1}^j < 0\}} \Delta A_{u-1}^{bid,j} + 1_{\{\phi_{u-1}^j \geq 0\}} \Delta A_{u-1}^{ask,j} \right).
\end{aligned}$$

We now show that sufficiency and necessity hold.

If ϕ is self-financing, then (2.2) reduces to

$$\begin{aligned}
V_t(\phi) &= V_0(\phi) + \sum_{u=1}^t \phi_u^0 \Delta B_u + \sum_{j=1}^N \phi_t^j \left(1_{\{\phi_t^j \geq 0\}} P_t^{bid,j} + 1_{\{\phi_t^j < 0\}} P_t^{ask,j} \right) \\
&\quad - \sum_{j=1}^N \sum_{u=1}^t \Delta \phi_u^j \left(1_{\{\Delta \phi_u^j \geq 0\}} P_{u-1}^{ask,j} + 1_{\{\Delta \phi_u^j < 0\}} P_{u-1}^{bid,j} \right) \\
&\quad + \sum_{j=1}^N \sum_{u=1}^t \phi_u^j \left(1_{\{\phi_u^j \geq 0\}} \Delta A_u^{ask,j} + 1_{\{\phi_u^j < 0\}} \Delta A_u^{bid,j} \right)
\end{aligned} \tag{2.3}$$

for $t \in \mathcal{T}^*$.

Conversely, assume that the value process satisfies

$$\begin{aligned}
V_t(\phi) &= V_0(\phi) + \sum_{u=1}^t \phi_u^0 \Delta B_u + \sum_{j=1}^N \phi_t^j \left(1_{\{\phi_t^j \geq 0\}} P_t^{bid,j} + 1_{\{\phi_t^j < 0\}} P_t^{ask,j} \right) \\
&\quad - \sum_{j=1}^N \sum_{u=1}^t \Delta \phi_u^j \left(1_{\{\Delta \phi_u^j \geq 0\}} P_{u-1}^{ask,j} + 1_{\{\Delta \phi_u^j < 0\}} P_{u-1}^{bid,j} \right) \\
&\quad + \sum_{j=1}^N \sum_{u=1}^t \phi_u^j \left(1_{\{\phi_u^j \geq 0\}} \Delta A_u^{ask,j} + 1_{\{\phi_u^j < 0\}} \Delta A_u^{bid,j} \right)
\end{aligned} \tag{2.4}$$

for $t \in \mathcal{T}^*$. Subtracting (2.4) from (2.2) yields

$$\begin{aligned}
0 &= \sum_{u=2}^t B_{u-1} \Delta \phi_u^0 + \sum_{j=1}^N \sum_{u=2}^t \Delta \phi_u^j \left(1_{\{\Delta \phi_u^j \geq 0\}} P_{u-1}^{ask,j} + 1_{\{\Delta \phi_u^j < 0\}} P_{u-1}^{bid,j} \right) \\
&\quad - \sum_{j=1}^N \sum_{u=2}^t \phi_{u-1}^j \left(1_{\{\phi_{u-1}^j < 0\}} \Delta A_{u-1}^{bid,j} + 1_{\{\phi_{u-1}^j \geq 0\}} \Delta A_{u-1}^{ask,j} \right)
\end{aligned}$$

for $t = 2, 3, \dots, T$. Applying the backwards difference Δ to both sides of the equation above gives us

$$\begin{aligned}
0 &= B_{t-1} \Delta \phi_t^0 + \sum_{j=1}^N \Delta \phi_t^j \left(1_{\{\Delta \phi_t^j \geq 0\}} P_{t-1}^{ask,j} + 1_{\{\Delta \phi_t^j < 0\}} P_{t-1}^{bid,j} \right) \\
&\quad - \sum_{j=1}^N \phi_{t-1}^j \left(1_{\{\phi_{t-1}^j < 0\}} \Delta A_{t-1}^{bid,j} + 1_{\{\phi_{t-1}^j \geq 0\}} \Delta A_{t-1}^{ask,j} \right)
\end{aligned}$$

for $t = 2, 3, \dots, T$. It follows that ϕ is self-financing. \square

The next lemma extends the previous result in terms of our numéraire B . For convenience, we let $V^*(\phi) := B^{-1}V(\phi)$ for all trading strategies ϕ .

Proposition 2.2.2. *A trading strategy ϕ is self-financing if and only if the discounted value process $V^*(\phi)$ satisfies*

$$\begin{aligned} V_t^*(\phi) &= V_0(\phi) + \sum_{j=1}^N \phi_t^j B_t^{-1} \left(1_{\{\phi_t^j \geq 0\}} P_t^{bid,j} + 1_{\{\phi_t^j < 0\}} P_t^{ask,j} \right) \\ &\quad - \sum_{j=1}^N \sum_{u=1}^t \Delta \phi_u^j B_{u-1}^{-1} \left(1_{\{\Delta \phi_u^j \geq 0\}} P_{u-1}^{ask,j} + 1_{\{\Delta \phi_u^j < 0\}} P_{u-1}^{bid,j} \right) \\ &\quad + \sum_{j=1}^N \sum_{u=1}^t \phi_u^j B_u^{-1} \left(1_{\{\phi_u^j \geq 0\}} \Delta A_u^{ask,j} + 1_{\{\phi_u^j < 0\}} \Delta A_u^{bid,j} \right) \end{aligned}$$

for all $t \in \mathcal{T}^*$.

Proof. Suppose that ϕ is self-financing. We may apply Proposition 2.2.1 to see that

$$\begin{aligned} \Delta(B_t^{-1} V_t(\phi)) &= B_{t-1}^{-1} \Delta V_t(\phi) + V_t(\phi) \Delta B_t^{-1} \\ &= B_{t-1}^{-1} \left(\phi_t^0 \Delta B_t + \sum_{j=1}^N \Delta \left(\phi_t^j (1_{\{\phi_t^j \geq 0\}} P_t^{bid,j} + 1_{\{\phi_t^j < 0\}} P_t^{ask,j}) \right) \right. \\ &\quad - \sum_{j=1}^N \Delta \phi_t^j \left(1_{\{\Delta \phi_t^j \geq 0\}} P_{t-1}^{ask,j} + 1_{\{\Delta \phi_t^j < 0\}} P_{t-1}^{bid,j} \right) \\ &\quad \left. + \sum_{j=1}^N \phi_t^j \left(1_{\{\phi_t^j \geq 0\}} \Delta A_t^{ask,j} + 1_{\{\phi_t^j < 0\}} \Delta A_t^{bid,j} \right) \right) \\ &\quad + \left(\phi_t^0 B_t + \sum_{j=1}^N \phi_t^j \left(1_{\{\phi_t^j \geq 0\}} P_t^{bid,j} + 1_{\{\phi_t^j < 0\}} P_t^{ask,j} \right) \right. \\ &\quad \left. + \sum_{j=1}^N \phi_t^j \left(1_{\{\phi_t^j \geq 0\}} \Delta A_t^{ask,j} + 1_{\{\phi_t^j < 0\}} \Delta A_t^{bid,j} \right) \right) \Delta B_t^{-1} \end{aligned}$$

for all $t \in \mathcal{T}^*$. We notice that

$$B_{t-1}^{-1} \Delta B_t + B_t \Delta B_t^{-1} = 0, \quad (2.5)$$

and

$$\begin{aligned} B_{t-1}^{-1} \Delta \left(\phi_t^j \left(1_{\{\phi_t^j \geq 0\}} P_t^{bid,j} + 1_{\{\phi_t^j < 0\}} P_t^{ask,j} \right) \right) &+ \Delta B_t^{-1} \phi_t^j \left(1_{\{\phi_t^j \geq 0\}} P_t^{bid,j} + 1_{\{\phi_t^j < 0\}} P_t^{ask,j} \right) \\ &= \Delta \left(\phi_t^j B_t^{-1} \left(1_{\{\phi_t^j \geq 0\}} P_t^{bid,j} + 1_{\{\phi_t^j < 0\}} P_t^{ask,j} \right) \right) \end{aligned} \quad (2.6)$$

by the product rule for Δ . Putting everything together, we obtain that

$$\begin{aligned} \Delta(B_t^{-1}V_t(\phi)) &= \sum_{j=1}^N \Delta \left(\phi_t^j B_t^{-1} \left(1_{\{\phi_t^j \geq 0\}} P_t^{bid,j} + 1_{\{\phi_t^j < 0\}} P_t^{ask,j} \right) \right) \\ &\quad - \sum_{j=1}^N \Delta \phi_t^j B_{t-1}^{-1} \left(1_{\{\Delta \phi_t^j \geq 0\}} P_{t-1}^{ask,j} + 1_{\{\Delta \phi_t^j < 0\}} P_{t-1}^{bid,j} \right) \\ &\quad + \sum_{j=1}^N \phi_t^j B_t^{-1} \left(1_{\{\phi_t^j \geq 0\}} \Delta A_t^{ask,j} + 1_{\{\phi_t^j < 0\}} \Delta A_t^{bid,j} \right) \end{aligned}$$

for all $t \in \mathcal{T}^*$. Summing both sides of the equation from $u = 1$ to $u = t$ proves necessity.

Conversely, if the value process $V(\phi)$ satisfies

$$\begin{aligned} B_t^{-1}V_t(\phi) &= V_0(\phi) + \sum_{j=1}^N \phi_t^j B_t^{-1} \left(1_{\{\phi_t^j \geq 0\}} P_t^{bid,j} + 1_{\{\phi_t^j < 0\}} P_t^{ask,j} \right) \\ &\quad - \sum_{j=1}^N \sum_{u=1}^t \Delta \phi_u^j B_{u-1}^{-1} \left(1_{\{\Delta \phi_u^j \geq 0\}} P_{u-1}^{ask,j} + 1_{\{\Delta \phi_u^j < 0\}} P_{u-1}^{bid,j} \right) \\ &\quad + \sum_{j=1}^N \sum_{u=1}^t \phi_u^j B_u^{-1} \left(1_{\{\phi_u^j \geq 0\}} \Delta A_u^{ask,j} + 1_{\{\phi_u^j < 0\}} \Delta A_u^{bid,j} \right) \end{aligned}$$

for all $t \in \mathcal{T}^*$, then

$$\begin{aligned} \Delta V_t(\phi) &= \Delta(B_t B_t^{-1} V_t(\phi)) = B_{t-1} \Delta(B_t^{-1} V_t(\phi)) + (B_t^{-1} V_t(\phi)) \Delta B_t \\ &= B_{t-1} \left(\sum_{j=1}^N \Delta \left(\phi_t^j B_t^{-1} \left(1_{\{\phi_t^j \geq 0\}} P_t^{bid,j} + 1_{\{\phi_t^j < 0\}} P_t^{ask,j} \right) \right) \right. \\ &\quad - \sum_{j=1}^N \Delta \phi_t^j B_{t-1}^{-1} \left(1_{\{\Delta \phi_t^j \geq 0\}} P_{t-1}^{ask,j} + 1_{\{\Delta \phi_t^j < 0\}} P_{t-1}^{bid,j} \right) \\ &\quad \left. + \sum_{j=1}^N \phi_t^j B_t^{-1} \left(1_{\{\phi_t^j \geq 0\}} \Delta A_t^{ask,j} + 1_{\{\phi_t^j < 0\}} \Delta A_t^{bid,j} \right) \right) \\ &\quad + \left(\phi_t^0 + \sum_{j=1}^N \phi_t^j B_t^{-1} \left(1_{\{\phi_t^j \geq 0\}} P_t^{bid,j} + 1_{\{\phi_t^j < 0\}} P_t^{ask,j} \right) \right. \\ &\quad \left. + 1_{\{\phi_t^j \geq 0\}} \Delta A_t^{ask,j} + 1_{\{\phi_t^j < 0\}} \Delta A_t^{bid,j} \right) \Delta B_t. \end{aligned}$$

From equations (2.5) and (2.6) we obtain

$$\begin{aligned} \Delta V_t(\phi) &= \phi_t^0 \Delta B_t + \sum_{j=1}^N \Delta \left(\phi_t^j \left(1_{\{\phi_t^j \geq 0\}} P_t^{bid,j} + 1_{\{\phi_t^j < 0\}} P_t^{ask,j} \right) \right) \\ &\quad - \sum_{j=1}^N \Delta \phi_t^j \left(1_{\{\Delta \phi_t^j \geq 0\}} P_{t-1}^{ask,j} + 1_{\{\Delta \phi_t^j < 0\}} P_{t-1}^{bid,j} \right) \\ &\quad + \sum_{j=1}^N \phi_t^j \left(1_{\{\phi_t^j \geq 0\}} \Delta A_t^{ask,j} + 1_{\{\phi_t^j < 0\}} \Delta A_t^{bid,j} \right). \end{aligned}$$

After summing both sides of the equation above from $u = 1$ to $u = t$ and applying Proposition 2.2.1, we see that ϕ is self-financing. \square

Remark 2.2.3. *If $P := P^{ask} = P^{bid}$ and $A := \Delta A^{ask} = \Delta A^{bid}$, then we recover the classic result: a trading strategy ϕ is self-financing if and only if the value process satisfies*

$$V_t^*(\phi) = V_0(\phi) + \sum_{j=1}^N \sum_{u=1}^t \phi_u^j \Delta \left(B_u^{-1} P_u^j + \sum_{w=1}^u B_w^{-1} A_w^j \right)$$

for all $t \in \mathcal{T}^*$.

For convenience, we define $P^{ask,*} := B^{-1} P^{ask}$, $P^{bid,*} := B^{-1} P^{bid}$, $A^{ask,*} := B^{-1} \Delta A^{ask}$, and $A^{bid,*} := B^{-1} \Delta A^{bid}$.

In frictionless markets, the set of all self-financing trading strategies is a linear space because securities' prices are not influenced by the direction of trading. This is no longer the case if the direction of trading matters: the trading strategy $\phi + \psi$ may not be self-financing whenever ϕ and ψ are self-financing. Intuitively this is true because transaction costs can be avoided whenever $\phi_t^j \psi_t^j < 0$ by combining orders. However, the trading strategy $(\theta^0, \phi^1 + \psi^1, \phi^2 + \psi^2, \dots, \phi^N + \psi^N)$ can enjoy the self-financing property if the units in the savings account θ^0 are properly adjusted. The next lemma shows that such θ^0 exists, is unique, and satisfies $\phi^0 + \psi^0 \leq \theta^0$.

Proposition 2.2.3. *Let ψ and ϕ be any two self-financing trading strategies with $V_0(\psi) = V_0(\phi) = 0$. Then there exists a unique predictable process θ^0 such that the*

trading strategy θ defined as $\theta := (\theta^0, \phi^1 + \psi^1, \dots, \phi^N + \psi^N)$ is self-financing with $V_0(\theta) = 0$. Moreover, $\phi^0 + \psi^0 \leq \theta^0$.

Proof. The trading strategies ϕ and ψ are self-financing, so by definition we have that

$$\begin{aligned} B_{t-1}\Delta\phi_t^0 + \sum_{j=1}^N \Delta\phi_t^j \left(1_{\{\Delta\phi_t^j \geq 0\}} P_{t-1}^{ask,j} + 1_{\{\Delta\phi_t^j < 0\}} P_{t-1}^{bid,j} \right) \\ = \sum_{j=1}^N \phi_{t-1}^j \left(1_{\{\phi_{t-1}^j \geq 0\}} \Delta A_{t-1}^{ask,j} + 1_{\{\phi_{t-1}^j < 0\}} \Delta A_{t-1}^{bid,j} \right) \end{aligned} \quad (2.7)$$

and

$$\begin{aligned} B_{t-1}\Delta\psi_t^0 + \sum_{j=1}^N \Delta\psi_t^j \left(1_{\{\Delta\psi_t^j \geq 0\}} P_{t-1}^{ask,j} + 1_{\{\Delta\psi_t^j < 0\}} P_{t-1}^{bid,j} \right) \\ = \sum_{j=1}^N \psi_{t-1}^j \left(1_{\{\psi_{t-1}^j \geq 0\}} \Delta A_{t-1}^{ask,j} + 1_{\{\psi_{t-1}^j < 0\}} \Delta A_{t-1}^{bid,j} \right) \end{aligned} \quad (2.8)$$

for $t = 2, 3, \dots, T$. By adding equations (2.7) and (2.8) we see that

$$\begin{aligned} B_{t-1}\Delta(\psi_t^0 + \phi_t^0) &= - \sum_{j=1}^N P_{t-1}^{ask,j} \left(1_{\{\Delta\phi_t^j \geq 0\}} \Delta\phi_t^j + 1_{\{\Delta\psi_t^j \geq 0\}} \Delta\psi_t^j \right) \\ &\quad - \sum_{j=1}^N P_{t-1}^{bid,j} \left(1_{\{\Delta\phi_t^j < 0\}} \Delta\phi_t^j + 1_{\{\Delta\psi_t^j < 0\}} \Delta\psi_t^j \right) \\ &\quad + \sum_{j=1}^N \left(\phi_{t-1}^j 1_{\{\phi_{t-1}^j \geq 0\}} + \psi_{t-1}^j 1_{\{\psi_{t-1}^j \geq 0\}} \right) \Delta A_{t-1}^{ask,j} \\ &\quad + \sum_{j=1}^N \left(\phi_{t-1}^j 1_{\{\phi_{t-1}^j < 0\}} + \psi_{t-1}^j 1_{\{\psi_{t-1}^j < 0\}} \right) \Delta A_{t-1}^{bid,j} \end{aligned}$$

for $t = 2, 3, \dots, T$. Using the equality $1_E + 1_{E^c} = 1$ for any set E , we have that

$$\begin{aligned} \psi_t^0 + \phi_t^0 &= \psi_{t-1}^0 + \phi_{t-1}^0 + B_{t-1}^{-1} \left(- \sum_{j=1}^N P_{t-1}^{ask,j} (\Delta\phi_t^j + \Delta\psi_t^j) \right. \\ &\quad + \sum_{j=1}^N (\phi_{t-1}^j + \psi_{t-1}^j) \Delta A_{t-1}^{ask,j} \\ &\quad + \sum_{j=1}^N (P_{t-1}^{ask,j} - P_{t-1}^{bid,j}) \left(1_{\{\Delta\phi_t^j < 0\}} \Delta\phi_t^j + 1_{\{\Delta\psi_t^j < 0\}} \Delta\psi_t^j \right) \\ &\quad \left. + \sum_{j=1}^N \left(1_{\{\phi_{t-1}^j < 0\}} \phi_{t-1}^j + 1_{\{\psi_{t-1}^j < 0\}} \psi_{t-1}^j \right) (\Delta A_{t-1}^{bid,j} - \Delta A_{t-1}^{ask,j}) \right) \end{aligned} \quad (2.9)$$

for $t = 2, 3, \dots, T$. Now, recursively define the process θ^0 as

$$\theta_1^0 := - \sum_{j=1}^N \left(1_{\{\phi_1^j \geq 0\}} \phi_1^j + 1_{\{\psi_1^j \geq 0\}} \psi_1^j \right) P_0^{ask,j} - \sum_{j=1}^N \left(1_{\{\phi_1^j < 0\}} \phi_1^j + 1_{\{\psi_1^j < 0\}} \psi_1^j \right) P_0^{bid,j},$$

and

$$\begin{aligned} \theta_t^0 := & \theta_{t-1}^0 + B_{t-1}^{-1} \left(- \sum_{j=1}^N \Delta(\phi_t^j + \psi_t^j) P_{t-1}^{ask,j} + \sum_{j=1}^N (\phi_{t-1}^j + \psi_{t-1}^j) \Delta A_{t-1}^{ask,j} \right. \\ & + \sum_{j=1}^N 1_{\{\Delta(\phi_t^j + \psi_t^j) < 0\}} \Delta(\phi_t^j + \psi_t^j) (P_{t-1}^{ask,j} - P_{t-1}^{bid,j}) \\ & \left. + \sum_{j=1}^N 1_{\{\phi_{t-1}^j + \psi_{t-1}^j < 0\}} (\phi_{t-1}^j + \psi_{t-1}^j) (\Delta A_{t-1}^{bid,j} - \Delta A_{t-1}^{ask,j}) \right) \end{aligned} \quad (2.10)$$

for $t = 2, 3, \dots, T$. It follows that θ^0 is unique and satisfies

$$\begin{aligned} & B_{t-1} \Delta \theta_t^0 + \sum_{j=1}^N \Delta(\phi_t^j + \psi_t^j) \left(1_{\{\Delta(\phi_t^j + \psi_t^j) \geq 0\}} P_{t-1}^{ask,j} + 1_{\{\Delta(\phi_t^j + \psi_t^j) < 0\}} P_{t-1}^{bid,j} \right) \\ & = \sum_{j=1}^N (\phi_{t-1}^j + \psi_{t-1}^j) \left(1_{\{\phi_{t-1}^j + \psi_{t-1}^j \geq 0\}} \Delta A_{t-1}^{ask,j} + 1_{\{\phi_{t-1}^j + \psi_{t-1}^j < 0\}} \Delta A_{t-1}^{bid,j} \right) \end{aligned}$$

for $t = 2, 3, \dots, T$. By definition, the trading strategy $\theta := (\theta^0, \phi^1 + \psi^1, \dots, \phi^N + \psi^N)$ is self-financing. Subtracting (2.9) from (2.10) yields

$$\begin{aligned} \theta_t^0 - (\phi_t^0 + \psi_t^0) = & \theta_{t-1}^0 - (\phi_{t-1}^0 + \psi_{t-1}^0) \\ & + \sum_{j=1}^N 1_{\{\Delta(\phi_t^j + \psi_t^j) < 0\}} \Delta(\phi_t^j + \psi_t^j) (P_{t-1}^{ask,j} - P_{t-1}^{bid,j}) \\ & - \sum_{j=1}^N \left(1_{\{\Delta \phi_t^j < 0\}} \Delta \phi_t^j + 1_{\{\Delta \psi_t^j < 0\}} \Delta \psi_t^j \right) (P_{t-1}^{ask,j} - P_{t-1}^{bid,j}) \\ & + \sum_{j=1}^N (\phi_{t-1}^j + \psi_{t-1}^j) 1_{\{\phi_{t-1}^j + \psi_{t-1}^j < 0\}} (\Delta A_{t-1}^{bid,j} - \Delta A_{t-1}^{ask,j}) \\ & - \sum_{j=1}^N (\phi_{t-1}^j 1_{\{\phi_{t-1}^j < 0\}} + \psi_{t-1}^j 1_{\{\psi_{t-1}^j < 0\}}) (\Delta A_{t-1}^{bid,j} - \Delta A_{t-1}^{ask,j}) \end{aligned} \quad (2.11)$$

for $t = 2, 3, \dots, T$. By Lemma A.0.6, the inequality

$$1_{\{X < 0\}} X + 1_{\{Y < 0\}} Y \leq 1_{\{X+Y < 0\}} (X + Y)$$

holds for any random variables X and Y . Moreover, the inequalities $P^{bid} \leq P^{ask}$ and $\Delta A^{ask} \leq \Delta A^{bid}$ hold by assumption. Hence, (2.11) reduces to

$$\theta_t^0 - (\phi_t^0 + \psi_t^0) \geq \theta_{t-1}^0 - (\phi_{t-1}^0 + \psi_{t-1}^0) \quad (2.12)$$

for $t = 2, 3, \dots, T$. Since $V_0(\phi) = V_0(\psi) = 0$, we have

$$\begin{aligned} \phi_1^0 &= - \sum_{j=1}^N \phi_1^j \left(1_{\{\phi_1^j \geq 0\}} P_0^{ask,j} + 1_{\{\phi_1^j < 0\}} P_0^{bid,j} \right), \\ \psi_1^0 &= - \sum_{j=1}^N \psi_1^j \left(1_{\{\psi_1^j \geq 0\}} P_0^{ask,j} + 1_{\{\psi_1^j < 0\}} P_0^{bid,j} \right). \end{aligned}$$

It follows that $\theta_1^0 = \phi_1^0 + \psi_1^0$ and $V_0(\theta) = V_0(\phi) + V_0(\psi) = 0$. After recursively solving (2.12), we conclude that $\theta_t^0 \geq \phi_t^0 + \psi_t^0$ for all $t \in \mathcal{T}^*$. \square

The next result is the natural extension of the previous proposition to value processes. It is intuitively true since some transaction costs may be avoided by combining orders.

Proposition 2.2.4. *Let ϕ and ψ be any two self-financing trading strategies such that $V_0(\phi) = V_0(\psi) = 0$. There exists a unique predictable process θ^0 such that the trading strategy defined as $\theta := (\theta^0, \phi^1 + \psi^1, \dots, \phi^N + \psi^N)$ is self-financing with $V_0(\theta) = 0$, and $V_T(\theta)$ satisfies*

$$V_T(\phi) + V_T(\psi) \leq V_T(\theta).$$

Proof. Let ϕ and ψ be self-financing trading strategies. Due to Proposition 2.2.1,

$$\begin{aligned}
V_T(\phi) + V_T(\psi) &= \sum_{u=1}^T (\phi_u^0 + \psi_u^0) \Delta B_u \\
&+ \sum_{j=1}^N \left(\left(1_{\{\phi_T^j \geq 0\}} \phi_T^j + 1_{\{\psi_T^j \geq 0\}} \psi_T^j \right) P_T^{bid,j} + \left(1_{\{\phi_T^j < 0\}} \phi_T^j + 1_{\{\psi_T^j < 0\}} \psi_T^j \right) P_T^{ask,j} \right) \\
&- \sum_{j=1}^N \sum_{u=1}^T \left(1_{\{\Delta \phi_u^j \geq 0\}} \Delta \phi_u^j + 1_{\{\Delta \psi_u^j \geq 0\}} \Delta \psi_u^j \right) P_{u-1}^{ask,j} \\
&- \sum_{j=1}^N \sum_{u=1}^T \left(1_{\{\Delta \phi_u^j < 0\}} \Delta \phi_u^j + 1_{\{\Delta \psi_u^j < 0\}} \Delta \psi_u^j \right) P_{u-1}^{bid,j} \\
&+ \sum_{j=1}^N \sum_{u=1}^T \left(\left(1_{\{\phi_u^j \geq 0\}} \phi_u^j + 1_{\{\psi_u^j \geq 0\}} \psi_u^j \right) \Delta A_u^{ask,j} + \left(1_{\{\phi_u^j < 0\}} \phi_u^j + 1_{\{\psi_u^j < 0\}} \psi_u^j \right) \Delta A_u^{bid,j} \right).
\end{aligned}$$

Using the equality $1_E + 1_{E^c} = 1$ for any set E , we may write

$$\begin{aligned}
V_T(\phi) + V_T(\psi) &= \sum_{u=1}^T (\phi_u^0 + \psi_u^0) \Delta B_u + \sum_{j=1}^N (\phi_T^j + \psi_T^j) (P_T^{bid,j} + P_T^{ask,j}) - C^1 \\
&- \sum_{j=1}^N \sum_{u=1}^T (\Delta \phi_u + \Delta \psi_u) (P_{u-1}^{ask,j} + P_{u-1}^{bid,j}) + \sum_{j=1}^N \sum_{u=1}^T (\phi_u^j + \psi_u^j) (\Delta A_u^{ask,j} + \Delta A_u^{bid,j}),
\end{aligned}$$

where C^1 is defined as

$$\begin{aligned}
C^1 &:= \sum_{j=1}^N \left(\left(1_{\{\phi_T^j \geq 0\}} \phi_T^j + 1_{\{\psi_T^j \geq 0\}} \psi_T^j \right) P_T^{ask,j} + \left(1_{\{\phi_T^j < 0\}} \phi_T^j + 1_{\{\psi_T^j < 0\}} \psi_T^j \right) P_T^{bid,j} \right) \\
&- \sum_{j=1}^N \sum_{u=1}^T \left(1_{\{\Delta \phi_u^j < 0\}} \Delta \phi_u^j + 1_{\{\Delta \psi_u^j < 0\}} \Delta \psi_u^j \right) P_{u-1}^{ask,j} \\
&- \sum_{j=1}^N \sum_{u=1}^T \left(1_{\{\Delta \phi_u^j \geq 0\}} \Delta \phi_u^j + 1_{\{\Delta \psi_u^j \geq 0\}} \Delta \psi_u^j \right) P_{u-1}^{bid,j} \\
&+ \sum_{j=1}^N \sum_{u=1}^T \left(\left(1_{\{\phi_u^j \geq 0\}} \phi_u^j + 1_{\{\psi_u^j \geq 0\}} \psi_u^j \right) \Delta A_u^{bid,j} + \left(1_{\{\phi_u^j < 0\}} \phi_u^j + 1_{\{\psi_u^j < 0\}} \psi_u^j \right) \Delta A_u^{ask,j} \right).
\end{aligned}$$

We use the equality $1_E + 1_{E^c} = 1$ again to arrive at

$$\begin{aligned}
& V_T(\phi) + V_T(\psi) \\
&= \sum_{u=1}^T (\phi_u^0 + \psi_u^0) \Delta B_u + \sum_{j=1}^N (\phi_T^j + \psi_T^j) \left(1_{\{\phi_T^j + \psi_T^j \geq 0\}} P_T^{bid,j} + 1_{\{\phi_T^j + \psi_T^j < 0\}} P_T^{ask,j} \right) \\
&- \sum_{j=1}^N \sum_{u=1}^T (\Delta \phi_u^j + \Delta \psi_u^j) \left(1_{\{\Delta \phi_u^j + \Delta \psi_u^j \geq 0\}} P_{u-1}^{ask,j} + 1_{\{\Delta \phi_u^j + \Delta \psi_u^j < 0\}} P_{u-1}^{bid,j} \right) \\
&+ \sum_{j=1}^N \sum_{u=1}^T (\phi_u^j + \psi_u^j) \left(1_{\{\phi_u^j + \psi_u^j \geq 0\}} \Delta A_u^{ask,j} + 1_{\{\phi_u^j + \psi_u^j < 0\}} \Delta A_u^{bid,j} \right) - C^1 - C^2,
\end{aligned} \tag{2.13}$$

where C^2 is defined as

$$\begin{aligned}
C^2 &:= - \sum_{j=1}^N (\phi_T^j + \psi_T^j) \left(1_{\{\phi_T^j + \psi_T^j < 0\}} P_T^{bid,j} + 1_{\{\phi_T^j + \psi_T^j \geq 0\}} P_T^{ask,j} \right) \\
&+ \sum_{j=1}^N \sum_{u=1}^T (\Delta \phi_u^j + \Delta \psi_u^j) \left(1_{\{\Delta \phi_u^j + \Delta \psi_u^j \geq 0\}} P_{u-1}^{bid,j} + 1_{\{\Delta \phi_u^j + \Delta \psi_u^j < 0\}} P_{u-1}^{ask,j} \right) \\
&- \sum_{j=1}^N \sum_{u=1}^T (\phi_u^j + \psi_u^j) \left(1_{\{\phi_u^j + \psi_u^j \geq 0\}} \Delta A_u^{bid,j} + 1_{\{\phi_u^j + \psi_u^j < 0\}} \Delta A_u^{ask,j} \right).
\end{aligned}$$

By Proposition 2.2.3, there exists a unique predictable process θ^0 such that the trading strategy defined as $\theta := (\theta^0, \phi^1 + \psi^1, \dots, \phi^N + \psi^N)$ is self-financing with $V_0(\theta) = 0$ and satisfies $\phi^0 + \psi^0 \leq \theta^0$. In view of Proposition 2.2.2, since θ is self-financing, it follows that

$$\begin{aligned}
V_T(\theta) &= \sum_{u=1}^T \theta_u^0 \Delta B_u + \sum_{j=1}^N (\phi_T^j + \psi_T^j) \left(1_{\{\phi_T^j + \psi_T^j \geq 0\}} P_T^{bid,j} + 1_{\{\phi_T^j + \psi_T^j < 0\}} P_T^{ask,j} \right) \\
&- \sum_{j=1}^N \sum_{u=1}^T (\Delta \phi_u^j + \Delta \psi_u^j) \left(1_{\{\Delta \phi_u^j + \Delta \psi_u^j \geq 0\}} P_{u-1}^{ask,j} + 1_{\{\Delta \phi_u^j + \Delta \psi_u^j < 0\}} P_{u-1}^{bid,j} \right) \\
&+ \sum_{j=1}^N \sum_{u=1}^T (\phi_u^j + \psi_u^j) \left(1_{\{\phi_u^j + \psi_u^j \geq 0\}} \Delta A_u^{ask,j} + \sum_{u=1}^T 1_{\{\phi_u^j + \psi_u^j < 0\}} \Delta A_u^{bid,j} \right).
\end{aligned} \tag{2.14}$$

Comparing equations (2.13) and (2.14) we see that

$$V_T(\phi) + V_T(\psi) = V_T(\theta) + \sum_{u=1}^T (\phi_u^0 + \psi_u^0 - \theta_u^0) \Delta B_u - C^1 - C^2. \tag{2.15}$$

According to Lemma A.0.6, the random variable $C^1 + C^2$ is nonnegative. Since $\phi^0 + \psi^0 \leq \theta^0$ and $\Delta B \geq 0$, it follows that

$$\sum_{u=1}^T (\phi_u^0 + \psi_u^0 - \theta_u^0) \Delta B_u \leq 0.$$

From (2.15), we conclude that $V_T(\phi) + V_T(\psi) \leq V_T(\theta)$. \square

The following technical lemma, which will be used in the next section, follows directly from Proposition 2.2.1 and Lemma A.0.2.

Lemma 2.2.1. *If a sequence of self-financing trading strategies ϕ^m converges a.s. to ϕ , then $V(\phi^m)$ converges a.s. to $V(\phi)$.*

2.2.1 Set of Values that can be Superhedged at Zero Cost. For all $t \in \mathcal{T}$, denote by $L^0(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R}^{(N+1)})$ the space of all (\mathbb{P} -equivalence classes of) $\mathbb{R}^{(N+1)}$ -valued, \mathcal{F}_t -measurable random variables. We equip $L^0(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R})$ with the topology of convergence in measure \mathbb{P} . Also, we denote by \mathcal{S} the set of all self-financing trading strategies. For the sake of conciseness, we will refer to sets that are closed with respect to convergence in measure \mathbb{P} simply as \mathbb{P} -closed.

We define the sets

$$\mathcal{K} := \{V_T^*(\phi) : \phi \in \mathcal{S}, V_0(\phi) = 0\},$$

$$L_+^0(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R}) := \{X \in L^0(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R}) : X \geq 0\},$$

$$\mathcal{K} - L_+^0(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R}) := \{Y - X : Y \in \mathcal{K} \text{ and } X \in L_+^0(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R})\}.$$

The set \mathcal{K} is *the set of attainable values at zero cost*, and has the interpretation of all possible terminal values associated with zero cost self-financing trading strategies. On the other hand, $\mathcal{K} - L_+^0(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R})$ is *the set of values that can be superhedged at zero cost*: for any $X \in \mathcal{K} - L_+^0(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R})$, there exists $\phi \in \mathcal{S}$ with $V_0(\phi) = 0$ so that $X \leq V_T(\phi)$.

The following lemma is needed to apply the Kreps-Yan Theorem (Theorem A.0.8), which will be used to prove the FFTAP.

Lemma 2.2.2. *The set $\mathcal{K} - L_+^0(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R})$ is a convex cone.*

Proof. Let $Y^1, Y^2 \in \mathcal{K} - L_+^0(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R})$. Then there exist $K^1, K^2 \in \mathcal{K}$ and $Z^1, Z^2 \in L_+^0(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R})$ such that $Y^1 = K^1 - Z^1$ and $Y^2 = K^2 - Z^2$. By definition of \mathcal{K} , there exist $\phi^1, \phi^2 \in \mathcal{S}$ with $V_0(\phi^1) = V_0(\phi^2) = 0$ such that $K^1 = V_T^*(\phi^1)$ and $K^2 = V_T^*(\phi^2)$. We will prove that for any positive scalars α_1 and α_2 the following holds

$$\alpha_1(V_T^*(\phi^1) - Z^1) + \alpha_2(V_T^*(\phi^2) - Z^2) \in \mathcal{K} - L_+^0(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R}),$$

or, equivalently, that there exists $K \in \mathcal{K}$ such that

$$\alpha_1 V_T^*(\phi^1) + \alpha_2 V_T^*(\phi^2) - \alpha_1 Z^1 - \alpha_2 Z^2 \leq K.$$

The value process is positive homogeneous, so $\alpha_1 V_T^*(\phi^1) + \alpha_2 V_T^*(\phi^2) = V_T^*(\alpha_1 \phi^1) + V_T^*(\alpha_2 \phi^2)$. According to Proposition 2.2.4, there exists a unique predictable process θ^0 such that the trading strategy defined as $\theta := (\theta^0, \alpha_1 \phi^{1,1} + \alpha_2 \phi^{2,1}, \dots, \alpha_1 \phi^{1,N} + \alpha_2 \phi^{2,N})$ is self-financing with $V_0(\theta) = 0$, and satisfies $V_T^*(\alpha_1 \phi^1) + V_T^*(\alpha_2 \phi^2) \leq V_T^*(\theta)$. By definition of \mathcal{K} , we have $V_T^*(\theta) \in \mathcal{K}$. Since

$$\begin{aligned} \alpha_1 V_T^*(\phi^1) + \alpha_2 V_T^*(\phi^2) - \alpha_1 Z^1 - \alpha_2 Z^2 &= V_T^*(\alpha_1 \phi^1) + V_T^*(\alpha_2 \phi^2) - \alpha_1 Z^1 - \alpha_2 Z^2 \\ &\leq V_T^*(\theta) - \alpha_1 Z^1 - \alpha_2 Z^2 \\ &\leq V_T^*(\theta), \end{aligned}$$

we may conclude that the claim holds. \square

Remark 2.2.4. *The set \mathcal{K} is not necessarily a convex cone. To see this, let us suppose that $T = 1$, $\mathcal{J} = \{0, 1\}$, and $r = 0$. Consider the trading strategies $\phi = \{\phi^0, 1\}$ and $\psi = \{\psi^0, -1\}$, where ϕ^0 and ψ^0 are chosen so that $V_0(\phi) = V_0(\psi) = 0$. By definition of \mathcal{K} , we have that $V_1(\phi), V_1(\psi) \in \mathcal{K}$. However, the random variable*

$$V_1^*(\phi) + V_1^*(\psi) = P_1^{bid} - P_1^{ask} + A_1^{ask} - A_1^{bid} + P_0^{bid} - P_0^{ask}$$

is generally not in the set \mathcal{K} . In frictionless markets, \mathcal{K} is a linear space since the value process is linear.

2.3 The No-Arbitrage Condition

We begin by introducing the definition of the no-arbitrage condition in our context.

Definition 2.3.1. *The no-arbitrage condition (NA) is satisfied if for each $\phi \in \mathcal{S}$ such that $V_0(\phi) = 0$ and $V_T(\phi) \geq 0$, we have $V_T(\phi) = 0$.*

In the present context, **NA** has the usual interpretation that “it is impossible to make something out of nothing.” The no-arbitrage condition can also be interpreted as: “if the zero payoff can be superhedged with a zero-cost, self-financing trading strategy, then this trading strategy must have zero terminal value.” Next, we provide equivalent conditions for **NA** to hold, which are more mathematically convenient.

Lemma 2.3.1. *The following conditions are equivalent:*

(i) **NA** is satisfied.

(ii) $(\mathcal{K} - L_+^0(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R})) \cap L_+^0(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R}) = \{0\}$.

(iii) $\mathcal{K} \cap L_+^0(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R}) = \{0\}$.

Proof. (ii) \implies (iii) If $(\mathcal{K} - L_+^0(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R})) \cap L_+^0(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R}) = \{0\}$, then

$$\{0\} \subset \mathcal{K} \cap L_+^0(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R}) \subset (\mathcal{K} - L_+^0(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R})) \cap L_+^0(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R}) = \{0\},$$

since $0 \in \mathcal{K} \cap L_+^0(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R})$. Therefore (iii) holds.

(iii) \implies (ii) Assume that $X \in \mathcal{K} \cap L_+^0(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R}) = \{0\}$ and $X \in (\mathcal{K} - L_+^0(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R})) \cap L_+^0(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R})$. Then there exists $Z \in L_+^0(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R})$ and $\phi \in \mathcal{S}$ with $V_0(\phi) = 0$ such that

$$X = V_T^*(\phi) - Z \geq 0.$$

It follows that $V_T^*(\phi) \geq 0$. However, since $\mathcal{K} \cap L_+^0(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R}) = \{0\}$, we have that $V_T^*(\phi) = 0$. Thus, $X \geq 0$. Since $\mathcal{K} \cap L_+^0(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R}) = \{0\}$, we obtain $X = 0$.

(i) \implies (iii) Suppose that $\mathcal{K} \cap L_+^0(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R}) \neq \{0\}$. This implies that there exists $X \in \mathcal{K}$ and $E \subseteq \Omega$ with $\mathbb{P}(E) > 0$ such that $X(\omega) \geq 0$ for $\omega \in \Omega$, and $X(\omega) > 0$ for $\omega \in E$. By the definition of \mathcal{K} , there exists $\phi \in \mathcal{S}$ with $V_0(\phi) = 0$ such that $X = V_T^*(\phi)$. This contradicts **NA**, so $\mathcal{K} \cap L_+^0(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R}) = \{0\}$.

(iii) \implies (i) Let $\phi \in \mathcal{S}$ so that $V_0(\phi) = 0$ and $V_T^*(\phi) \geq 0$, and assume that $\mathcal{K} \cap L_+^0(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R}) = \{0\}$. Since $V_T^*(\phi) \in \mathcal{K}$, we have that $V_T(\phi) = 0$. Hence, **NA** holds. \square

We proceed by defining The Efficient Friction Assumption in our context (cf. Kabanov et al. [KRS02]).

The Efficient Friction Assumption (EF):

$$\{\phi \in \mathcal{S} : V_0(\phi) = V_T(\phi) = 0\} = \{0\}. \quad (2.16)$$

Note that if (2.16) is satisfied, then for each $\phi \in \mathcal{S}$, we have $V_0(\phi) = V_T(\phi) = 0$ if and only if $\phi = 0$. Consequently, if **NA** holds under **EF**, then for each $\phi \in \mathcal{S}$ such that $V_0(\phi) = 0$ and $V_T(\phi) \geq 0$, we have $\phi = 0$.

Next, we exemplify the importance of **EF** in the context of markets with transaction costs.

Example 2.3.1. Let $\mathcal{T} = \{0, 1\}$ and $\mathcal{J} = \{0, 1\}$, and assume that $P_t^{bid,1} < P_t^{ask,1}$, for $t = 0, 1$, and $A_1^{ask,1} < A_1^{bid,1}$. Suppose that there exists a trading strategy ϕ that is nonzero a.s. so that $V_0(\phi) = V_1(\phi) = 0$. By the definition of the value process,

$$\phi_1^0 + 1_{\{\phi_1^1 \geq 0\}} \phi_1^1 P_0^{ask,1} - 1_{\{\phi_1^1 < 0\}} \phi_1^j P_0^{bid,1} = 0,$$

and

$$\begin{aligned} V_1^*(\phi) &= 1_{\{\phi_1^1 \geq 0\}} \phi_1^1 \left(P_1^{bid,1} + A_1^{ask,1} - P_0^{ask,1} \right) \\ &\quad + 1_{\{\phi_1^1 < 0\}} \phi_1^1 \left(P_1^{ask,1} + A_1^{bid,1} - P_0^{bid,1} \right) = 0. \end{aligned}$$

Suppose that there exists another market $(B, \tilde{P}^{bid,1}, \tilde{P}^{ask,1}, \tilde{A}^{ask,1}, \tilde{A}^{bid,1})$ satisfying $P_1^{bid,1} < \tilde{P}_1^{bid,1} \leq \tilde{P}_1^{ask,1} < P_1^{ask,1}$, and $\tilde{A}_1^{ask,1} = A_1^{ask,1}$, $\tilde{A}_1^{bid,1} = A_1^{bid,1}$, $\tilde{P}_0^{bid,1} = P_0^{bid,1}$, and $\tilde{P}_0^{ask,1} = P_0^{ask,1}$. If we denote by $\tilde{V}(\phi)$ the value process corresponding to the market $(B, \tilde{P}^{bid}, \tilde{P}^{ask}, \tilde{A}^{ask}, \tilde{A}^{bid})$, we then have $\tilde{V}_0(\phi) = V_0(\phi) = 0$, $\tilde{V}_1(\phi) \geq 0$, and $\mathbb{P}(\tilde{V}_1(\phi) > 0) > 0$, which violates **NA** for the market $(B, \tilde{P}^{bid}, \tilde{P}^{ask}, \tilde{A}^{ask}, \tilde{A}^{bid})$.

We will denote by **NAEF** the no-arbitrage condition under the efficient friction assumption.

In what follows, we denote by \mathcal{P} the set of all \mathbb{R}^N -valued, \mathbb{F} -predictable processes. Also, we define the mapping

$$\begin{aligned} F(\phi) &:= \sum_{j=1}^N \phi_T^j \left(1_{\{\phi_T^j \geq 0\}} P_T^{bid,j,*} + 1_{\{\phi_T^j < 0\}} P_T^{ask,j,*} \right) \\ &\quad - \sum_{j=1}^N \sum_{u=1}^T \Delta \phi_u^j \left(1_{\{\Delta \phi_u^j \geq 0\}} P_{u-1}^{ask,j,*} + 1_{\{\Delta \phi_u^j < 0\}} P_{u-1}^{bid,j,*} \right) \\ &\quad + \sum_{j=1}^N \sum_{u=1}^T \phi_u^j \left(1_{\{\phi_u^j \geq 0\}} A_u^{ask,j,*} + 1_{\{\phi_u^j < 0\}} A_u^{bid,j,*} \right) \end{aligned} \tag{2.17}$$

for all \mathbb{R}^N -valued stochastic processes

$$(\phi_s)_{s=1}^T \in L^0(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R}^N) \times \cdots \times L^0(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R}^N),$$

and let $\mathbb{K} := \{F(\phi) : \phi \in \mathcal{P}\}$.

Remark 2.3.1.

- (i) Note that F is defined on the set of all \mathbb{R}^N -valued stochastic processes. On the contrary, the value process is defined on the set of trading strategies, which are \mathbb{R}^{N+1} -valued predictable processes.
- (ii) The set \mathbb{K} has the same financial interpretation as the set \mathcal{K} . We introduce the set \mathbb{K} because it is more convenient to work with from the mathematical point of view.
- (iii) $F(\alpha\phi) = \alpha F(\phi)$ for any nonnegative random variable α .

The next result provides an equivalent condition for **EF** to hold.

Lemma 2.3.2. *The efficient friction assumption (**EF**) is satisfied if and only if $\{\psi \in \mathcal{P} : F(\psi) = 0\} = \{0\}$.*

Proof. Let $\psi \in \mathcal{P}$ such that $F(\psi) = 0$. Let us define the trading strategy $\phi := (\phi^0, \psi^1, \dots, \psi^N)$, where ϕ^0 is recursively defined as

$$\phi_t^0 := \begin{cases} -\sum_{j=1}^N \psi_1^j (1_{\{\psi_1^j \geq 0\}} P_0^{ask,j} + 1_{\{\psi_1^j < 0\}} P_0^{bid,j}), & t = 1, \\ \phi_{t-1}^0 + B_{t-1}^{-1} \left(-\sum_{j=1}^N \Delta\psi_t^j (1_{\{\Delta\psi_t^j \geq 0\}} P_{t-1}^{ask,j} + 1_{\{\Delta\psi_t^j < 0\}} P_{t-1}^{bid,j}) \right. \\ \quad \left. + \sum_{j=1}^N \psi_{t-1}^j (1_{\{\psi_{t-1}^j \geq 0\}} \Delta A_{t-1}^{ask,j} + 1_{\{\psi_{t-1}^j < 0\}} \Delta A_{t-1}^{bid,j}) \right), & t = 2, \dots, T. \end{cases}$$

By definition, $\phi \in \mathcal{S}$ and $V_0(\phi) = 0$. According to Proposition 2.2.2,

$$\begin{aligned} V_T^*(\phi) &= \sum_{j=1}^N \psi_T^j (1_{\{\psi_T^j \geq 0\}} P_T^{bid,j,*} + 1_{\{\psi_T^j < 0\}} P_T^{ask,j,*}) \\ &\quad - \sum_{j=1}^N \sum_{u=1}^T \Delta\psi_u^j (1_{\{\Delta\psi_u^j \geq 0\}} P_{u-1}^{ask,j,*} + 1_{\{\Delta\psi_u^j < 0\}} P_{u-1}^{bid,j,*}) \\ &\quad + \sum_{j=1}^N \sum_{u=1}^T \psi_u^j (1_{\{\psi_u^j \geq 0\}} A_u^{ask,j,*} + 1_{\{\psi_u^j < 0\}} A_u^{bid,j,*}), \end{aligned}$$

so $V_T^*(\phi) = F(\psi)$. Thus, $V_T^*(\phi) = 0$. **EF** is satisfied, so $\phi^j = 0$ for $j = 0, \dots, N$, which in particular implies that $\psi^j = 0$ for $j = 1, \dots, N$.

Conversely, suppose **EF** holds, and fix $\phi \in \mathcal{S}$ so that $V_0(\phi) = V_T^*(\phi) = 0$. By Proposition 2.2.2, we have

$$\begin{aligned} V_T^*(\phi) &= \sum_{j=1}^N \phi_T^j (1_{\{\phi_T^j \geq 0\}} P_T^{bid,j,*} + 1_{\{\phi_T^j < 0\}} P_T^{ask,j,*}) \\ &\quad - \sum_{j=1}^N \sum_{u=1}^T \Delta \phi_u^j (1_{\{\Delta \phi_u^j \geq 0\}} P_{u-1}^{ask,j,*} + 1_{\{\Delta \phi_u^j < 0\}} P_{u-1}^{bid,j,*}) \\ &\quad + \sum_{j=1}^N \sum_{u=1}^T \phi_u^j (1_{\{\phi_u^j \geq 0\}} A_u^{ask,j,*} + 1_{\{\phi_u^j < 0\}} A_u^{bid,j,*}). \end{aligned}$$

Define the predictable process $\psi^j := \phi^j$ for $j = 1, \dots, N$. From the equation above we see that $F(\psi) = V_T^*(\phi)$, so $F(\psi) = 0$. By assumption, we have that $\psi^j = 0$ for $j = 1, \dots, N$, which implies $\phi^j = 0$ for $j = 1, \dots, N$. From the definition of $V_0(\phi)$,

$$V_0(\phi) = \phi_1^0 + \sum_{j=1}^N \phi_1^j (1_{\{\phi_1^j \geq 0\}} P_0^{ask,j} + 1_{\{\phi_1^j < 0\}} P_0^{bid,j}).$$

It follows that $\phi_1^0 = 0$ because $V_0(\phi) = 0$. Since $\phi \in \mathcal{S}$,

$$\begin{aligned} \phi_t^0 &= \phi_{t-1}^0 + B_{t-1}^{-1} \left(- \sum_{j=1}^N \Delta \phi_t^j (1_{\{\Delta \phi_t^j \geq 0\}} P_{t-1}^{ask,j} + 1_{\{\Delta \phi_t^j < 0\}} P_{t-1}^{bid,j}) \right. \\ &\quad \left. + \sum_{j=1}^N \phi_{t-1}^j (1_{\{\phi_{t-1}^j \geq 0\}} \Delta A_{t-1}^{ask,j} + 1_{\{\phi_{t-1}^j < 0\}} \Delta A_{t-1}^{bid,j}) \right), \quad t = 2, \dots, T. \end{aligned}$$

By recursively solving for $\phi_2^0, \dots, \phi_T^0$, we deduce $\phi_t^0 = 0$ for $t = 2, \dots, T$. Hence, $\phi^j = 0$ for $j = 0, 1, \dots, N$. \square

Lemma 2.3.3. *We have that $\mathcal{K} = \mathbb{K}$.*

Proof. Suppose $K \in \mathcal{K}$. Then $K = V_T^*(\phi)$ for some $\phi \in \mathcal{S}$ with $V_0(\phi) = 0$. According

to Proposition 2.2.2,

$$\begin{aligned}
K &= \sum_{j=1}^N \phi_T^j (1_{\{\phi_T^j \geq 0\}} P_T^{bid,j,*} + 1_{\{\phi_T^j < 0\}} P_T^{ask,j,*}) \\
&\quad - \sum_{j=1}^N \sum_{u=1}^T \Delta \phi_u^j (1_{\{\Delta \phi_u^j \geq 0\}} P_{u-1}^{ask,j,*} + 1_{\{\Delta \phi_u^j < 0\}} P_{u-1}^{bid,j,*}) \\
&\quad + \sum_{j=1}^N \sum_{u=1}^T \phi_u^j (1_{\{\phi_u^j \geq 0\}} A_u^{ask,j,*} + 1_{\{\phi_u^j < 0\}} A_u^{bid,j,*}).
\end{aligned}$$

Define $\psi^j := \phi^j$ for $j = 1, \dots, N$, we obtain $K = F(\psi) \in \mathbb{K}$.

If $W \in \mathbb{K}$, then there exists $\psi \in \mathcal{P}$ so that $W = F(\psi)$. Let $\phi^j := \psi^j$ for $j = 1, \dots, N$, and define ϕ^0 so that $\phi \in \mathcal{S}$ and $V_0(\phi) = 0$ (as in the proof of Lemma 2.3.2). Then, from Proposition 2.2.2,

$$\begin{aligned}
V_T^*(\phi) &= \sum_{j=1}^N \psi_T^j (1_{\{\psi_T^j \geq 0\}} P_T^{bid,j,*} + 1_{\{\psi_T^j < 0\}} P_T^{ask,j,*}) \\
&\quad - \sum_{j=1}^N \sum_{u=1}^T \Delta \psi_u^j (1_{\{\Delta \psi_u^j \geq 0\}} P_{u-1}^{ask,j,*} + 1_{\{\Delta \psi_u^j < 0\}} P_{u-1}^{bid,j,*}) \\
&\quad + \sum_{j=1}^N \sum_{u=1}^T \psi_u^j (1_{\{\psi_u^j \geq 0\}} A_u^{ask,j,*} + 1_{\{\psi_u^j < 0\}} A_u^{bid,j,*}),
\end{aligned}$$

which gives us $W = V_T^*(\phi) \in \mathcal{K}$. □

2.3.1 A Key Closedness Property of the Set of Attainable Values at Zero

Cost. In this section, we prove that $\mathcal{K} - L_+^0(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R})$ is \mathbb{P} -closed whenever **NAEF** is satisfied. This plays a central role in this study because it allows us to apply the Kreps-Yan theorem, which essentially implies the FFTAP (Theorem 2.4.1).

We will denote by $\|\cdot\|$ the Euclidean norm on \mathbb{R}^N .

Let us first recall the following lemma from Schachermayer [Sch04], which is a version of Lemma 2 in Kabanov and Stricker [KS01b].

Lemma 2.3.4. *For a sequence of random variables $X^m \in L^0(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^N)$ there is a strictly increasing sequence of positive, integer-valued, \mathcal{F} -measurable random variables*

τ^m such that X^{τ^m} converges a.s. in the one-point-compactification $\mathbb{R}^N \cup \{\infty\}$ to some random variable $X \in L^0(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^N \cup \{\infty\})$. Moreover, we may find the subsequence such that $\|X\| = \limsup_m \|X^m\|$, where $\|\infty\| = \infty$.

The next result extends Lemma 2.3.4 to processes, which will be a key ingredient for proving Theorem 2.3.1.

Lemma 2.3.5. *Let \mathcal{F}^i be a σ -algebra, and $Y_i^m \in L^0(\Omega, \mathcal{F}^i, \mathbb{P}; \mathbb{R}^N)$ for $i = 1, \dots, M$. Suppose that $\mathcal{F}^i \subseteq \mathcal{F}^j$ for all $i \leq j$, and that Y_i^m satisfies $\limsup_m \|Y_i^m\| < \infty$ for $i = 1, \dots, M$. Then there is a strictly increasing sequence of positive, integer-valued, \mathcal{F}^M -measurable random variables τ^m such that, for $i = 1, \dots, M$, the sequence $Y_i^{\tau^m}$ converges a.s. to some $Y_i \in L^0(\Omega, \mathcal{F}^i, \mathbb{P}; \mathbb{R}^N)$.*

Proof. We first apply Lemma 2.3.4 to the random variable Y_1^m : there exists a strictly increasing sequence of positive, integer-valued, \mathcal{F}^1 -measurable random variables τ_1^m such that $\{\tau_1^1(\omega), \tau_1^2(\omega), \dots\} \subseteq \mathbb{N}$ for $\omega \in \Omega$, and $Y_1^{\tau_1^m}$ converges a.s. to some $Y_1 \in L^0(\Omega, \mathcal{F}^1, \mathbb{P}; \mathbb{R}^N)$. Since $\limsup_m \|Y_2^m\| < \infty$, we also have that $\limsup_m \|Y_2^{\tau_1^m}\| < \infty$. Moreover, $Y_2^{\tau_1^m} \in L^0(\Omega, \mathcal{F}^2, \mathbb{P}; \mathbb{R}^N)$ since $\mathcal{F}^1 \subseteq \mathcal{F}^2$. Therefore, we may apply Lemma 2.3.4 to the sequence $Y_2^{\tau_1^m}$ to find a strictly increasing sequence of positive, integer-valued, \mathcal{F}^2 -measurable random variables τ_2^m such that

$$\{\tau_2^1(\omega), \tau_2^2(\omega), \dots\} \subseteq \{\tau_1^1(\omega), \tau_1^2(\omega), \dots\} \subseteq \mathbb{N}, \quad \text{a.e. } \omega \in \Omega, \quad (2.18)$$

and $Y_2^{\tau_2^m}$ converges a.s. to some $Y_2 \in L^0(\Omega, \mathcal{F}^2, \mathbb{P}; \mathbb{R}^N)$. From (2.18), the sequence $Y_1^{\tau_2^m}$ converges a.s. to Y_1 .

We may continue by recursively repeating the argument above to the sequences Y_i^m , for $i = 3, \dots, M$, to find strictly increasing sequences of positive, integer-valued, \mathcal{F}^i -measurable random variables τ_i^m such that

$$\{\tau_i^1(\omega), \tau_i^2(\omega), \dots\} \subseteq \dots \subseteq \{\tau_1^1(\omega), \tau_1^2(\omega), \dots\} \subseteq \mathbb{N}, \quad \text{a.e. } \omega \in \Omega, \quad (2.19)$$

and $Y_i^{\tau_i^m}$ converges a.s. to some $Y_i \in L^0(\Omega, \mathcal{F}^i, \mathbb{P}; \mathbb{R}^N)$. Because of (2.19), we see that $Y_i^{\tau_M^m}$ converges a.s. to Y_i for $i = 1, \dots, M$. Therefore, $\tau^m := \tau_M^m$ defines the desired sequence. \square

Lemma 2.3.6. *Let \mathcal{F}^i be a σ -algebra, and $Y_i^m \in L^0(\Omega, \mathcal{F}^i, \mathbb{P}; \mathbb{R}^N)$ for $i = 1, \dots, M$. Suppose that $\mathcal{F}^i \subseteq \mathcal{F}^j$ for all $i \leq j$, and that there exists $k \in \{1, \dots, M\}$ and $\Omega' \subseteq \Omega$ with $\mathbb{P}(\Omega') > 0$ such that $\limsup_m \|Y_k^m(\omega)\| = \infty$ for a.e. $\omega \in \Omega'$, and $\limsup_m \|Y_i^m(\omega)\| < \infty$ for $i = 1, \dots, k-1$ and for a.e. $\omega \in \Omega$. Then there exists a strictly increasing sequence of positive, integer-valued, \mathcal{F}^k -measurable random variables τ^m such that $\lim_m \|Y_k^{\tau^m}(\omega)\| = \infty$, for a.e. $\omega \in \Omega'$, and ⁷*

$$X_i^m(\omega) := 1_{\Omega'}(\omega) \frac{Y_i^{\tau^m(\omega)}(\omega)}{\|Y_k^{\tau^m(\omega)}(\omega)\|}, \quad \omega \in \Omega, \quad i = 1, \dots, M,$$

satisfies $\lim_m X_i^m(\omega) = 0$, for $i = 1, \dots, k-1$ and for a.e. $\omega \in \Omega$

Proof. Since $\limsup_m \|Y_k^m(\omega)\| = \infty$ for a.e. $\omega \in \Omega'$, we may apply Lemma 2.3.4 to the sequence Y_k^m to find a strictly increasing sequence of positive, integer-valued, \mathcal{F}^k -measurable random variables τ^m so that $\|Y_k^{\tau^m(\omega)}(\omega)\|$ diverges for a.e. $\omega \in \Omega'$.

Because $\limsup_m \|Y_i^m\| < \infty$ for $i = 1, \dots, k-1$, we have $\limsup_m \|Y_i^{\tau^m}\| < \infty$ for $i = 1, \dots, k-1$. Now since $\|Y_k^{\tau^m(\omega)}(\omega)\|$ diverges for a.e. $\omega \in \Omega'$,

$$\lim_{m \rightarrow \infty} \|X_i^m(\omega)\| = 1_{\Omega'}(\omega) \lim_{m \rightarrow \infty} \frac{\|Y_i^{\tau^m(\omega)}(\omega)\|}{\|Y_k^{\tau^m(\omega)}(\omega)\|} = 0, \quad \text{a.e. } \omega \in \Omega, \quad i = 1, \dots, k-1.$$

Thus, $\|X_i^m\|$ converges a.s. to 0 for $i = 1, \dots, k-1$, which implies that X_i^m converges a.s. to 0 for $i = 1, \dots, k-1$. Hence, the claim holds. \square

We are now ready to prove a crucial result in this chapter.

⁷We take $X_i^m(\omega) = 0$ whenever $\|Y_k^{\tau^m(\omega)}(\omega)\| = 0$. We will take the convention $x/0 = 0$ throughout this section.

Theorem 2.3.1. *If the no-arbitrage condition under the efficient friction assumption (NAEF) is satisfied, then the set $\mathcal{K} - L_+^0(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R})$ is \mathbb{P} -closed.*

Proof. According to Lemma 2.3.3, we may equivalently prove that $\mathbb{K} - L_+^0(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R})$ is \mathbb{P} -closed. Suppose that $X^m \in \mathbb{K} - L_+^0(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R})$ converges in probability to X . Then there exists a subsequence X^{k_m} of X^k so that X^{k_m} converges a.s. to X . With a slight abuse of notation, we will denote by X^m the sequence X^{k_m} in what follows. By the definition of $\mathbb{K} - L_+^0(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R})$, there exists $Z^m \in L_+^0(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R})$ and $\phi^m \in \mathcal{P}$ so that

$$X^m = F(\phi^m) - Z^m. \quad (2.20)$$

We proceed the proof in two steps. In the first step, we show by contradiction that $\limsup_m \|\phi_s^m\| < \infty$ for all $s \in \mathcal{T}^*$.

Step 1a: Let us assume that $\limsup_m \|\phi_s^m\| < \infty$ for all $s \in \mathcal{T}^*$ does not hold. Then

$$\mathcal{I}^0 := \left\{ s \in \mathcal{T}^* : \exists \Omega' \subseteq \Omega \text{ s.t. } \mathbb{P}(\Omega') > 0, \limsup_{m \rightarrow \infty} \|\phi_s^m(\omega)\| = \infty \text{ for a.e. } \omega \in \Omega' \right\}$$

is nonempty. Let $t_0 := \min \mathcal{I}^0$, and define the \mathcal{F}_{t_0-1} -measurable set

$$E^0 := \left\{ \omega \in \Omega : \limsup_{m \rightarrow \infty} \|\phi_{t_0}^m(\omega)\| = \infty \right\}.$$

Note that $\mathbb{P}(E^0) > 0$ by assumption. We now apply Lemma 2.3.6 to ϕ^m : there exists a strictly increasing sequence of positive, integer-valued, \mathcal{F}_{t_0-1} -measurable random variables τ_0^m such that

$$\lim_{m \rightarrow \infty} \|\phi_{t_0}^{\tau_0^m(\omega)}(\omega)\| = \infty, \quad \text{a.e. } \omega \in E^0, \quad (2.21)$$

and

$$\psi_s^{m,(0)} := 1_{E^0} \frac{\phi_s^{\tau_0^m}}{\|\phi_{t_0}^{\tau_0^m}\|}, \quad s \in \mathcal{T}^*, \quad (2.22)$$

satisfies $\lim_m \psi_s^{m,(0)}(\omega) = 0$, for $s = 1, \dots, t_0 - 1$, for a.e. $\omega \in \Omega$.

If $\limsup_m \|\psi_s^{m,(0)}\| < \infty$ for all $s \in \{t_0 + 1, \dots, T\}$, then define $k := 1$ and the sequence $\varphi^m := \psi^{m,(0)}$. Otherwise we repeat the procedure above to $\psi^{m,(0)}$ on the set E^0 . Towards this, we define

$$t_1 := \min \left\{ s \in \{t_0 + 1, \dots, T\} : \exists \Omega' \subseteq E^0 \text{ such that } \mathbb{P}(\Omega') > 0, \right. \\ \left. \limsup_{m \rightarrow \infty} \|\psi_s^{m,(0)}(\omega)\| = \infty \text{ for a.e. } \omega \in \Omega' \right\},$$

and define the \mathcal{F}_{t_1-1} -measurable set

$$E^1 := \left\{ \omega \in E^0 : \limsup_{m \rightarrow \infty} \|\psi_{t_1}^{m,(0)}(\omega)\| = \infty \right\}.$$

Next, we apply Lemma 2.3.6 to $\psi^{m,(0)}$ to find a strictly increasing sequence of positive, integer-valued, \mathcal{F}_{t_1-1} -measurable random variables τ_1^m such that

$$\lim_{m \rightarrow \infty} \|\psi_{\tau_1^m}^{\tau_1^m(\omega), (0)}(\omega)\| = \infty, \quad \text{a.e. } \omega \in E^1, \quad (2.23)$$

and

$$\psi_s^{m,(1)} := 1_{E^1} \frac{\psi_s^{\tau_1^m, (0)}}{\|\psi_{\tau_1^m}^{\tau_1^m, (0)}\|}, \quad s \in \mathcal{T}^*, \quad (2.24)$$

satisfies $\lim_m \psi_s^{m,(1)}(\omega) = 0$, for $s = 1, \dots, t_1 - 1$, for a.e. $\omega \in \Omega$. Observe that the sequence $\psi^{m,(1)}$ satisfies

$$\psi_s^{m,(1)} = 1_{E^0 \cap E^1} \frac{\phi_s^{\tau_0 \circ \tau_1^m}}{\|\psi_{\tau_1^m}^{\tau_1^m, (0)}\| \cdot \|\phi_{\tau_0}^{\tau_0 \circ \tau_1^m}\|}, \quad s \in \mathcal{T}^*.$$

where we denote by $\tau_0 \circ \tau_1^m$ the composition $\tau_0^{\tau_1^m}$.

As above, we proceed as follows.

Recursively for $i = 2, \dots, T$

If $\limsup_m \|\psi_s^{m,(i-1)}\| < \infty$ for all $s \in \{t_{i-1} + 1, \dots, T\}$, then define $k := i$ and $\varphi^m := \psi^{m,(k-1)}$, and proceed to Step 1b.

Else, define

$$t_i := \min \left\{ s \in \{t_{i-1} + 1, \dots, T\} : \exists \Omega' \subseteq E^{i-1} \text{ s.t. } \mathbb{P}(\Omega') > 0, \right. \\ \left. \limsup_{m \rightarrow \infty} \|\psi_s^{m, (i-1)}(\omega)\| = \infty \text{ for a.e. } \omega \in \Omega' \right\},$$

and

$$E^i := \{\omega \in E^{i-1} : \limsup_{m \rightarrow \infty} \|\psi_{t_i}^{m, (i-1)}(\omega)\| = \infty\}.$$

Next, apply Lemma 2.3.6 to $\psi^{m, (i)}$: there exists a strictly increasing sequence of positive, integer-valued, \mathcal{F}_{t_i-1} -measurable random variables τ_i^m such that

$$\{\tau_i^1(\omega), \tau_i^2(\omega), \dots\} \subseteq \dots \subseteq \{\tau_0^1(\omega), \tau_0^2(\omega), \dots\}, \quad \text{a.e. } \omega \in \Omega, \quad (2.25)$$

the sequence $\psi_{t_i}^{\tau_i^m, (i-1)}$ satisfies

$$\lim_{m \rightarrow \infty} \|\psi_{t_i}^{\tau_i^m(\omega), (i-1)}(\omega)\| = \infty, \quad \text{a.e. } \omega \in E^i, \quad (2.26)$$

and the sequence $\psi^{m, (i)}$ defined as

$$\psi_s^{m, (i)} := 1_{E^i} \frac{\psi_s^{\tau_i^m, (i-1)}}{\|\psi_{t_i}^{\tau_i^m, (i-1)}\|}, \quad s \in \mathcal{T}^*, \quad (2.27)$$

satisfies $\lim_m \psi_s^{m, (i)}(\omega) = 0$ for $s = 1, \dots, t_i - 1$, for a.e. $\omega \in \Omega$.

Repeat: $i \rightarrow i + 1$.

Given this construction, we define

$$\beta_i^m(\omega) := \tau_i \circ \tau_{i+1} \circ \dots \circ \tau_k^m(\omega), \quad i \in \{0, \dots, k\}, \omega \in \Omega, \\ U^m(\omega) := \|\phi_{t_0}^{\beta_0^m(\omega)}(\omega)\| \prod_{i=1}^k \|\psi_{t_i}^{\beta_i^m(\omega), (i-1)}(\omega)\|, \quad \omega \in \Omega.$$

We make the following observations on this construction:

- (i) The construction always produces a sequence φ^m such that $\limsup_m \|\varphi_s^m\| < \infty$ for all $s \in \mathcal{T}^*$. Indeed, if $t_i = T$ for some $i = 1, \dots, T$, then $\lim_m \psi_s^{m, (i)}(\omega) = 0$

for $s = 1, \dots, T - 1$, for a.e. $\omega \in \Omega$, and $\lim_m \|\psi_T^{m,(i)}(\omega)\| = 1_{E^i}(\omega)$, for a.e. $\omega \in \Omega$. The sequence $\psi^{m,(i)}$ clearly satisfies $\limsup_m \|\psi_s^{m,(i)}\| < \infty$ for all $s \in \mathcal{T}^*$.

(ii) We have that $\varphi_s^m \in L^0(\Omega, \mathcal{F}_{t_k-1}, \mathbb{P}, \mathbb{R}^N)$ for $s = 1, \dots, t_k - 1$, and $\varphi_s^m \in L^0(\Omega, \mathcal{F}_{s-1}, \mathbb{P}, \mathbb{R}^N)$ for $s = t_k, \dots, T$. Hence, the sequence φ^m is *not* a sequence of predictable processes. However, the limit of any a.s. convergent subsequence of φ^m is predictable because φ_s^m converges a.s. to 0 for $s = 1, \dots, t_k - 1$.

(iii) $E^k \subseteq \dots \subseteq E^0$, and $\mathbb{P}(E^k) > 0$.

(iv) Any a.s. convergent subsequence of φ^m converges a.s. to a nonzero process since $\|\varphi_{t_k}^m\|$ converges a.s. to 1_{E^k} , which is nonzero a.s. since $\mathbb{P}(E^k) > 0$.

(v) From (2.22), (2.24), and (2.27), we have $\varphi_s^m = 1_E \phi_s^{\beta_0^m} / U^m$ for all $s \in \mathcal{T}^*$, where $E := \bigcap_{i=1}^k E^i$. Because $E^k \subseteq \dots \subseteq E^0$,

$$\varphi_s^m = 1_{E^k} \frac{\phi_s^{\beta_0^m}}{U^m}, \quad s \in \mathcal{T}^*. \quad (2.28)$$

(vi) $U^m(\omega)$ diverges for a.e. $\omega \in E^k$ since (2.21), (2.23), (2.25), and (2.26) hold.

Step 1b: By the previous step, $\limsup_m \|\varphi_s^m\| < \infty$ for all $s \in \mathcal{T}^*$. We apply Lemma 2.3.5 to φ^m to find a strictly increasing sequence of positive, integer-valued, \mathcal{F}_{T-1} -measurable random variables ρ^m so that φ^{ρ^m} converges a.s. to some process φ such that⁸ $\varphi_s \in L^0(\Omega, \mathcal{F}_{t_k-1}, \mathbb{P}; \mathbb{R}^N)$ for $s = 1, \dots, t_k - 1$, and $\varphi_s \in L^0(\Omega, \mathcal{F}_{s-1}, \mathbb{P}; \mathbb{R}^N)$ for $s = t_k, \dots, T$. By observation (ii) in Step 1a, we have that φ is predictable.

Step 1c: We proceed by showing that **NAEF** implies $\mathbb{P}(E^0) = 0$, which contradicts the assumption that $\mathbb{P}(E^0) > 0$.

⁸See observation (ii) in Step 1a.

Towards this, we first show that the process φ constructed in Step 1b satisfies $F(\varphi) \in \mathbb{K}$. For the sake of notation, we define $\eta^m := \beta_0^m$. From (2.28), we have $\varphi^{\rho^m} = 1_{E^k} \phi^{\eta^m} / U^{\rho^m}$. Since 1_{E^k} and U^{ρ^m} are nonnegative, \mathbb{R} -valued random variables,

$$1_{E^k} \frac{F(\phi^{\eta^m})}{U^{\rho^m}} = F\left(1_{E^k} \frac{\phi^{\eta^m}}{U^{\rho^m}}\right) = F(\varphi^{\rho^m}). \quad (2.29)$$

Because φ^{ρ^m} converges a.s. to φ , we may apply Lemma A.0.2 to see that $F(\varphi^{\rho^m})$ converges a.s. to $F(\varphi)$. Since φ is predictable, we have from the definition of \mathbb{K} that $F(\varphi) \in \mathbb{K}$.

We proceed by showing that $F(\varphi) \in L_+^0(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R})$. Towards this, we define $\tilde{X}^m := X^{\eta^m} / U^{\rho^m}$ and $\tilde{Z}^m := Z^{\eta^m} / U^{\rho^m}$. From (2.20), we see that

$$F(\phi^{\eta^m}) = X^{\eta^m} + Z^{\eta^m}. \quad (2.30)$$

By multiplying both sides of (2.30) by $1_{E^k} / U^{\rho^m}$, we see from (2.29) that

$$F(\varphi^{\rho^m}) = 1_{E^k} (\tilde{X}^m + \tilde{Z}^m). \quad (2.31)$$

The sequence X^m converges a.s. by assumption, so the sequence X^{η^m} also converges a.s. Recall that the sequence $U^m(\omega)$ diverges⁹ for a.e. $\omega \in E^k$, so $U^{\rho^m}(\omega)$ diverges for a.e. $\omega \in E^k$ since $\{\rho^1(\omega), \rho^2(\omega), \dots\} \subseteq \mathbb{N}$ for a.e. $\omega \in \Omega$. Hence, $1_{E^k} \tilde{X}^m$ converges a.s. to 0. Since $F(\varphi^{\rho^m})$ and $1_{E^k} \tilde{X}^m$ converge a.s., the sequence $1_{E^k} \tilde{Z}^m$ also converges a.s. to some $Z \in L_+^0(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R})$. Thus, $F(\varphi^{\rho^m})$ converges a.s. to Z , which implies $F(\varphi) \in L_+^0(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R})$.

Since $F(\varphi) \in \mathbb{K}$, we immediately see that $F(\varphi) \in \mathbb{K} \cap L_+^0(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R})$. It is assumed that **NA** is satisfied, so by Lemmas 2.3.1 and 2.3.3 we deduce that $F(\varphi) = 0$. We are supposing that **EF** holds, so according to Lemma 2.3.2 we have $\varphi = 0$. This cannot happen given our assumption that $\mathbb{P}(E^k) > 0$ because¹⁰ $\|\varphi_{t_k}\| = 1_{E^k}$.

⁹See observations (vi) in Step 1a.

¹⁰See observations (iv) in Step 1a.

Therefore, we must have that $\mathbb{P}(E^k) = 0$. This contradicts the construction in Step 1a, so $\mathbb{P}(E^0) = 0$.

Step 2: By the conclusion in Step 1, we obtain that $\limsup_m \|\phi_s^m\| < \infty$ for $s \in \mathcal{T}^*$. By applying Lemma 2.3.5 to ϕ^m , we may find a strictly increasing sequence of positive, integer-valued, \mathcal{F}_{T-1} -measurable random variables σ^m such that ϕ^{σ^m} converges a.s. to some predictable process ϕ .

By Lemma A.0.2, the sequence $F(\phi^{\sigma^m})$ converges a.s. to $F(\phi)$. Since $\phi \in \mathcal{P}$, we have $F(\phi) \in \mathbb{K}$. Because X^m converges a.s. to X , the sequence X^{σ^m} also converges a.s. to X . From (2.20), it is true that $X^{\sigma^m} = F(\phi^{\sigma^m}) - Z^{\sigma^m}$. Since X^{σ^m} and $F(\phi^{\sigma^m})$ converges a.s., the sequence Z^{σ^m} also converges a.s. Thus, $F(\phi^{\sigma^m}) - X^{\sigma^m}$ converges a.s. to some nonnegative random variable $Z := F(\phi) - X$, which gives us that $X = F(\phi) - Z$. We conclude that $X \in \mathbb{K} - L_+^0(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R})$. This finishes the proof. \square

2.4 The First Fundamental Theorem of Asset Pricing

In this section, we formulate and prove a version of the First Fundamental Theorem of Asset Pricing (FFTAP). We define the following set for convenience:

$$\mathcal{Z} := \{\mathbb{Q} : \mathbb{Q} \sim \mathbb{P}, \text{ and } P^{ask,*}, P^{bid,*}, A^{ask,*}, A^{bid,*} \text{ are } \mathbb{Q}\text{-integrable}\}.$$

We now define a risk-neutral measure in our context.

Definition 2.4.1. *A probability measure \mathbb{Q} is a risk-neutral measure if $\mathbb{Q} \in \mathcal{Z}$, and if $\mathbb{E}_{\mathbb{Q}}[V_T^*(\phi)] \leq 0$ for all $\phi \in \mathcal{S}$ such that ϕ^j is bounded a.s., for $j \in \mathcal{J}^*$, and $V_0(\phi) = 0$.*

A natural question to ask is whether the expectation appearing in the definition above exists. The following lemma shows that, indeed, it does.

Lemma 2.4.1. *Suppose that $\mathbb{Q} \in \mathcal{Z}$, and let $\phi \in \mathcal{S}$ be such that ϕ^j is bounded a.s., for $j \in \mathcal{J}^*$, and $V_0(\phi) = 0$. Then $V_T^*(\phi)$ is \mathbb{Q} -integrable.*

Proof. From the definition of \mathcal{Z} , the processes $P^{ask,*}$, $P^{bid,*}$, $A^{ask,*}$, $A^{bid,*}$ are \mathbb{Q} -integrable. Since ϕ is self-financing, we have by Proposition 2.2.2 that

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}}[|V_T^*(\phi)|] &= \mathbb{E}_{\mathbb{Q}} \left[\left| \sum_{j=1}^N \phi_T^j (1_{\{\phi_T^j \geq 0\}} P_T^{bid,j,*} + 1_{\{\phi_T^j < 0\}} P_T^{ask,j,*}) \right. \right. \\ &\quad - \sum_{j=1}^N \sum_{u=1}^T \Delta \phi_u^j (1_{\{\Delta \phi_u^j \geq 0\}} P_{u-1}^{ask,j,*} + 1_{\{\Delta \phi_u^j < 0\}} P_{u-1}^{bid,j,*}) \\ &\quad \left. \left. + \sum_{j=1}^N \sum_{u=1}^T \phi_u^j (1_{\{\phi_u^j \geq 0\}} A_u^{ask,j,*} + 1_{\{\phi_u^j < 0\}} A_u^{bid,j,*}) \right| \right]. \end{aligned}$$

Because \mathbb{Q} is equivalent to \mathbb{P} , and since ϕ^j is bounded \mathbb{P} -a.s. for $j \in \mathcal{J}^*$, we have that ϕ^j is bounded \mathbb{Q} -a.s. for $j \in \mathcal{J}^*$. Therefore, from the equation above, we deduce from the triangle inequality that $\mathbb{E}_{\mathbb{Q}}[|V_T^*(\phi)|] < \infty$. Hence $V_T^*(\phi)$ is \mathbb{Q} -integrable. \square

The next lemma provides a mathematically convenient condition that is equivalent to **NA**.

Lemma 2.4.2. *The no-arbitrage condition (**NA**) is satisfied if and only if for each $\phi \in \mathcal{S}$ such that ϕ^j is bounded a.s. for $j \in \mathcal{J}^*$, $V_0(\phi) = 0$, and $V_T(\phi) \geq 0$, we have $V_T(\phi) = 0$.*

Proof. Necessity holds immediately, so we only show sufficiency. Let $\phi \in \mathcal{S}$ be a trading strategy so that $V_0(\phi) = 0$ and $V_T(\phi) \geq 0$. We will show that $V_T(\phi) = 0$.

First, define the \mathcal{F}_{t-1} measurable set $\Omega_t^{m,j} := \{\omega \in \Omega : |\phi_t^j(\omega)| \leq m\}$ for $m \in \mathbb{N}$, $t \in \mathcal{T}^*$, and $j \in \mathcal{J}^*$, and define the sequence of trading strategies ψ^m as $\psi_t^{m,j} := 1_{\Omega_t^{m,j}} \phi_t^j$ for $t \in \mathcal{T}^*$ and $j \in \mathcal{J}^*$, where $\psi^{m,0}$ is chosen so that ψ^m is self-financing and $V_0(\psi^m) = 0$. By Lemma A.0.3, the sequence $1_{\Omega_t^{m,j}}$ converges a.s. to 1 for all $t \in \mathcal{T}^*$ and $j \in \mathcal{J}^*$. Thus, $\psi_t^{m,j}$ converges a.s. to ϕ_t^j for all $t \in \mathcal{T}^*$ and $j \in \mathcal{J}^*$.

Now we prove that $V_0(\psi^m)$ converges a.s. to $V_0(\phi)$. Towards this, we first show that $\psi_1^{m,j}$ converges a.s. to ϕ_1^j for all $j \in \mathcal{J}$. By the definition of $V_0(\psi^m)$,

$$V_0(\psi^m) = \psi_1^{m,0} + \sum_{j=1}^N \psi_1^{m,j} (1_{\{\psi_1^{m,j} \geq 0\}} P_0^{ask,j} + 1_{\{\psi_1^{m,j} < 0\}} P_0^{bid,j}).$$

Since $\psi^{m,0}$ is chosen so that $V_0(\psi^m) = 0$, we have

$$\psi_1^{m,0} = - \sum_{j=1}^N \psi_1^{m,j} (1_{\{\psi_1^{m,j} \geq 0\}} P_0^{ask,j} + 1_{\{\psi_1^{m,j} < 0\}} P_0^{bid,j}).$$

The sequence $\psi_1^{m,j}$ converges a.s. to ϕ_1^j for all $j \in \mathcal{J}^*$, so by Lemma A.0.2 the sequence $\psi_1^{m,0}$ converges a.s. to

$$- \sum_{j=1}^N \phi_1^j (1_{\{\phi_1^j \geq 0\}} P_0^{ask,j} + 1_{\{\phi_1^j < 0\}} P_0^{bid,j}).$$

However, $V_0(\phi) = 0$, so

$$\phi_1^0 = - \sum_{j=1}^N \phi_1^j (1_{\{\phi_1^j \geq 0\}} P_0^{ask,j} + 1_{\{\phi_1^j < 0\}} P_0^{bid,j}).$$

Hence, $\psi_1^{m,0}$ converges a.s. to ϕ_1^0 . Thus, $\psi_1^{m,j}$ converges a.s. to ϕ_1^j for all $j \in \mathcal{J}$. Thus, $V_0(\psi^m)$ converges to $V_0(\phi)$.

According to Lemma 2.2.1, $V_T^*(\psi^m)$ converges a.s. to $V_T^*(\phi)$ since $\psi_t^{m,j}$ converges a.s. to ϕ_t^j for all $t \in \mathcal{T}^*$ and $j \in \mathcal{J}^*$.

Next, since ψ^m is self-financing and $\psi^{m,j}$ is bounded a.s. for all $j \in \mathcal{J}^*$ and all $m \in \mathbb{N}$, we obtain

$$V_0(\psi^m) = 0, V_T(\psi^m) \geq 0 \implies V_T(\psi^m) = 0, \quad m \in \mathbb{N}. \quad (2.32)$$

Because $B > 0$, it immediately follows that

$$V_0(\psi^m) = 0, V_T^*(\psi^m) \geq 0 \implies V_T^*(\psi^m) = 0, \quad m \in \mathbb{N}.$$

Since $V_0(\psi^m)$ converges a.s. to $V_0(\phi)$, and $V_T^*(\psi^m)$ converges a.s. to $V_T^*(\phi)$ we have

$$V_0(\phi) = 0, V_T^*(\phi) \geq 0 \implies V_T^*(\phi) = 0.$$

Hence,

$$V_0(\phi) = 0, V_T(\phi) \geq 0 \implies V_T(\phi) = 0.$$

Since $V_0(\phi) = 0$ and $V_T(\phi) \geq 0$, we conclude that $V_T(\phi) = 0$, so **NA** holds. \square

We are now ready to prove the FFTAP.

Theorem 2.4.1 (First Fundamental Theorem of Asset Pricing). *The following conditions are equivalent:*

(i) *The no-arbitrage condition under the efficient friction assumption (NAEF) is satisfied.*

(ii) *There exists a risk-neutral measure.*

(iii) *There exists a risk-neutral measure \mathbb{Q} so that $d\mathbb{Q}/d\mathbb{P} \in L^\infty(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R})$.*

Proof. In order to prove these equivalences, we show that (ii) \Rightarrow (i), (i) \Rightarrow (iii), and (iii) \Rightarrow (ii). The implication (iii) \Rightarrow (ii) is immediate, so we only show the remaining two.

(ii) \Rightarrow (i): We prove by contradiction. Let \mathbb{Q} be a risk-neutral measure and assume that **NA** does not hold. By Lemma 2.4.2, there exists $\phi \in \mathcal{S}$ so that ϕ^j is bounded a.s., for $j \in \mathcal{J}^*$, $V_0(\phi) = 0$, $V_T^*(\phi) \geq 0$, and $\mathbb{P}(V_T^*(\phi)(\omega) > 0) > 0$. Since \mathbb{Q} is equivalent to \mathbb{P} , we have $V_0(\phi) = 0$, $V_T^*(\phi) \geq 0$ \mathbb{Q} -a.s., and $\mathbb{Q}(V_T^*(\phi)(\omega) > 0) > 0$. So $\mathbb{E}_{\mathbb{Q}}[V_T^*(\phi)] > 0$, which contradicts that \mathbb{Q} is risk-neutral. Hence, **NA** holds.

(i) \Rightarrow (iii): We first construct a probability measure $\tilde{\mathbb{P}}$ satisfying $\tilde{\mathbb{P}} \in \mathcal{Z}$ and $d\tilde{\mathbb{P}}/d\mathbb{P} \in L^\infty(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R})$. Towards this, let us define the \mathcal{F}_T -measurable weight function

$$w := 1 + \sum_{u=0}^T \|P_u^{ask,*}\| + \sum_{u=0}^T \|P_u^{bid,*}\| + \sum_{u=1}^T \|A_u^{ask,*}\| + \sum_{u=1}^T \|A_u^{bid,*}\|, \quad (2.33)$$

and let $\tilde{\mathbb{P}}$ be the measure on \mathcal{F}_T with Radon-Nikodým derivative $d\tilde{\mathbb{P}}/d\mathbb{P} = \tilde{c}/w$, where $\tilde{c} := 1/\mathbb{E}_{\mathbb{P}}[1/w]$. Since $w \geq 1$, we have $d\tilde{\mathbb{P}}/d\mathbb{P} \in L^\infty(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R})$. Since $1 \leq c$, the measure $\tilde{\mathbb{P}}$ is equivalent to \mathbb{P} . Next, because $\mathbb{E}_{\mathbb{P}}[d\tilde{\mathbb{P}}/d\mathbb{P}] = 1$, we deduce that the measure $\tilde{\mathbb{P}}$ is a probability measure. By the choice of the weight function w , the processes $P^{ask,*}$, $P^{bid,*}$, $A^{ask,*}$, $A^{bid,*}$ are $\tilde{\mathbb{P}}$ -integrable:

$$\mathbb{E}_{\tilde{\mathbb{P}}}[\|P_t^{ask,*}\|] = \tilde{c} \mathbb{E}_{\mathbb{P}}[\|P_t^{ask,*}\|/w] \leq \tilde{c} < \infty,$$

$$\mathbb{E}_{\tilde{\mathbb{P}}}[\|P_t^{bid,*}\|] = \tilde{c} \mathbb{E}_{\mathbb{P}}[\|P_t^{bid,*}\|/w] \leq \tilde{c} < \infty,$$

$$\mathbb{E}_{\tilde{\mathbb{P}}}[\|A_t^{ask,*}\|] = \tilde{c} \mathbb{E}_{\mathbb{P}}[\|A_t^{ask,*}\|/w] \leq \tilde{c} < \infty,$$

$$\mathbb{E}_{\tilde{\mathbb{P}}}[\|A_t^{bid,*}\|] = \tilde{c} \mathbb{E}_{\mathbb{P}}[\|A_t^{bid,*}\|/w] \leq \tilde{c} < \infty,$$

for all $t \in \mathcal{T}$. It follows that $\tilde{\mathbb{P}} \in \mathcal{Z}$.

Since $\tilde{\mathbb{P}}$ is equivalent to \mathbb{P} , we have that

$$(\mathcal{K} - L_+^0(\Omega, \mathcal{F}_T, \tilde{\mathbb{P}}; \mathbb{R})) \cap L_+^0(\Omega, \mathcal{F}_T, \tilde{\mathbb{P}}; \mathbb{R}) = \{0\}$$

by Lemma 2.3.1, the set $\mathcal{K} - L_+^0(\Omega, \mathcal{F}_T, \tilde{\mathbb{P}}; \mathbb{R})$ is $\tilde{\mathbb{P}}$ -closed according to Theorem 2.3.1, and $\mathcal{K} - L_+^0(\Omega, \mathcal{F}_T, \tilde{\mathbb{P}}; \mathbb{R})$ is a convex cone by Lemma 2.2.2.

Let us now consider the set $\mathcal{C} := (\mathcal{K} - L_+^0(\Omega, \mathcal{F}_T, \tilde{\mathbb{P}}; \mathbb{R})) \cap L^1(\Omega, \mathcal{F}_T, \tilde{\mathbb{P}}; \mathbb{R})$. We observe the following:

- Convergence in $L^1(\Omega, \mathcal{F}_T, \tilde{\mathbb{P}}; \mathbb{R})$ implies convergence in measure $\tilde{\mathbb{P}}$, so we have that \mathcal{C} is closed in $L^1(\Omega, \mathcal{F}_T, \tilde{\mathbb{P}}; \mathbb{R})$.
- Since $0 \in L_+^1(\Omega, \mathcal{F}_T, \tilde{\mathbb{P}}; \mathbb{R}) \subseteq L_+^0(\Omega, \mathcal{F}_T, \tilde{\mathbb{P}}; \mathbb{R})$ and $0 \in \mathcal{C}$, it is true that

$$0 \subseteq \mathcal{C} \cap L_+^1(\Omega, \mathcal{F}_T, \tilde{\mathbb{P}}; \mathbb{R}) \subseteq (\mathcal{K} - L_+^0(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R})) \cap L_+^0(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R}) = \{0\}.$$

Hence, $\mathcal{C} \cap L_+^1(\Omega, \mathcal{F}_T, \tilde{\mathbb{P}}; \mathbb{R}) = \{0\}$.

- The set $L^1(\Omega, \mathcal{F}_T, \tilde{\mathbb{P}}; \mathbb{R})$ is a linear space (and thus a convex cone), so \mathcal{C} is also a convex cone since the intersection of two convex cones is a convex cone.

- Since $0 \in \mathcal{K}$, we have that $\mathcal{C} \supseteq L_-^1(\Omega, \mathcal{F}_T, \tilde{\mathbb{P}}; \mathbb{R})$, where we denote by $L_-^1(\Omega, \mathcal{F}_T, \tilde{\mathbb{P}}; \mathbb{R})$ the set of non positive $\tilde{\mathbb{P}}$ -integrable random variables.

Thus, according to Lemma A.0.8, there exists a strictly positive functional¹¹ $f \in L^\infty(\Omega, \mathcal{F}_T, \tilde{\mathbb{P}}; \mathbb{R})$ such that $\mathbb{E}_{\tilde{\mathbb{P}}}[Kf] \leq 0$ for all $K \in \mathcal{C}$. Because $0 \in L_+^0(\Omega, \mathcal{F}_T, \tilde{\mathbb{P}}; \mathbb{R})$, it follows from the definition of \mathcal{C} that

$$\mathbb{E}_{\tilde{\mathbb{P}}}[Kf] \leq 0, \quad K \in \mathcal{K} \cap L^1(\Omega, \mathcal{F}_T, \tilde{\mathbb{P}}; \mathbb{R}).$$

By the definition of \mathcal{K} , this implies that $\mathbb{E}_{\tilde{\mathbb{P}}}[V_T^*(\phi)f] \leq 0$ for all $\phi \in \mathcal{S}$ such that $V_0(\phi) = 0$ and $V_T^*(\phi) \in L^1(\Omega, \mathcal{F}_T, \tilde{\mathbb{P}}; \mathbb{R})$. In particular, $\mathbb{E}_{\tilde{\mathbb{P}}}[V_T^*(\phi)f] \leq 0$ for all $\phi \in \mathcal{S}$ such that ϕ^j is bounded a.s., for $j \in \mathcal{J}^*$, $V_0(\phi) = 0$, and $V_T^*(\phi) \in L^1(\Omega, \mathcal{F}_T, \tilde{\mathbb{P}}; \mathbb{R})$. Since $\tilde{\mathbb{P}} \in \mathcal{Z}$, we obtain from Lemma 2.4.1 that $V_T^*(\phi)$ is $\tilde{\mathbb{P}}$ -integrable. Thus, $\mathbb{E}_{\tilde{\mathbb{P}}}[V_T^*(\phi)f] \leq 0$ for all $\phi \in \mathcal{S}$ such that ϕ^j is bounded a.s., for $j \in \mathcal{J}^*$, and $V_0(\phi) = 0$.

We proceed by constructing a risk-neutral measure. Let \mathbb{Q} be the measure on \mathcal{F}_T with Radon-Nikodým derivative $d\mathbb{Q}/d\tilde{\mathbb{P}} := cf$, where $c := 1/\mathbb{E}_{\tilde{\mathbb{P}}}[f]$. Because f is a strictly positive functional in $L^\infty(\Omega, \mathcal{F}_T, \tilde{\mathbb{P}}; \mathbb{R})$, we have that \mathbb{Q} is equivalent to $\tilde{\mathbb{P}}$. Since $\tilde{\mathbb{P}}$ is equivalent to \mathbb{P} , it follows that \mathbb{Q} is equivalent to \mathbb{P} . Moreover, $\mathbb{E}_{\tilde{\mathbb{P}}}[d\mathbb{Q}/d\tilde{\mathbb{P}}] = 1$, so \mathbb{Q} is a probability measure. Also,

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \frac{d\mathbb{Q}}{d\tilde{\mathbb{P}}} \frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} = c\tilde{c} \frac{f}{w}. \quad (2.34)$$

Thus, since $w \geq 1$ and $f \in L^\infty(\Omega, \mathcal{F}_T, \tilde{\mathbb{P}}; \mathbb{R})$, we have $d\mathbb{Q}/d\mathbb{P} \in L^\infty(\Omega, \mathcal{F}_T, \tilde{\mathbb{P}}; \mathbb{R})$. This gives us $d\mathbb{Q}/d\mathbb{P} \in L^\infty(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R})$ since $\tilde{\mathbb{P}}$ is equivalent to \mathbb{P} . Moreover, the processes

¹¹For each $h \in L_+^1(\Omega, \mathcal{F}_T, \tilde{\mathbb{P}}; \mathbb{R})$ with $h \neq 0$, we have $\mathbb{E}_{\tilde{\mathbb{P}}}[fh] > 0$.

$P^{ask,*}$, $P^{bid,*}$, $A^{ask,*}$, $A^{bid,*}$ are \mathbb{Q} -integrable:

$$\mathbb{E}_{\mathbb{Q}}[\|P_t^{ask,*}\|] = c\tilde{c} \mathbb{E}_{\mathbb{P}}[\|P_t^{ask,*}\|(f/w)] \leq c\tilde{c} \mathbb{E}_{\mathbb{P}}[f] < \infty,$$

$$\mathbb{E}_{\mathbb{Q}}[\|P_t^{bid,*}\|] = c\tilde{c} \mathbb{E}_{\mathbb{P}}[\|P_t^{bid,*}\|(f/w)] \leq c\tilde{c} \mathbb{E}_{\mathbb{P}}[f] < \infty,$$

$$\mathbb{E}_{\mathbb{Q}}[\|A_t^{ask,*}\|] = c\tilde{c} \mathbb{E}_{\mathbb{P}}[\|A_t^{ask,*}\|(f/w)] \leq c\tilde{c} \mathbb{E}_{\mathbb{P}}[f] < \infty,$$

$$\mathbb{E}_{\mathbb{Q}}[\|A_t^{bid,*}\|] = c\tilde{c} \mathbb{E}_{\mathbb{P}}[\|A_t^{bid,*}\|(f/w)] \leq c\tilde{c} \mathbb{E}_{\mathbb{P}}[f] < \infty,$$

for all $t \in \mathcal{T}$. Hence, $\mathbb{Q} \in \mathcal{Z}$. We conclude that \mathbb{Q} is a risk-neutral measure since $\mathbb{E}_{\mathbb{Q}}[V_T^*(\phi)] = c \mathbb{E}_{\mathbb{P}}[V_T^*(\phi)f] \leq 0$ for all $\phi \in \mathcal{S}$ such that ϕ^j is bounded a.s., for $j \in \mathcal{J}^*$, and $V_0(\phi) = 0$. Thus, (iii) holds. \square

Remark 2.4.1. (i) Note that **EF** is not needed to prove the implication (ii) \Rightarrow (i).

(ii) In practice, it is typically required for a market model to satisfy **NA**. According to Theorem 2.4.1, it is enough to check that there exists a risk-neutral measure. However, this is not straightforward because it has to be verified whether there exists a probability measure $\mathbb{Q} \in \mathcal{Z}$ so that $\mathbb{E}_{\mathbb{Q}}[V_T^*(\phi)] \leq 0$ for all $\phi \in \mathcal{S}$ so that ϕ^j is bounded a.s., for $j \in \mathcal{J}^*$, and $V_0(\phi) = 0$. We will show in the following section that consistent pricing systems help solve this issue (see Proposition 2.5.1 and Theorem 2.5.1).

2.5 Consistent Pricing Systems

Consistent pricing systems (CPSs) are instrumental in no-arbitrage pricing in markets with transaction costs—they provide a bridge between martingale theory in no-arbitrage pricing in frictionless markets and more general concepts in no-arbitrage pricing in markets with transaction costs. Essentially, CPSs are interpreted as corresponding auxiliary frictionless markets. They are very useful from the practical point of view because they provide a straightforward way to verify whether a financial market model satisfies **NA**. In this section, we explore the relationship between CPSs and **NA**.

We begin by defining a CPS in our context.

Definition 2.5.1. *A consistent pricing system (CPS) corresponding to the market $(B, P^{ask}, P^{bid}, A^{ask}, A^{bid})$ is a quadruplet $\{\mathbb{Q}, P, A, M\}$ consisting of*

- (i) *a probability measure $\mathbb{Q} \in \mathcal{Z}$,*
- (ii) *an adapted process P satisfying $P^{bid,*} \leq P \leq P^{ask,*}$,*
- (iii) *an adapted process A satisfying $A^{ask,*} \leq A \leq A^{bid,*}$,*
- (iv) *a martingale M under \mathbb{Q} satisfying $M_t = P_t + \sum_{u=1}^t A_u$ for all $t \in \mathcal{T}$.*

Remark 2.5.1. *Since our market is fixed throughout the paper, we shall simply refer to $\{\mathbb{Q}, P, A, M\}$ as a CPS, rather than a CPS corresponding to the market $(B, P^{ask}, P^{bid}, A^{ask}, A^{bid})$.*

For a CPS $\{\mathbb{Q}, P, A, M\}$, the process P is interpreted as the corresponding auxiliary frictionless ex-dividend price process, and the process A has the interpretation of the corresponding auxiliary frictionless cumulative dividend process, whereas M is viewed as the corresponding auxiliary frictionless cumulative price process.

The next result establishes a relationship between **NA** and CPSs in our context.

Proposition 2.5.1. *If there exists a consistent pricing system (CPS), then the no-arbitrage condition (**NA**) is satisfied.*

Proof. Suppose there exists a CPS, call it $\{\mathbb{Q}, P, A, M\}$, and suppose $\phi \in \mathcal{S}$ is a trading strategy such that ϕ^j is bounded a.s., for $j \in \mathcal{J}^*$, and $V_0(\phi) = 0$. By

Proposition 2.2.2, a trading strategy ϕ is self-financing if and only if

$$\begin{aligned} V_T^*(\phi) &= \sum_{j=1}^N \phi_T^j (1_{\{\phi_T^j \geq 0\}} P_T^{bid,j,*} + 1_{\{\phi_T^j < 0\}} P_T^{ask,j,*}) \\ &\quad - \sum_{j=1}^N \sum_{u=1}^T \Delta \phi_u^j (1_{\{\Delta \phi_u^j \geq 0\}} P_{u-1}^{ask,j,*} + 1_{\{\Delta \phi_u^j < 0\}} P_{u-1}^{bid,j,*}) \\ &\quad + \sum_{j=1}^N \sum_{u=1}^T \phi_u^j (1_{\{\phi_u^j \geq 0\}} A_u^{ask,j,*} + 1_{\{\phi_u^j < 0\}} A_u^{bid,j,*}). \end{aligned}$$

Because $P^{bid} \leq P \leq P^{ask}$ and $A^{ask} \leq A \leq A^{bid}$, we have that

$$V_T^*(\phi) \leq \sum_{j=1}^N \left(\phi_T^j P_T^j + \sum_{u=1}^T (-\Delta \phi_u^j P_{u-1}^j + \phi_u^j A_u^j) \right).$$

Since $M = P + \sum_{u=1}^T A_u$ is a martingale under \mathbb{Q} , and because ϕ^j is bounded a.s., for $j \in \mathcal{J}^*$, we have

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}}[V_T^*(\phi)] &\leq \sum_{j=1}^N \mathbb{E}_{\mathbb{Q}} \left[\phi_T^j P_T^j + \sum_{u=1}^T (-\Delta \phi_u^j P_{u-1}^j + \phi_u^j A_u^j) \right] \\ &= \sum_{j=1}^N \sum_{u=1}^T \mathbb{E}_{\mathbb{Q}} \left[\Delta \phi_u^j \mathbb{E}_{\mathbb{Q}} \left[P_T^j - P_{u-1}^j + \sum_{w=u}^T A_w^j \middle| \mathcal{F}_{u-1} \right] \right] \\ &= \sum_{j=1}^N \sum_{u=1}^T \mathbb{E}_{\mathbb{Q}} \left[\Delta \phi_u^j \mathbb{E}_{\mathbb{Q}} \left[P_T^j + \sum_{w=1}^T A_w^j - P_{u-1}^j - \sum_{w=1}^{u-1} A_w^j \middle| \mathcal{F}_{u-1} \right] \right] \\ &= \sum_{j=1}^N \sum_{u=1}^T \mathbb{E}_{\mathbb{Q}} \left[\Delta \phi_u^j \mathbb{E}_{\mathbb{Q}} \left[M_T^j - M_{u-1}^j \middle| \mathcal{F}_{u-1} \right] \right] \\ &= 0. \end{aligned}$$

Therefore, \mathbb{Q} is a risk-neutral measure. According to Theorem 2.4.1, we may conclude that **NA** holds. \square

At this point, a natural question to ask is whether there exists a CPS whenever **NA** is satisfied. In general, this is still an open question. However, for the special case in which there are no transaction costs in the dividends paid by the securities, $A^{ask} = A^{bid}$, will show in Theorem 2.5.1 that there exists a CPS if and only if **NAEF** is satisfied.

Proposition 2.5.1 is important from the modeling point of view because it provides a sufficient condition for a model to satisfy **NA**. As the next example below illustrates, it is usually straightforward to check whether there exists a CPS.

Example 2.5.1. *Lets consider the CDS specified in Example 2.1.1. Recall that the cumulative dividend processes A^{ask} and A^{bid} corresponding to the CDS are defined as*

$$A_t^{ask} := 1_{\{\tau \leq t\}}\delta - \kappa^{ask} \sum_{u=1}^t 1_{\{u < \tau\}},$$

$$A_t^{bid} := 1_{\{\tau \leq t\}}\delta - \kappa^{bid} \sum_{u=1}^t 1_{\{u < \tau\}}$$

for all $t \in \mathcal{T}^*$. Let us fix any probability measure \mathbb{Q} equivalent to \mathbb{P} . We postulate that the ex-dividend prices P^{ask} and P^{bid} satisfy

$$P_t^{ask,*} = \mathbb{E}_{\mathbb{Q}} \left[\sum_{u=t+1}^T A_u^{bid,*} \middle| \mathcal{F}_t \right],$$

$$P_t^{bid,*} = \mathbb{E}_{\mathbb{Q}} \left[\sum_{u=t+1}^T A_u^{ask,*} \middle| \mathcal{F}_t \right],$$

for all $t \in \mathcal{T}^*$. By substituting $A^{ask,*}$ and $A^{bid,*}$ into the equations for $P^{ask,*}$ and $P^{bid,*}$ above, we see that

$$P_t^{ask,*} = \mathbb{E}_{\mathbb{Q}} \left[1_{\{t < \tau \leq T\}} B_{\tau}^{-1} \delta - \kappa^{bid} \sum_{u=t+1}^T B_u^{-1} 1_{\{u < \tau\}} \middle| \mathcal{F}_t \right],$$

$$P_t^{bid,*} = \mathbb{E}_{\mathbb{Q}} \left[1_{\{t < \tau \leq T\}} B_{\tau}^{-1} \delta - \kappa^{ask} \sum_{u=t+1}^T B_u^{-1} 1_{\{u < \tau\}} \middle| \mathcal{F}_t \right].$$

For a fixed $\kappa \in [\kappa^{bid}, \kappa^{ask}]$, we define¹²

$$A_t := B_t^{-1} (1_{\{\tau=t\}}\delta - \kappa 1_{\{t < \tau\}}), \quad t \in \mathcal{T}^*,$$

$$P_t := \mathbb{E}_{\mathbb{Q}} \left[\sum_{u=t+1}^T A_u \middle| \mathcal{F}_t \right] = \mathbb{E}_{\mathbb{Q}} \left[1_{\{t < \tau \leq T\}} B_{\tau}^{-1} \delta - \kappa \sum_{u=t+1}^T B_u^{-1} 1_{\{u < \tau\}} \middle| \mathcal{F}_t \right], \quad t \in \mathcal{T},$$

$$M_t := P_t + \sum_{u=1}^t A_u, \quad t \in \mathcal{T}.$$

¹²We take the convention $A_0 = 0$.

The quadruplet $\{\mathbb{Q}, P, A, M\}$ is a CPS. To see this, first observe that A and P are \mathbb{Q} -integrable since A is bounded \mathbb{Q} -a.s. Thus, $\mathbb{Q} \in \mathcal{Z}$. Next, M satisfies

$$M_t = \mathbb{E}_{\mathbb{Q}} \left[1_{\{\tau \leq T\}} B_{\tau}^{-1} \delta - \kappa \sum_{u=1}^T B_u^{-1} 1_{\{u < \tau\}} \middle| \mathcal{F}_t \right], \quad t \in \mathcal{T},$$

so M is a Doob martingale under \mathbb{Q} . Also, since $\kappa \in [\kappa^{bid}, \kappa^{ask}]$, we have $A^{ask,*} \leq A \leq A^{bid,*}$ and $P^{bid,*} \leq P \leq P^{ask,*}$. Thus, $\{\mathbb{Q}, P, A, M\}$ is a CPS. According to Proposition 2.5.1, we may additionally conclude that the financial market model $\{B, P^{ask}, P^{bid}, A^{ask}, A^{bid}\}$ satisfies NA.

2.5.1 Consistent Pricing Systems Under the Assumption $A^{ask} = A^{bid}$. In this section, we investigate the relationship between risk-neutral measures and CPSs under the assumption $A^{ask} = A^{bid}$. Let us denote by A the process A^{ask} . We begin by proving two preliminary lemmas that hold in general (without the assumption $A^{ask} = A^{bid}$).

Lemma 2.5.1. *If \mathbb{Q} is a risk-neutral measure, then*

$$\begin{aligned} P_{\sigma_1}^{bid,j,*} &\leq \mathbb{E}_{\mathbb{Q}} \left[P_{\sigma_2}^{ask,j,*} + \sum_{u=\sigma_1+1}^{\sigma_2} A_u^{bid,j,*} \middle| \mathcal{F}_{\sigma_1} \right], \\ P_{\sigma_1}^{ask,j,*} &\geq \mathbb{E}_{\mathbb{Q}} \left[P_{\sigma_2}^{bid,j,*} + \sum_{u=\sigma_1+1}^{\sigma_2} A_u^{ask,j,*} \middle| \mathcal{F}_{\sigma_1} \right], \end{aligned}$$

for all $j \in \mathcal{J}^*$ and stopping times $0 \leq \sigma_1 < \sigma_2 \leq T$.

Proof. Suppose \mathbb{Q} is a risk-neutral measure. For stopping times $0 \leq \sigma_1 < \sigma_2 \leq T$ and random variables $\xi_{\sigma_1} \in L^{\infty}(\Omega, \mathcal{F}_{\sigma_1}, \mathbb{P}; \mathbb{R}^N)$, we define the trading strategy

$$\theta(\sigma_1, \sigma_2, \xi_{\sigma_1}) := \left((\theta_t^0(\sigma_1, \sigma_2, \xi_{\sigma_1}), 1_{\{\sigma_1+1 \leq t \leq \sigma_2\}} \xi_{\sigma_1}^1, \dots, 1_{\{\sigma_1+1 \leq t \leq \sigma_2\}} \xi_{\sigma_1}^N) \right)_{t=1}^T,$$

where $\theta^0(\sigma_1, \sigma_2, \xi_{\sigma_1})$ is chosen such that $\theta(\sigma_1, \sigma_2, \xi_{\sigma_1})$ is self-financing and

$V_0(\theta(\sigma_1, \sigma_2, \xi_{\sigma_1})) = 0$. Due to Proposition 2.2.2, the value process associated with θ

satisfies

$$\begin{aligned} V_T^*(\theta(\sigma_1, \sigma_2, \xi_{\sigma_1})) &= \sum_{j=1}^N 1_{\{\xi_{\sigma_1}^j \geq 0\}} \xi_{\sigma_1}^j \left(P_{\sigma_2}^{bid,j,*} + \sum_{u=\sigma_1+1}^{\sigma_2} A_u^{ask,j,*} - P_{\sigma_1}^{ask,j,*} \right) \\ &\quad + \sum_{j=1}^N 1_{\{\xi_{\sigma_1}^j < 0\}} \xi_{\sigma_1}^j \left(P_{\sigma_2}^{ask,j,*} + \sum_{u=\sigma_1+1}^{\sigma_2} A_u^{bid,j,*} - P_{\sigma_1}^{bid,j,*} \right). \end{aligned}$$

Since \mathbb{Q} is a risk-neutral measure, we have $\mathbb{E}_{\mathbb{Q}}[V_T^*(\theta(\sigma_1, \sigma_2, \xi_{\sigma_1}))] \leq 0$ for all stopping times $0 \leq \sigma_1 < \sigma_2 \leq T$ and random variables $\xi_{\sigma_1} \in L^\infty(\Omega, \mathcal{F}_{\sigma_1}, \mathbb{P}; \mathbb{R}^N)$. Hence, we are able to obtain

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}} \left[\sum_{j=1}^N 1_{\{\xi_{\sigma_1}^j \geq 0\}} \xi_{\sigma_1}^j \left(P_{\sigma_2}^{bid,j,*} + \sum_{u=\sigma_1+1}^{\sigma_2} A_u^{ask,j,*} - P_{\sigma_1}^{ask,j,*} \right) \right. \\ \left. + \sum_{j=1}^N 1_{\{\xi_{\sigma_1}^j < 0\}} \xi_{\sigma_1}^j \left(P_{\sigma_2}^{ask,j,*} + \sum_{u=\sigma_1+1}^{\sigma_2} A_u^{bid,j,*} - P_{\sigma_1}^{bid,j,*} \right) \right] \leq 0, \end{aligned}$$

for all stopping times $0 \leq \sigma_1 < \sigma_2 \leq T$ and random variables

$\xi_{\sigma_1} \in L^\infty(\Omega, \mathcal{F}_{\sigma_1}, \mathbb{P}; \mathbb{R}^N)$. Therefore, we get that

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}} \left[\sum_{j=1}^N 1_{\{\xi_{\sigma_1}^j \geq 0\}} \xi_{\sigma_1}^j \mathbb{E}_{\mathbb{Q}} \left[P_{\sigma_2}^{bid,j,*} + \sum_{u=\sigma_1+1}^{\sigma_2} A_u^{ask,j,*} - P_{\sigma_1}^{ask,j,*} \middle| \mathcal{F}_{\sigma_1} \right] \right. \\ \left. + \sum_{j=1}^N 1_{\{\xi_{\sigma_1}^j < 0\}} \xi_{\sigma_1}^j \mathbb{E}_{\mathbb{Q}} \left[P_{\sigma_2}^{ask,j,*} + \sum_{u=\sigma_1+1}^{\sigma_2} A_u^{bid,j,*} - P_{\sigma_1}^{bid,j,*} \middle| \mathcal{F}_{\sigma_1} \right] \right] \leq 0. \end{aligned}$$

for all stopping times $0 \leq \sigma_1 < \sigma_2 \leq T$ and $\xi_{\sigma_1} \in L^\infty(\Omega, \mathcal{F}_{\sigma_1}, \mathbb{P}; \mathbb{R}^N)$. By

Lemma A.0.7, we are able to conclude that

$$\begin{aligned} P_{\sigma_1}^{bid,j,*} &\leq \mathbb{E}_{\mathbb{Q}} \left[P_{\sigma_2}^{ask,j,*} + \sum_{u=\sigma_1+1}^{\sigma_2} A_u^{bid,j,*} \middle| \mathcal{F}_{\sigma_1} \right], \\ P_{\sigma_1}^{ask,j,*} &\geq \mathbb{E}_{\mathbb{Q}} \left[P_{\sigma_2}^{bid,j,*} + \sum_{u=\sigma_1+1}^{\sigma_2} A_u^{ask,j,*} \middle| \mathcal{F}_{\sigma_1} \right], \end{aligned}$$

for all $j \in \mathcal{J}^*$, and all stopping times $0 \leq \sigma_1 < \sigma_2 \leq T$. \square

The next result is motivated by Theorem 4.5 in Cherny [Che07b]. We will denote by \mathcal{T}_t the set of stopping times in $\{t, t+1, \dots, T\}$, for all $t \in \mathcal{T}$.

Lemma 2.5.2. *Suppose \mathbb{Q} is a risk-neutral measure, and let*

$$X_s^{b,j} := \operatorname{ess\,sup}_{\sigma \in \mathcal{T}_s} \mathbb{E}_{\mathbb{Q}} \left[P_{\sigma}^{bid,j,*} + \sum_{u=1}^{\sigma} A_u^{ask,j,*} \middle| \mathcal{F}_s \right],$$

$$X_s^{a,j} := \operatorname{ess\,inf}_{\sigma \in \mathcal{T}_s} \mathbb{E}_{\mathbb{Q}} \left[P_{\sigma}^{ask,j,*} + \sum_{u=1}^{\sigma} A_u^{bid,j,*} \middle| \mathcal{F}_s \right],$$

for all $j \in \mathcal{J}^*$ and $s \in \mathcal{T}$. Then X^b is a supermartingale and X^a is a submartingale, both under \mathbb{Q} , and satisfy $X^b \leq X^a$.

Proof. Let us fix $j \in \mathcal{J}^*$. The processes $X^{b,j}$ and $X^{a,j}$ are Snell envelopes, so $X^{a,j}$ is a supermartingale and $X^{b,j}$ is a submartingale, both under \mathbb{Q} (see for instance, Föllmer and Schied [FS04]).

We now show that $X^{b,j} \leq X^{a,j}$. Let us define the process

$$X_t^j := \mathbb{E}_{\mathbb{Q}} \left[P_{\tau_1}^{bid,j,*} + \sum_{u=1}^{\tau_1} A_u^{ask,j,*} \middle| \mathcal{F}_t \right] - \mathbb{E}_{\mathbb{Q}} \left[P_{\tau_2}^{ask,j,*} + \sum_{u=1}^{\tau_2} A_u^{bid,j,*} \middle| \mathcal{F}_t \right], \quad t \in \mathcal{T}.$$

For any stopping times $\tau_1, \tau_2 \in \mathcal{T}_t$, we see that

$$\begin{aligned} X_t^j &= \mathbb{E}_{\mathbb{Q}} \left[\mathbb{E}_{\mathbb{Q}} \left[P_{\tau_1}^{bid,j,*} + \sum_{u=1}^{\tau_1} A_u^{ask,j,*} - P_{\tau_2}^{ask,j,*} - \sum_{u=1}^{\tau_2} A_u^{bid,j,*} \middle| \mathcal{F}_{\tau_1 \wedge \tau_2} \right] \middle| \mathcal{F}_t \right] \\ &= \mathbb{E}_{\mathbb{Q}} \left[1_{\{\tau_1 \leq \tau_2\}} \left(P_{\tau_1}^{bid,j,*} + \sum_{u=1}^{\tau_1} A_u^{ask,j,*} - \mathbb{E}_{\mathbb{Q}} \left[P_{\tau_2}^{ask,j,*} + \sum_{u=1}^{\tau_2} A_u^{bid,j,*} \middle| \mathcal{F}_{\tau_1} \right] \right) \middle| \mathcal{F}_t \right] \\ &\quad + \mathbb{E}_{\mathbb{Q}} \left[1_{\{\tau_1 > \tau_2\}} \left(\mathbb{E}_{\mathbb{Q}} \left[P_{\tau_1}^{bid,j,*} + \sum_{u=1}^{\tau_1} A_u^{ask,j,*} \middle| \mathcal{F}_{\tau_2} \right] - P_{\tau_2}^{ask,j,*} - \sum_{u=1}^{\tau_2} A_u^{bid,j,*} \right) \middle| \mathcal{F}_t \right]. \end{aligned}$$

After rearranging terms, we deduce that

$$\begin{aligned} X_t^j &= \mathbb{E}_{\mathbb{Q}} \left[1_{\{\tau_1 \leq \tau_2\}} P_{\tau_1}^{bid,j,*} + \sum_{u=1}^{\tau_1} (A_u^{ask,j,*} - A_u^{bid,j,*}) \right. \\ &\quad \left. - 1_{\{\tau_1 \leq \tau_2\}} \mathbb{E}_{\mathbb{Q}} \left[P_{\tau_2}^{ask,j,*} + \sum_{u=\tau_1+1}^{\tau_2} A_u^{bid,j,*} \middle| \mathcal{F}_{\tau_1} \right] \middle| \mathcal{F}_t \right] \\ &\quad + \mathbb{E}_{\mathbb{Q}} \left[1_{\{\tau_1 > \tau_2\}} \mathbb{E}_{\mathbb{Q}} \left[P_{\tau_1}^{bid,j,*} + \sum_{u=\tau_2+1}^{\tau_1} A_u^{ask,j,*} \middle| \mathcal{F}_{\tau_2} \right] \right. \\ &\quad \left. - 1_{\{\tau_1 > \tau_2\}} (P_{\tau_2}^{ask,j,*} + \sum_{u=1}^{\tau_2} (A_u^{bid,j,*} - A_u^{ask,j,*})) \middle| \mathcal{F}_t \right]. \end{aligned}$$

Because $A^{ask,*} \leq A^{bid,*}$, we are able to obtain

$$\begin{aligned} X_t^j &\leq \mathbb{E}_{\mathbb{Q}} \left[1_{\{\tau_1 \leq \tau_2\}} \left(P_{\tau_1}^{bid,j,*} - \mathbb{E}_{\mathbb{Q}} \left[P_{\tau_2}^{ask,j,*} + \sum_{u=\tau_1+1}^{\tau_2} A_u^{bid,j,*} \middle| \mathcal{F}_{\tau_1} \right] \right) \middle| \mathcal{F}_t \right] \\ &\quad + \mathbb{E}_{\mathbb{Q}} \left[1_{\{\tau_1 > \tau_2\}} \left(\mathbb{E}_{\mathbb{Q}} \left[P_{\tau_1}^{bid,j,*} + \sum_{u=\tau_2+1}^{\tau_1} A_u^{ask,j,*} \middle| \mathcal{F}_{\tau_2} \right] - P_{\tau_2}^{ask,j,*} \right) \middle| \mathcal{F}_t \right]. \end{aligned} \quad (2.35)$$

Since \mathbb{Q} is a risk-neutral measure, we see from Lemma 2.5.1 and (2.35) that $X_t^j \leq 0$. The stopping times τ_1 and τ_2 are arbitrary in the definition of X^j , so we conclude that $X^{b,j} \leq X^{a,j}$. \square

The next theorem gives sufficient and necessary conditions for there to exist a CPS (cf. Cherny [Che07b]; Denis, Guasoni, and Rásonyi [DGR11]).

Theorem 2.5.1. *Under the assumption that $A^{ask} = A^{bid}$, there exists a consistent pricing system (CPS) if and only if the no-arbitrage condition under the efficient condition (**NAEF**) is satisfied.*

Proof. Necessity is shown in Proposition 2.5.1, so we only prove sufficiency. Suppose that **NAEF** is satisfied. According to Theorem 2.4.1, there exists a risk-neutral measure \mathbb{Q} . By Lemma 2.5.1,

$$\begin{aligned} P_{\sigma_1}^{bid,j,*} &\leq \mathbb{E}_{\mathbb{Q}} \left[P_{\sigma_2}^{ask,j,*} + \sum_{u=\sigma_1+1}^{\sigma_2} A_u^{j,*} \middle| \mathcal{F}_{\sigma_1} \right], \\ P_{\sigma_1}^{ask,j,*} &\geq \mathbb{E}_{\mathbb{Q}} \left[P_{\sigma_2}^{bid,j,*} + \sum_{u=\sigma_1+1}^{\sigma_2} A_u^{j,*} \middle| \mathcal{F}_{\sigma_1} \right], \end{aligned}$$

for all $j \in \mathcal{J}^*$ and stopping times $0 \leq \sigma_1 < \sigma_2 \leq T$. Now, let us define the processes

$$\begin{aligned}
Y_t^{b,j} &:= \operatorname{ess\,sup}_{\sigma \in \mathcal{T}_t} \mathbb{E}_{\mathbb{Q}} \left[P_{\sigma}^{bid,j,*} + \sum_{u=t+1}^{\sigma} A_u^{j,*} \middle| \mathcal{F}_t \right], \\
Y_t^{a,j} &:= \operatorname{ess\,inf}_{\sigma \in \mathcal{T}_t} \mathbb{E}_{\mathbb{Q}} \left[P_{\sigma}^{ask,j,*} + \sum_{u=t+1}^{\sigma} A_u^{j,*} \middle| \mathcal{F}_t \right], \\
X_t^{b,j} &:= Y_t^{b,j} + \sum_{u=1}^t A_u^{j,*}, \\
X_t^{a,j} &:= Y_t^{a,j} + \sum_{u=1}^t A_u^{j,*},
\end{aligned} \tag{2.36}$$

for all $t \in \mathcal{T}$ and $j \in \mathcal{J}^*$. From Lemma 2.5.2, we know that under \mathbb{Q} the process X^a is a submartingale and the process X^b is a supermartingale, and that they satisfy $X^b \leq X^a$.

For $t = 0, 1, \dots, T-1$ and $j \in \mathcal{J}^*$, recursively define

$$\begin{aligned}
M_0^j &:= Y_0^{a,j}, \\
P_0^j &:= Y_0^{a,j}, \\
P_{t+1}^j &:= \lambda_t^j Y_{t+1}^{a,j} + (1 - \lambda_t^j) Y_{t+1}^{b,j}, \\
M_{t+1}^j &:= P_{t+1}^j + \sum_{u=1}^{t+1} A_u^j,
\end{aligned} \tag{2.37}$$

where λ_t^j satisfies

$$\lambda_t^j = \begin{cases} \frac{M_t^j - \mathbb{E}_{\mathbb{Q}}[X_{t+1}^{b,j} | \mathcal{F}_t]}{\mathbb{E}_{\mathbb{Q}}[X_{t+1}^{a,j} - X_{t+1}^{b,j} | \mathcal{F}_t]}, & \text{if } \mathbb{E}_{\mathbb{Q}}[X_{t+1}^{a,j} | \mathcal{F}_t] \neq \mathbb{E}_{\mathbb{Q}}[X_{t+1}^{b,j} | \mathcal{F}_t], \\ \frac{1}{2}, & \text{otherwise.} \end{cases} \tag{2.38}$$

Lets fix $j \in \mathcal{J}^*$ for the rest of the proof.

Step 1: In this step, we show that the processes above are well defined and adapted.

First, note that P_0 and M_0 are well defined, and that, by (2.38),

$$\lambda_0^j = \frac{M_0^j - \mathbb{E}_{\mathbb{Q}}[X_1^{b,j} | \mathcal{F}_0]}{\mathbb{E}_{\mathbb{Q}}[X_1^{a,j} - X_1^{b,j} | \mathcal{F}_0]}, \quad \text{or} \quad \lambda_0^j = \frac{1}{2}.$$

Thus, λ_0^j is well defined and \mathcal{F}_0 -measurable. Next, we compute P_1^j and M_1^j , and consequently we compute λ_1^j ; all of them being \mathcal{F}_1 -measurable. Inductively, we see that P_t^j , M_t^j , and λ_t^j , for $t = 2, \dots, T$ are well defined and \mathcal{F}_t -measurable.

Step 2: We inductively show that $\lambda_t^j \in [0, 1]$ for $t = 0, 1, \dots, T - 1$. We first show that $\lambda_0^j \in [0, 1]$. If $\mathbb{E}_{\mathbb{Q}}[X_1^{a,j} - X_1^{b,j} | \mathcal{F}_0] = 0$, then $\lambda_0^j \in [0, 1]$ automatically, so suppose that $\mathbb{E}_{\mathbb{Q}}[X_1^{a,j} - X_1^{b,j} | \mathcal{F}_0] > 0$. Now, by the definition of M^j , we have that $M_0^j = X_0^{a,j}$, so (2.38) gives that

$$\lambda_0^j = \frac{X_0^{a,j} - \mathbb{E}_{\mathbb{Q}}[X_1^{b,j} | \mathcal{F}_0]}{\mathbb{E}_{\mathbb{Q}}[X_1^{a,j} - X_1^{b,j} | \mathcal{F}_0]}. \quad (2.39)$$

The process $X^{a,j}$ is a submartingale under \mathbb{Q} , so it immediately follows that $\lambda_0^j \leq 1$. On the other hand, since $X^{b,j}$ is a supermartingale under \mathbb{Q} ,

$$\lambda_0^j \geq \frac{X_0^{a,j} - X_0^{b,j}}{\mathbb{E}_{\mathbb{Q}}[X_1^{a,j} - X_1^{b,j} | \mathcal{F}_0]}.$$

Because $X_0^{a,j} \geq X_0^{b,j}$, we deduce that $\lambda_0^j \geq 0$.

Suppose that $\lambda_t^j \in [0, 1]$ for $t = 0, 1, \dots, T-2$. We now prove that $\lambda_{T-1}^j \in [0, 1]$. If $\mathbb{E}_{\mathbb{Q}}[X_T^{a,j} - X_T^{b,j} | \mathcal{F}_{T-1}] = 0$, then $\lambda_{T-1}^j = 1/2$, so assume that $\mathbb{E}_{\mathbb{Q}}[X_T^{a,j} - X_T^{b,j} | \mathcal{F}_{T-1}] > 0$. According to (2.38) and the definition of M^j , we have that

$$\lambda_{T-1}^j = \frac{\lambda_{T-2}^j X_{T-1}^{a,j} + (1 - \lambda_{T-2}^j) X_{T-1}^{b,j} - \mathbb{E}_{\mathbb{Q}}[X_T^{b,j} | \mathcal{F}_{T-1}]}{\mathbb{E}_{\mathbb{Q}}[X_T^{a,j} - X_T^{b,j} | \mathcal{F}_{T-1}]}. \quad (2.40)$$

Since $\lambda_{T-2}^j \leq 1$, and because $X^{b,j}$ is a supermartingale under \mathbb{Q} , we have that

$$\lambda_{T-1}^j \geq \frac{\lambda_{T-2}^j (X_{T-1}^{a,j} - X_{T-1}^{b,j})}{\mathbb{E}_{\mathbb{Q}}[X_T^{a,j} - X_T^{b,j} | \mathcal{F}_{T-1}]}$$

Because $X^{a,j} \geq X^{b,j}$, we arrive at $\lambda_{T-1}^j \geq 0$. Now, since $X_{T-1}^{a,j} \geq X_{T-1}^{b,j}$ and $\lambda_{T-2}^j \leq 1$, we see from (2.40) that

$$\lambda_{T-1}^j \leq \frac{X_{T-1}^{a,j} - \mathbb{E}_{\mathbb{Q}}[X_T^{b,j} | \mathcal{F}_{T-1}]}{\mathbb{E}_{\mathbb{Q}}[X_T^{a,j} - X_T^{b,j} | \mathcal{F}_{T-1}]}$$

The process $X^{a,j}$ is a submartingale under \mathbb{Q} , so it follows that $\lambda_{T-1}^j \leq 1$.

Thus, we conclude that $\lambda_t^j \in [0, 1]$ for $t = 0, 1, \dots, T - 1$.

Step 3: Next, we show that M is a martingale under \mathbb{Q} . First we note that by (2.36) and (2.37) we have

$$M_{t+1}^j = \lambda_t^j X_{t+1}^{a,j} + (1 - \lambda_t^j) X_{t+1}^{b,j}. \quad (2.41)$$

From here, the \mathbb{Q} -integrability of M^j follows from \mathbb{Q} -integrability of $X^{a,j}$, $X^{b,j}$ and boundedness of λ^j . From (2.38) and (2.41), we get that $\mathbb{E}_{\mathbb{Q}}[M_{t+1}^j | \mathcal{F}_t] = M_t^j$, for $t = 0, 1, \dots, T - 1$. Hence, M^j is a martingale under \mathbb{Q} .

Step 4: We continue by showing that P^j satisfies $P^{bid,*,j} \leq P^j \leq P^{ask,*,j}$. Let us first show that $P_0^{bid,j} \leq P_0^j \leq P_0^{ask,j}$. By definition of P_0^j , we have that $P_0^j = Y_0^{a,j}$, and by (2.36) we see that $Y_0^{a,j} = X_0^{a,j}$. Therefore, the claim holds since $P_0^{bid,j} \leq X_0^{a,j} \leq P_0^{ask,j}$.

We proceed by proving that $P_t^{bid,j} \leq P_t^j \leq P_t^{ask,j}$ for all $t \in \{1, \dots, T\}$. Towards this, let $t \in \{1, \dots, T\}$. By the definition of P_t^j , we have $P_t^j = \lambda_{t-1}^j Y_t^{a,j} + (1 - \lambda_{t-1}^j) Y_t^{b,j}$. From (2.36), it is true that $X_t^{a,j} \geq X_t^{b,j}$ if and only if $Y_t^{a,j} \geq Y_t^{b,j}$. Also, since $t \in \mathcal{T}_t$, we see from (2.36) that $Y_t^{b,j} \geq P_t^{bid,j,*}$ and $Y_t^{a,j} \leq P_t^{ask,j}$. According to Step 1, $\lambda_{t-1}^j \in [0, 1]$. So, putting everything together, we obtain

$$P_t^{bid,j} \leq Y_t^{b,j} \leq P_t^j \leq Y_t^{a,j} \leq P_t^{ask,j}.$$

We conclude that $\{\mathbb{Q}, P, A, M\}$ is a CPS. \square

2.6 The Superhedging and Subhedging Theorem

In this section, we formulate the superhedging ask price and subhedging bid price of a dividend-paying derivative contract, and then we provide an important representation theorem for these prices.

For results related to this topic, both for discrete-time and continuous-time markets with transaction costs, we refer to, among others, Soner, Shreve, and Cvitanic [SSC95]; Levental and Skorohod [LS97]; Cvitanic, Pham, and Touzi [CPT99];

Touzi [Tou99]; Bouchard and Touzi [BT00]; Kabanov, Rásonyi, and Stricker [KRS02]; Schachermayer [Sch04]; Campi and Schachermayer [CS06]; Cherny [Che07b]; Pennanen [Pen11d, Pen11a, Pen11b, Pen11c]. Our contribution to this literature is that we consider dividend-paying securities, such as swap contracts, as hedging securities.

A *derivative contract* D is any a.s. bounded, \mathbb{R} -valued, \mathbb{F} -adapted process. Here, D is interpreted as the spot cash flow process (not the cumulative cash flow process). We remark that the boundedness restriction on derivative contracts is natural for fixed income securities.

Let us now define the set of self-financing trading strategies initiated at time $t \in \{0, 1, \dots, T-1\}$ with bounded components ($j = 1, \dots, N$) as

$$\mathcal{S}^*(t) := \left\{ \phi \in \mathcal{S} : \phi^j \text{ is bounded a.s. for } j \in \mathcal{J}^*, \phi_s = 1_{\{t+1 \leq s\}} \phi_s \text{ for all } s \in \mathcal{T}^* \right\},$$

and the set of attainable values at zero cost initiated at time $t \in \{0, 1, \dots, T-1\}$ as

$$\mathcal{K}(t) := \left\{ V_T^*(\phi) : \phi \in \mathcal{S}^*(t) \text{ such that } V_0(\phi) = 0 \right\}.$$

Remark 2.6.1.

(i) $\mathcal{S}^*(t)$ and $\mathcal{K}(t)$ are closed with respect to multiplication by random variables in

$$L_+^\infty(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R}).^{13}$$

(ii) $\mathcal{S} \supset \mathcal{S}^*(0) \supset \mathcal{S}^*(1) \supset \dots \supset \mathcal{S}^*(T-1)$ and $\mathcal{K} \supset \mathcal{K}(0) \supset \mathcal{K}(1) \supset \dots \supset \mathcal{K}(T-1)$.

Moreover, if \mathbb{Q} is a risk-neutral measure, then $\mathbb{E}_{\mathbb{Q}}[K] \leq 0$ for all $K \in \mathcal{K}(t)$, for $t = 0, 1, \dots, T-1$.

We proceed by defining the main objects of this section.

¹³ $L_+^\infty(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R}) := \{X \in L^\infty(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R}) : X \geq 0\}$.

Definition 2.6.1. *The discounted superhedging ask and subhedging bid prices of a derivative contract D at time $t \in \{0, \dots, T-1\}$ are defined as*

$\pi_t^{ask}(D) := \text{ess inf } \mathcal{W}^a(t, D)$ and $\pi_t^{bid}(D) := \text{ess sup } \mathcal{W}^b(t, D)$, where

$$\mathcal{W}^a(t, D) := \left\{ W \in L^0(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R}) : -W + \sum_{u=t+1}^T D_u^* \in \mathcal{K}(t) - L_+^0(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R}) \right\},$$

$$\mathcal{W}^b(t, D) := \left\{ W \in L^0(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R}) : W - \sum_{u=t+1}^T D_u^* \in \mathcal{K}(t) - L_+^0(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R}) \right\}.$$

Note that $\pi_t^{ask}(D) = -\pi_t^{bid}(-D)$, and

$$\mathcal{W}^a(t, D) = \left\{ W \in L^0(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R}) : \exists K \in \mathcal{K}(t) \text{ such that } \sum_{u=t+1}^T D_u^* \leq K + W \right\},$$

$$\mathcal{W}^b(t, D) = \left\{ W \in L^0(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R}) : \exists K \in \mathcal{K}(t) \text{ such that } -\sum_{u=t+1}^T D_u^* \leq K - W \right\}.$$

Remark 2.6.2.

- (i) *For each $t \in \{0, 1, \dots, T-1\}$, the prices $\pi_t^{ask}(D)$ and $\pi_t^{bid}(D)$ have the following interpretations: The price $\pi_t^{ask}(D)$ is interpreted as the least discounted cash amount W at time t so that the gain $-W + \sum_{u=t+1}^T D_u^*$ can be superhedged at zero cost. On the other hand, the random variable $\pi_t^{bid}(D)$ is interpreted as the greatest discounted cash amount W at time t so that the gain $W - \sum_{u=t+1}^T D_u^*$ can be superhedged at zero cost.*
- (ii) *In view of (i) above, it is unreasonable for the discounted ex-dividend ask price at time $t \in \{0, 1, \dots, T-1\}$ of a derivative D to be a.s. greater than $\pi_t^{ask}(D)$, and for the ex-dividend bid price at time $t \in \{0, 1, \dots, T-1\}$ of a derivative D to be a.s. less than $\pi_t^{bid}(D)$.*
- (iii) *Direction of trade matters: a market participant can buy a derivative D at price $\pi_t^{ask}(D)$ and sell D at price $\pi_t^{bid}(D)$. This is in contrast to frictionless markets, where a derivative contract can be bought and sold at the same price.*

(iv) The prices $\pi_t^{ask}(D)$ and $\pi_t^{bid}(D)$ satisfy $\pi_t^{ask}(D) < \infty$ and $\pi_t^{bid}(D) > -\infty$. Indeed, since $0 \in \mathcal{K}(t)$, $1 \in L_{++}^0(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R})$, and $\sum_{u=t+1}^T D_u^*$ is a.s. bounded, say by M , we have that $-M + \sum_{u=t+1}^T D_u^* \in L_-^0(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R})$. Thus, $\pi_t^{ask}(D) \leq M$. Similarly, $\pi_t^{bid}(D) \geq -M$.

Next, we define the sets of extended attainable values initiated at time $t \in \{0, 1, \dots, T-1\}$ associated with cash amount $W \in L^0(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R})$:

$$\begin{aligned} \mathcal{K}^a(t, W) &:= \mathcal{K}(t) + \left\{ \xi \left(-W + \sum_{u=t+1}^T D_u^* \right) : \xi \in L_+^\infty(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R}) \right\}, \\ \mathcal{K}^b(t, W) &:= \mathcal{K}(t) + \left\{ \xi \left(W - \sum_{u=t+1}^T D_u^* \right) : \xi \in L_+^\infty(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R}) \right\}. \end{aligned}$$

Remark 2.6.3.

(i) The sets $\mathcal{K}^a(t, W)$ and $\mathcal{K}^b(t, W)$ are closed with respect to multiplication by random variables in the set $L_+^\infty(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R})$, and in view of Lemma 2.2.2 they are convex cones. Also, $\mathcal{K}(t) \subset \mathcal{K}^a(t, W) \cap \mathcal{K}^b(t, W)$ since $0 \in L_+^\infty(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R})$.

(ii) In view of Proposition 2.2.2,

$$\left\{ \xi \left(-\pi_t^{ask}(D) + \sum_{u=t+1}^T D_u^* \right) : \xi \in L_+^\infty(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R}) \right\} \quad (2.42)$$

is the set of all discounted terminal values associated with zero-cost, self-financing, buy-and-hold trading strategies in the derivative contract D with discounted ex-dividend ask price $\pi_t^{ask}(D)$. On the other hand, the convex cone

$$\left\{ \xi \left(\pi_t^{bid}(D) - \sum_{u=t+1}^T D_u^* \right) : \xi \in L_+^\infty(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R}) \right\} \quad (2.43)$$

is the set of all discounted terminal values associated with zero-cost, self-financing, sell-and-hold trading strategies in the derivative contract D with discounted ex-dividend bid price $\pi_t^{bid}(D)$.

We will now introduce definitions related to the sets of extended attainable values. For each $t \in \{0, 1, \dots, T-1\}$ and $X \in L^0(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R})$, a probability measure \mathbb{Q} is *risk-neutral for $\mathcal{K}^a(t, X)$ ($\mathcal{K}^b(t, X)$)* if $\mathbb{Q} \in \mathcal{Z}$ and X is \mathbb{Q} -integrable, and if $\mathbb{E}_{\mathbb{Q}}[K] \leq 0$ for all $K \in \mathcal{K}^a(t, X)$ ($K \in \mathcal{K}^b(t, X)$). We denote by $\mathcal{R}^a(t, X)$ ($\mathcal{R}^b(t, X)$) the set of all risk-neutral measures \mathbb{Q} for $\mathcal{K}^a(t, X)$ ($\mathcal{K}^b(t, X)$). We say that **NA** holds for $\mathcal{K}^a(t, X)$ if $\mathcal{K}^a(t, X) \cap L_+^0(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R}) = \{0\}$, and likewise we say that **NA** holds for $\mathcal{K}^b(t, X)$ if $\mathcal{K}^b(t, X) \cap L_+^0(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R}) = \{0\}$.

We will say that $\mathcal{K}^a(t, X)$ satisfies **EF** if

$$\begin{aligned} & \left\{ (\phi, \xi) \in \mathcal{S}^*(t) \times L_+^\infty(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R}) : V_0(\phi) = 0, V_T^*(\phi) + \xi \left(-X + \sum_{u=t+1}^T D_u^* \right) = 0 \right\} \\ & = \{(0, 0)\}, \end{aligned}$$

and say that $\mathcal{K}^b(t, X)$ satisfies **EF** if

$$\begin{aligned} & \left\{ (\phi, \xi) \in \mathcal{S}^*(t) \times L_+^\infty(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R}) : V_0(\phi) = 0, V_T^*(\phi) + \xi \left(X - \sum_{u=t+1}^T D_u^* \right) = 0 \right\} \\ & = \{(0, 0)\}. \end{aligned}$$

Remark 2.6.4. According to Lemma 2.6.2, for any $t \in \{0, 1, \dots, T-1\}$ and $X \in L^0(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{P})$, **NAEF** holds for $\mathcal{K}^a(t, X)$ ($\mathcal{K}^b(t, X)$) if and only if $\mathcal{R}^a(t, X) \neq \emptyset$ ($\mathcal{R}^b(t, X) \neq \emptyset$).

For each $t \in \{0, 1, \dots, T-1\}$, we denote by $\mathcal{R}(t)$ the set of all risk-neutral measures for $\mathcal{K}(t)$. Specifically, we define $\mathcal{R}(t)$ as

$$\mathcal{R}(t) := \left\{ \mathbb{Q} \in \mathcal{Z} : \mathbb{E}_{\mathbb{Q}}[K] \leq 0 \text{ for all } K \in \mathcal{K}(t) \right\}.$$

We note that $\mathcal{R}^a(t, X) \cup \mathcal{R}^b(t, X) \subseteq \mathcal{R}(t)$ for any $X \in L^0(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R})$ since $\mathcal{K}(t) \subseteq \mathcal{K}^a(t, X) \cap \mathcal{K}^b(t, X)$. Also, by the definition of a risk-neutral measure, it immediately follows that any risk-neutral measure \mathbb{Q} (as in Definition 2.4.1) satisfies $\mathbb{Q} \in \mathcal{R}(t)$ for any $t \in \{0, 1, \dots, T-1\}$.

The next technical lemma is needed to derive the dual representations of the superhedging ask and subhedging bid prices.

Lemma 2.6.1.

- (i) For each $t \in \{0, 1, \dots, T-1\}$, if $\mathcal{R}(t) \neq \emptyset$ and $\mathbb{Q} \in \mathcal{R}(t)$, then we have that $\mathbb{E}_{\mathbb{Q}}[K|\mathcal{F}_t] \leq 0$ \mathbb{Q} -a.s. for all $K \in \mathcal{K}(t)$.
- (ii) For each $t \in \{0, 1, \dots, T-1\}$ and $X \in L^0(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R})$, if $\mathcal{R}^a(t, X) \neq \emptyset$ and $\mathbb{Q} \in \mathcal{R}^a(t, X)$, then we have that $\mathbb{E}_{\mathbb{Q}}[K^a|\mathcal{F}_t] \leq 0$ \mathbb{Q} -a.s. for all $K^a \in \mathcal{K}^a(t, X)$.
- (iii) For each $t \in \{0, 1, \dots, T-1\}$ and $X \in L^0(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R})$, if $\mathcal{R}^b(t, X) \neq \emptyset$ and $\mathbb{Q} \in \mathcal{R}^b(t, X)$, then we have that $\mathbb{E}_{\mathbb{Q}}[K^b|\mathcal{F}_t] \leq 0$ \mathbb{Q} -a.s. for all $K^b \in \mathcal{K}^b(t, X)$.

Proof. We only prove (i) and (ii). The proof of (iii) is very similar to the proof of (ii). We fix $t \in \{1, \dots, T-1\}$ throughout the proof. Observe that in view of Lemma 2.4.1, we have that for each $\mathbb{Q} \in \mathcal{R}(t)$, any $K \in \mathcal{K}$ is \mathbb{Q} -integrable. Moreover, because any derivative contract is bounded a.s., for each $\mathbb{Q} \in \mathcal{R}^a(t, X)$ ($\mathbb{Q} \in \mathcal{R}^b(t, X)$), any $K^a \in \mathcal{K}^a(t, X)$ ($K^b \in \mathcal{K}^b(t, X)$) is \mathbb{Q} -integrable.

(i): We prove by contradiction. Let $\mathbb{Q} \in \mathcal{R}(t)$, and suppose that there exists and $K \in \mathcal{K}(t)$ such that $\mathbb{E}_{\mathbb{Q}}[K|\mathcal{F}_t](\omega) > 0$ for $\omega \in \Omega^t$, where $\Omega^t \subseteq \Omega$ and $\mathbb{P}(\Omega^t) > 0$. Note that $\Omega^t \in \mathcal{F}_t$ since $\mathbb{E}_{\mathbb{Q}}[K|\mathcal{F}_t]$ is \mathcal{F}_t -measurable. By definition of $\mathcal{K}(t)$, there exists $\phi \in \mathcal{S}^*(t)$ with $V_0(\phi) = 0$ such that $K = V_T^*(\phi)$. Define the process $\psi := 1_{\Omega^t}\phi$. Since Ω^t is \mathcal{F}_t -measurable and $\mathcal{S}^*(t)$ is closed with respect to multiplication by random variables in the set $L_+^\infty(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R})$, we have that $\psi \in \mathcal{S}^*(t)$. Moreover, $V_0(\psi) = 1_{\Omega^t}V_0(\phi) = 0$ because 1_{Ω^t} is nonnegative. Therefore, $V_T^*(\psi) \in \mathcal{K}(t)$. Since $V_T^*(\psi) = 1_{\Omega^t}V_T^*(\phi) = 1_{\Omega^t}K$, we have that $\mathbb{E}_{\mathbb{Q}}[V_T^*(\psi)] = \mathbb{E}_{\mathbb{Q}}[1_{\Omega^t}\mathbb{E}_{\mathbb{Q}}[K|\mathcal{F}_t]] > 0$, which contradicts that $\mathbb{Q} \in \mathcal{R}(t)$.

(ii): As in (i), we will prove by contradiction. Let $X \in L^0(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R})$ and $\mathbb{Q} \in \mathcal{R}^a(t, X)$, and assume that there exist $K \in \mathcal{K}(t)$ and $\xi \in L_+^\infty(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R})$ such that

$$-\xi(\omega)X(\omega) + \mathbb{E}_{\mathbb{Q}} \left[K + \xi \sum_{u=t+1}^T D_u^* \middle| \mathcal{F}_t \right] (\omega) > 0, \quad \omega \in \Omega^t,$$

where $\Omega^t \subseteq \Omega$ and $\mathbb{P}(\Omega^t) > 0$. Since $\mathcal{R}^a(t, X) \subset \mathcal{R}(t)$, we have that $\mathbb{Q} \in \mathcal{R}(t)$. In view of (i) above, it follows that $\mathbb{E}_{\mathbb{Q}}[K | \mathcal{F}_t] \leq 0$. Thus,

$$-\xi(\omega)X(\omega) + \xi(\omega) \mathbb{E}_{\mathbb{Q}} \left[\sum_{u=t+1}^T D_u^* \middle| \mathcal{F}_t \right] (\omega) > 0, \quad \omega \in \Omega^t.$$

We proceed by defining the \mathcal{F}_t -measurable random variable $\vartheta := 1_{\Omega^t} \xi$. Because $\Omega^t \in \mathcal{F}_t$, it is true that $\vartheta \in L_+^\infty(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R})$. Now, by the tower property of conditional expectations we obtain

$$\mathbb{E}_{\mathbb{Q}} \left[\vartheta \left(-X + \sum_{u=t+1}^T D_u^* \right) \right] = \mathbb{E}_{\mathbb{Q}} \left[1_{\Omega^t} \left(-\xi X + \xi \mathbb{E}_{\mathbb{Q}} \left[\sum_{u=t+1}^T D_u^* \middle| \mathcal{F}_t \right] \right) \right] > 0.$$

This contradicts that $\mathbb{Q} \in \mathcal{R}^a(t, X)$ since $\vartheta \in L_+^\infty(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R})$ and $0 \in \mathcal{K}(t)$. \square

Next result is a version of the FFTAP for the extended markets.

Lemma 2.6.2. *For each $t \in \{0, 1, \dots, T-1\}$ and $W \in L^0(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R})$, if the no-arbitrage condition under the efficient friction assumption is satisfied for $\mathcal{K}^a(t, W)$ ($\mathcal{K}^b(t, W)$), then $\mathcal{R}^a(t, W) \neq \emptyset$ ($\mathcal{R}^b(t, W) \neq \emptyset$).*

Proof. Let us first fix $t \in \{0, 1, \dots, T-1\}$ and $W \in L^0(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R})$. We only prove the lemma for $\mathcal{K}^a(t, W)$, because the proof for $\mathcal{K}^b(t, W)$ is similar. Instead of working with $\mathcal{K}^a(t, X)$, we will work with the more mathematically convenient set

$$\mathbb{K}^a(t, W) := \{G(\phi, \xi, t, W) : \phi \in \mathcal{P}(t), \xi \in L_+^\infty(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R})\},$$

where $\mathcal{P}(t)$ is the set

$$\mathcal{P}(t) := \{\phi \in \mathcal{P} : \phi^j \text{ is a.s. bounded for } j \in \mathcal{J}^*, \phi_s = 1_{\{t+1 \leq s\}} \phi_s \text{ for all } s \in \mathcal{T}^*\},$$

and

$$\begin{aligned}
G(\phi, \xi, t, W) &:= \sum_{j=1}^N \phi_T^j (1_{\{\phi_T^j \geq 0\}} P_T^{bid,j,*} + 1_{\{\phi_T^j < 0\}} P_T^{ask,j,*}) \\
&\quad - \sum_{j=1}^N \sum_{u=t+1}^T \Delta \phi_u^j (1_{\{\Delta \phi_u^j \geq 0\}} P_{u-1}^{ask,j,*} + 1_{\{\Delta \phi_u^j < 0\}} P_{u-1}^{bid,j,*}) \\
&\quad + \sum_{j=1}^N \sum_{u=t+1}^T \phi_u^j (1_{\{\phi_u^j \geq 0\}} A_u^{ask,j,*} + 1_{\{\phi_u^j < 0\}} A_u^{bid,j,*}) + \xi \left(-W + \sum_{u=t+1}^T D_u^* \right)
\end{aligned} \tag{2.44}$$

for all for all \mathbb{R}^N -valued stochastic processes

$$(\phi_s)_{s=1}^T \in L^0(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R}^N) \times \cdots \times L^0(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R}^N),$$

and random variables $\xi \in L_+^\infty(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R})$.

Since $\mathbb{K}^a(t, W) = \mathcal{K}^a(t, W)$, we may equivalently prove that

$\mathbb{K}^a(t, W) - L_+^0(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R})$ is \mathbb{P} -closed whenever **NAEF** is satisfied for $\mathcal{K}^a(t, W)$. Let $X^m \in \mathbb{K}^a(t, W) - L_+^0(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R})$ be a sequence converging in probability to some X . We may find a subsequence X^{k_m} that converges a.s. to X . With an abuse of notation we denote this subsequence by X^m . By the definition of $\mathbb{K}^a(t, W)$, we may find $\phi^m \in \mathcal{P}(t)$, $\xi^m \in L_+^0(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R})$, and $Z^m \in L_+^0(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R})$ so that $X^m = G(\phi^m, \xi^m, t, W) - Z^m$. Using the same arguments as in Step 1 in the proof of Theorem 2.3.1, we prove that $\limsup_m \|\phi_s^m\| < \infty$ for all $t \in \mathcal{T}^*$ and $\limsup_m \xi^m < \infty$. Then, we apply Lemma 2.3.5 to show that we may find a strictly increasing set of positive, integer-valued, \mathcal{F}_{T-1} measurable random variables σ^m such that ϕ^{σ^m} converges a.s. to some bounded a.s. predictable process ϕ , and ξ^{σ^m} converges a.s. to some $\xi \in L_+^\infty(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R})$. This gives us that $G(\phi^{\sigma^m}, \xi^{\sigma^m}, t, W) - X^{\sigma^m}$ converges a.s. to some nonnegative random variable. Therefore $\mathbb{K}^a(t, W) - L_+^0(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R})$ is \mathbb{P} -closed.

We now argue that there exists a risk-neutral measure for $\mathcal{K}^a(t, W)$. Towards this, we define the convex cone $\mathcal{C}^a := (\mathcal{K}^a(t, W) - L_+^0(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R})) \cap L^1(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R})$.

Due to the closedness property of $\mathbb{K}^a(t, W) - L_+^0(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R})$, we have that the set \mathcal{C}^a is closed in $L^1(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R})$. As in the proof of Theorem 2.4.1, we may construct a measure $\mathbb{Q} \in \mathcal{Z}$ such that W is \mathbb{Q} -integrable, and $\mathbb{E}_{\mathbb{Q}}[K^a] \leq 0$ for all $K^a \in \mathcal{K}^a(t, X)$. This completes the proof. \square

We are ready to prove the main result of this section: the dual representations of the superhedging ask price and subhedging bid price.

Theorem 2.6.1. *Suppose that the no-arbitrage condition under the efficient friction assumption (NAEF) is satisfied. Let $t \in \{0, 1, \dots, T-1\}$ and D be a derivative contract. Then the following hold:*

- (i) *The essential infimum of $\mathcal{W}^a(t, D)$ and the essential supremum of $\mathcal{W}^b(t, D)$ are attained.*
- (ii) *Suppose that for each $t \in \{0, 1, \dots, T-1\}$ and $X \in L^0(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R})$, the efficient friction assumption (EF) holds for $\mathcal{K}^a(t, X)$ and $\mathcal{K}^b(t, X)$. Then the discounted superhedging ask and subhedging bid prices for derivative contract D at time t satisfy*

$$\pi_t^{ask}(D) = \operatorname{ess\,sup}_{\mathbb{Q} \in \mathcal{R}(t)} \mathbb{E}_{\mathbb{Q}} \left[\sum_{u=t+1}^T D_u^* \middle| \mathcal{F}_t \right], \quad (2.45)$$

$$\pi_t^{bid}(D) = \operatorname{ess\,inf}_{\mathbb{Q} \in \mathcal{R}(t)} \mathbb{E}_{\mathbb{Q}} \left[\sum_{u=t+1}^T D_u^* \middle| \mathcal{F}_t \right]. \quad (2.46)$$

Proof. Since $\pi_t^{ask}(D) = -\pi_t^{bid}(-D)$ holds for all $t \in \{0, \dots, T-1\}$ and derivative contracts D , it suffices to show that the essential infimum of $\mathcal{W}^a(t, D)$ is attained and (2.45) holds. Let us fix $t \in \{0, 1, \dots, T-1\}$ throughout the proof.

We first prove (i). Let W^m be a sequence decreasing a.s. to $\pi_t^{ask}(D)$, and fix $K^m \in \mathcal{K}(t)$ and $Z^m \in L_+^0(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R})$ so that $-W^m + \sum_{u=t+1}^T D_u^* = K^m - Z^m$. Since

a.s. converges implies convergence in probability, we see that the sequence $K^m - Z^m$ converges in probability to some Y . Due to Theorem 2.3.1, we have that $\mathcal{K}(t) - L_+^0(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R})$ is \mathbb{P} -closed. Therefore, $Y \in \mathcal{K}(t) - L_+^0(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R})$. This proves that $-\pi_t^{ask}(D) + \sum_{u=t+1}^T D_u^* \in \mathcal{K}(t) - L_+^0(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R})$.

Next, we show that (ii) holds. We begin by showing that

$$\pi_t^{ask}(D) \geq \operatorname{ess\,sup}_{\mathbb{Q} \in \mathcal{R}(t)} \mathbb{E}_{\mathbb{Q}} \left[\sum_{u=t+1}^T D_u^* \middle| \mathcal{F}_t \right]. \quad (2.47)$$

By (i), we have that $\pi_t^{ask}(D) \in \mathcal{W}^a(t, D)$, so there exists $K^* \in \mathcal{K}(t)$ so that

$$K^* + \pi_t^{ask}(D) - \sum_{u=t+1}^T D_u^* \geq 0. \quad (2.48)$$

We are assuming that **NAEF** is satisfied, so according to Theorem 2.4.1 there exists a risk-neutral measure \mathbb{Q}^* . Because any risk-neutral measure \mathbb{Q} satisfies $\mathbb{Q} \in \mathcal{R}(t)$, we obtain that $\mathbb{Q}^* \in \mathcal{R}(t)$. By taking the conditional expectation with respect to \mathcal{F}_t under \mathbb{Q}^* of both sides of the last inequality we deduce that

$$\pi_t^{ask}(D) + \mathbb{E}_{\mathbb{Q}^*} [K^* | \mathcal{F}_t] \geq \mathbb{E}_{\mathbb{Q}^*} \left[\sum_{u=t+1}^T D_u^* \middle| \mathcal{F}_t \right].$$

According to part (i) of Lemma 2.6.1, we have that $\mathbb{E}_{\mathbb{Q}^*} [K^* | \mathcal{F}_t] \leq 0$. As a result, $\pi_t^{ask}(D) \geq \mathbb{E}_{\mathbb{Q}^*} \left[\sum_{u=t+1}^T D_u^* \middle| \mathcal{F}_t \right]$. Taking the essential supremum of both sides of the last inequality over $\mathcal{R}(t)$ proves that (2.47) holds.

Next, we show that

$$\pi_t^{ask}(D) \leq \operatorname{ess\,sup}_{\mathbb{Q} \in \mathcal{R}(t)} \mathbb{E}_{\mathbb{Q}} \left[\sum_{u=t+1}^T D_u^* \middle| \mathcal{F}_t \right]. \quad (2.49)$$

By (i), we have that $\pi_t^{ask}(D) > -\infty$, so we may take $X \in L^0(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R})$ so that $\pi_t^{ask}(D) > X$. We now prove by contradiction that **NA** holds for $\mathcal{K}^b(t, X)$. Towards this aim, we assume that there exist $K \in \mathcal{K}(t)$, $\xi \in L_+^\infty(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R})$, and $\Omega^0 \subseteq \Omega$ with $\mathbb{P}(\Omega^0) > 0$ so that

$$K + \xi \left(X - \sum_{u=t+1}^T D_u^* \right) \geq 0 \quad \text{a.s.}, \quad K + \xi \left(X - \sum_{u=t+1}^T D_u^* \right) > 0 \quad \text{a.s. on } \Omega^0. \quad (2.50)$$

Since **NA** is satisfied for underlying market \mathcal{K} , we have from (2.50) that there exists $\Omega^1 \subseteq \Omega^0$ with $\mathbb{P}(\Omega^1) > 0$ such that $\Omega^1 \in \mathcal{F}_t$ and $\xi > 0$ a.s. on Ω^1 . Otherwise, our assumption that **NA** holds is contradicted. Of course, if $\Omega^1 \subseteq \Omega^0$ is any set such that $\Omega^1 \in \mathcal{F}_t$, $\mathbb{P}(\Omega^1) > 0$, and $\xi = 0$ a.s. on Ω^1 , then $1_{\Omega^1}K \in \mathcal{K}(t) \in \mathcal{K}$ satisfies $1_{\Omega^1}K \geq 0$ a.s., and $1_{\Omega^1}K > 0$ a.s. on Ω^1 , which violates that **NA** is satisfied.

Moreover, we observe that $X - \sum_{u=t+1}^T D_u^* \geq 0$ a.s. on Ω^1 . If there exists $\Omega^2 \subseteq \Omega^1$ with $\mathbb{P}(\Omega^2) > 0$ such that $\Omega^2 \in \mathcal{F}_t$ and $X - \sum_{u=t+1}^T D_u^* < 0$ a.s. on Ω^2 , then from (2.50) we see that $K \geq 0$ a.s., and $K > 0$ a.s. on Ω^2 , which contradicts that **NA** holds for \mathcal{K} .

Now, let us define

$$\tilde{X} := 1_{\Omega^1}X + 1_{(\Omega^1)^c}\pi_t^{ask}(D), \quad \tilde{K} := 1_{\Omega^1}\frac{K}{\sup_{\omega \in \Omega^1}\{\xi(\omega)\}} + 1_{(\Omega^1)^c}K^*.$$

From (2.48) we immediately have that

$$\tilde{K} + \tilde{X} - \sum_{u=t+1}^T D_u^* = K^* + \pi_t^{ask}(D) - \sum_{u=t+1}^T D_u^* \geq 0 \quad \text{a.s. on } (\Omega^1)^c.$$

On the other hand, from (2.50) and since $X - \sum_{u=t+1}^T D_u^* \geq 0$ a.s. on Ω^1 , we see that

$$\tilde{K} + \tilde{X} - \sum_{u=t+1}^T D_u^* = \frac{K}{\sup_{\omega \in \Omega^1}\{\xi(\omega)\}} + X - \sum_{u=t+1}^T D_u^* \geq 0 \quad \text{a.s. on } \Omega^1.$$

Consequently, $\tilde{K} + \tilde{X} - \sum_{u=t+1}^T D_u^* \geq 0$ a.s. on Ω . Now, since $0 \leq 1/\sup_{\omega \in \Omega^1}\{\xi(\omega)\} < \infty$, and because $\mathcal{K}(t)$ is a convex cone that is closed with respect to multiplication by random variables in $L_+^\infty(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R})$, we have that $\tilde{K} \in \mathcal{K}(t)$. Therefore $\tilde{X} \in \mathcal{W}^a(t, D)$. However, since \tilde{X} satisfies $\tilde{X} \leq \pi_t^{ask}(D)$ and $\mathbb{P}(\tilde{X} < \pi_t^{ask}(D)) > 0$, we have that $\tilde{X} \in \mathcal{W}^a(t, D)$ contradicts $\pi_t^{ask}(D) = \text{ess inf } \mathcal{W}^a(t, D)$. Thus, **NA** holds for $\mathcal{K}^b(t, X)$.

By assumption, **EF** holds for $\mathcal{K}^b(t, X)$, so **NAEF** is satisfied for $\mathcal{K}^b(t, X)$. According to Lemma 2.6.2 there exists $\hat{\mathbb{Q}} \in \mathcal{R}^b(t, X)$. In view of (iii) in Lemma 2.6.1,

we see that

$$\zeta X + \mathbb{E}_{\hat{\mathbb{Q}}}[K|\mathcal{F}_t] \leq \zeta \mathbb{E}_{\hat{\mathbb{Q}}}\left[\sum_{u=t+1}^T D_u^*|\mathcal{F}_t\right], \quad K \in \mathcal{K}(t), \zeta \in L_+^\infty(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R}).$$

Since $0 \in \mathcal{K}(t)$ and $1 \in L_+^\infty(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R})$, we obtain that $X \leq \mathbb{E}_{\hat{\mathbb{Q}}}\left[\sum_{u=t+1}^T D_u^*|\mathcal{F}_t\right]$.

Now, because $\mathcal{R}^b(t, X) \subseteq \mathcal{R}(t)$, we have that $\hat{\mathbb{Q}} \in \mathcal{R}(t)$. Hence,

$$X \leq \mathbb{E}_{\hat{\mathbb{Q}}}\left[\sum_{u=t+1}^T D_u^*|\mathcal{F}_t\right] \leq \sup_{\mathbb{Q} \in \mathcal{R}(t)} \mathbb{E}_{\mathbb{Q}}\left[\sum_{u=t+1}^T D_u^*|\mathcal{F}_t\right]. \quad (2.51)$$

The random variable $X < \pi_t^{ask}(D)$ is arbitrary, so for any scalar $\epsilon > 0$ we may take

$X := \pi_t^{ask}(D) - \epsilon$. From (2.51), we see that

$$\pi_t^{ask}(D) \leq \sup_{\mathbb{Q} \in \mathcal{R}(t)} \mathbb{E}_{\mathbb{Q}}\left[\sum_{u=t+1}^T D_u^*|\mathcal{F}_t\right] + \epsilon, \quad \epsilon > 0.$$

Letting ϵ approach zero shows that (2.49) holds. This completes the proof of (i). \square

CHAPTER 3

DYNAMIC CONIC FINANCE IN MARKETS WITH TRANSACTION COSTS

This chapter studies *dynamic conic finance* in the financial market model introduced in Chapter 2. We first formulate the no-good-deal condition and prove a version of the Fundamental Theorem of No-Good-Deal Pricing (FTNGDP). Then, we introduce the *no-good-deal prices ask and bid prices* of a derivative, and compute the no-good-deal ask and bid prices of European-style Asian options in a market with transaction costs.

We extensively use the results on dynamic acceptability indices that were obtained in Bielecki et al. [BCZ11]. Thus, we adopt the mathematical set-up that was used therein. In particular, we assume that the underlying probability space is finite, say $\Omega = \{\omega_1, \dots, \omega_N\}$. This is an assumption that indeed is made so to simplify the presentation, but the results can be extended to the case of a general probability space.

In Chapter 2, we worked with the set of values that can be superhedged at zero cost which we denoted by \mathcal{K} . In the context of DCAIs, it is more convenient to work with processes instead. With this in mind, we define the sets

$$\mathcal{S}(t) := \begin{cases} \{\phi : \phi \in \mathcal{S}, V_0(\phi) = 0\}, & t = 0 \\ \{\phi : \phi \in \mathcal{S}, \phi_s = 1_{\{s \geq t+1\}} \phi_s \text{ for all } s = 1, 2, \dots, T\}, & t \in \{1, \dots, T-1\}. \end{cases}$$

$$\mathcal{L}_+(t) := \left\{ (Z_s)_{s=0}^T : Z_s \in L_+(\Omega, \mathcal{F}_s, \mathbb{P}), Z_s = \mathbb{I}_{\{s \geq t+1\}} Z_s, s = 0, \dots, T \right\}, \quad (3.1)$$

$$\mathcal{H}(t) := \left\{ \left(0, \dots, 0, \Delta(V_{t+1}^*(\phi) - Z_{t+1}), \dots, \Delta(V_T^*(\phi) - Z_T) \right) : \phi \in \mathcal{S}(t), Z \in \mathcal{L}_+(t) \right\}, \quad (3.2)$$

for $t \in \{0, \dots, T-1\}$. Note that for any $H \in \mathcal{H}(t)$ there exists $\phi \in \mathcal{S}(t)$ and

$Z \in \mathcal{L}_+(t)$ so that

$$\sum_{s=t}^T H_s = V_T^*(\phi) - Z_T.$$

Therefore, the set $\mathcal{H}(t)$ is analogous to the set $\mathcal{K}(t) - L_+^0(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R})$ used in Chapter 2.

We proceed by defining the no-arbitrage condition for $\mathcal{H}(t)$.

Definition 3.0.2. *The no-arbitrage condition (NA) at time $t \in \{0, \dots, T-1\}$ for $\mathcal{H}(t)$ is satisfied if for each $H \in \mathcal{H}(t)$ satisfying $\sum_{s=t+1}^T H_s \geq 0$, we have $\mathbb{E}_{\mathbb{Q}}[\sum_{s=t+1}^T H_s | \mathcal{F}_t] = 0$.*

Next, we define a risk-neutral measure for $\mathcal{H}(t)$.

Definition 3.0.3. *For any fixed $t \in \{0, \dots, T-1\}$, we say that a probability measure \mathbb{Q} is risk-neutral for $\mathcal{H}(t)$ if $\mathbb{Q} \sim \mathbb{P}$, and if $\mathbb{E}_{\mathbb{Q}}[\sum_{s=t+1}^T H_s | \mathcal{F}_t] \leq 0$ for all $H \in \mathcal{H}(t)$. The set of all risk-neutral measures for $\mathcal{H}(t)$ will be denoted by $\mathcal{R}(\mathcal{H}(t))$.*

We will refer to $\mathcal{H}(t)$ as *set of hedging cash flows initiated at time t* . We proceed by proving that $\mathcal{H}(t)$ is a convex cone, which follows from Lemma 2.2.2.

Lemma 3.0.3. *The set $\mathcal{H}(t)$ is a convex cone for all $t \in \{0, \dots, T-1\}$.*

Proof. Fix $t \in \{0, \dots, T-1\}$, $H^1, H^2 \in \mathcal{H}(t)$ and $\lambda_1, \lambda_2 \geq 0$. By the definition of $\mathcal{H}(t)$, there exists $\phi, \psi \in \mathcal{S}(t)$ and $Z^1, Z^2 \in \mathcal{L}_+(t)$ such that

$$\begin{aligned} H^1 &= \left(0, \dots, 0, \Delta(V_{t+1}^*(\phi) - Z_{t+1}^1), \Delta(V_{t+2}^*(\phi) - Z_{t+2}^1), \dots, \Delta(V_T^*(\phi) - Z_T^1)\right), \\ H^2 &= \left(0, \dots, 0, \Delta(V_{t+1}^*(\psi) - Z_{t+1}^2), \Delta(V_{t+2}^*(\psi) - Z_{t+2}^2), \dots, \Delta(V_T^*(\psi) - Z_T^2)\right). \end{aligned}$$

According to Lemma 2.2.2, the set

$$\mathcal{K}(s) := \{V_s^*(\phi) - X : \phi \text{ is self-financing, } X \text{ is an } \mathcal{F}_s\text{-measurable r.v., and } X \geq 0\}$$

is a convex cone for all $s \in \mathcal{T}$. Let us define the process

$$Y := \lambda_1 \left(0, \dots, 0, V_{t+1}^*(\phi) - Z_{t+1}^1, V_{t+2}^*(\phi) - Z_{t+2}^1, \dots, V_T^*(\phi) - Z_T^1 \right) \\ + \lambda_2 \left(0, \dots, 0, V_{t+1}^*(\psi) - Z_{t+1}^2, V_{t+2}^*(\psi) - Z_{t+2}^2, \dots, V_T^*(\psi) - Z_T^2 \right).$$

Note that $\lambda_1 H^1 + \lambda_2 H^2 = \Delta Y$. By Proposition 2.2.4 there exists a unique predictable process θ^0 so that

- $\theta := (\theta^0, \lambda_1 \phi^1 + \lambda_2 \psi^1, \dots, \lambda_1 \phi^N + \lambda_2 \psi^N) \in \mathcal{S}(t)$,
- $V_0(\theta) = 0$,
- $\lambda_1 V^*(\phi) + \lambda_2 V^*(\psi) = V^*(\theta) - Z^3$, for some $Z^3 \in \mathcal{L}_+(t)$.

Hence,

$$\lambda_1 (V^*(\phi) - Z^1) + \lambda_2 (V^*(\psi) - Z^2) = V^*(\theta) - Z,$$

where $Z := -\lambda_1 Z^1 - \lambda_2 Z^2 - Z^3$.

Therefore, we have that

$$Y = (0, \dots, 0, V_{t+1}^*(\theta) - Z_{t+1}, V_{t+2}^*(\theta) - Z_{t+2}, \dots, V_T^*(\theta) - Z_T),$$

and hence

$$\Delta Y = (0, \dots, 0, \Delta(V_{t+1}^*(\theta) - Z_{t+1}), \Delta(V_{t+2}^*(\theta) - Z_{t+2}), \dots, \Delta(V_T^*(\theta) - Z_T))$$

for some $\theta \in \mathcal{S}(t)$ and some $Z \in \mathcal{L}_+(t)$. We conclude that $\lambda_1 H^1 + \lambda_2 H^2 \in \mathcal{H}(t)$. \square

The next result follows from Theorem 2.4.1.

Proposition 3.0.1. *If $\mathcal{R}(\mathcal{H}(t)) \neq \emptyset$, then the no-arbitrage condition holds true at time $t \in \{0, \dots, T-1\}$ for $\mathcal{H}(t)$.*

Next, we define the notion of no-arbitrage bounds.

Definition 3.0.4. Let $t \in \{0, \dots, T-1\}$.

- A set of extended cash flows associated with an \mathcal{F}_t -measurable random variable S_t and a process $D \in L^0$ is defined as

$$\begin{aligned} \tilde{\mathcal{H}}(t, S_t) := & \left\{ \left(0, \dots, 0, \xi_t S_t, H_{t+1} - \xi_t D_{t+1}^*, \dots, H_T - \xi_t D_T^* \right) \right. \\ & \left. : H \in \mathcal{H}(t), \xi_t \text{ is an } \mathcal{F}_t\text{-measurable r.v.} \right\}. \end{aligned}$$

- The pricing interval associated with a process $D \in L^0$ and a set of probability measures \mathcal{X} is defined as

$$\mathcal{I}(t, D; \mathcal{R}(\mathcal{H}(t))) := \left\{ \mathbb{E}^{\mathbb{Q}} \left[\sum_{s=t+1}^T D_s^* \middle| \mathcal{F}_t \right] : \mathbb{Q} \in \mathcal{X} \right\}.$$

A cash flow in $\tilde{\mathcal{H}}(t, S_t)$ is interpreted as the sum of a position in $\mathcal{H}(t)$ and a *static* position of ξ_t units in the discounted cash flow $(0, \dots, 0, S_t, -D_{t+1}^*, \dots, -D_T^*)$.

We will say that $\mathcal{I}(t, D; \mathcal{X})$ is a *risk-neutral pricing interval* if it is nonempty, and if for each $S_t \in \mathcal{I}(t, D; \mathcal{X})$ the no-arbitrage condition is satisfied for $\tilde{\mathcal{H}}(t, S_t)$. That is, $\mathcal{I}(t, D; \mathcal{X})$ is a risk-neutral pricing interval if it is nonempty, and if for each $S_t \in \mathcal{I}(t, D; \mathcal{X})$ and each $\tilde{H} \in \tilde{\mathcal{H}}(t, S_t)$ such that $\sum_{s=t+1}^T \tilde{H}_s \geq 0$, we have $\sum_{s=t+1}^T \tilde{H}_s = 0$. If $\mathcal{I}(t, D; \mathcal{X})$ is a risk-neutral pricing interval, we call any $S_t \in \mathcal{I}(t, D; \mathcal{X})$ a *risk-neutral price*, $\sup_{\mathbb{Q} \in \mathcal{R}(\mathcal{H}(t))} \mathbb{E}_t^{\mathbb{Q}} \left[\sum_{s=t+1}^T D_s^* \right]$ the *upper no-arbitrage bound*, and $\inf_{\mathbb{Q} \in \mathcal{R}(\mathcal{H}(t))} \mathbb{E}_t^{\mathbb{Q}} \left[\sum_{s=t+1}^T D_s^* \right]$ the *lower no-arbitrage bound*.

The following lemma gives a necessary condition for $\mathcal{I}(t, D, \mathcal{X})$ to be a risk-neutral pricing interval.

Lemma 3.0.4. Let $t \in \{0, \dots, T-1\}$ and $D \in L^0$. If $\mathcal{R}(\mathcal{H}(t)) \neq \emptyset$, then $\mathcal{I}(t, D; \mathcal{R}(\mathcal{H}(t)))$ is a risk-neutral pricing interval.

Proof. Fix $t \in \{0, \dots, T-1\}$, $D \in L^0$, and $S_t \in \mathcal{I}(t, D; \mathcal{R}(\mathcal{H}(t)))$. Let $\tilde{H} \in \tilde{\mathcal{H}}(t, S_t)$

be a cash flow such that $\sum_{s=t+1}^T \tilde{H}_s \geq 0$. By definition of $\tilde{\mathcal{H}}(t, S_t)$, we have that

$$\xi_t S_t + \sum_{s=t+1}^T (H_s - \xi_t D_s^*) \geq 0 \quad (3.3)$$

for some $H \in \mathcal{H}(t)$ and some \mathcal{F}_t -measurable random variable ξ_t .

Now, since $\mathcal{R}(\mathcal{H}(t)) \neq \emptyset$ and $S_t \in \mathcal{I}(t, D; \mathcal{R}(\mathcal{H}(t)))$, there exists $\mathbb{Q} \in \mathcal{R}(\mathcal{H}(t))$ such that $S_t = \mathbb{E}_t^{\mathbb{Q}}[\sum_{s=t+1}^T D_s^*]$. It follows that $\xi_t \mathbb{E}_t^{\mathbb{Q}}[\sum_{s=t+1}^T D_s^*] - \xi_t S_t = 0$. From (3.3) we see that $\mathbb{E}_t^{\mathbb{Q}}[\sum_{s=t+1}^T H_s] \geq 0$ holds. Since $\mathbb{Q} \in \mathcal{R}(\mathcal{H}(t))$, we have that $\mathbb{E}_t^{\mathbb{Q}}[\sum_{s=t+1}^T H_s] = 0$, which gives us that

$$\xi_t S_t + \mathbb{E}_t^{\mathbb{Q}}\left[\sum_{s=t+1}^T (H_s - \xi_t D_s^*)\right] = 0. \quad (3.4)$$

Equations (3.3) and (3.4) allow us to conclude that $\xi_t S_t + \sum_{s=t+1}^T (H_s - \xi_t D_s^*) = 0$, which implies that the no-arbitrage condition holds for $\tilde{\mathcal{H}}(t, S_t)$. \square

3.1 The No-Good-Deal Condition

It was shown in [BCZ11] that any Dynamic Coherent Acceptability Index α can be associated with a left-continuous, increasing family of Dynamic Coherent Risk Measures (DCRMs) $(\rho^\gamma)_{\{\gamma>0\}}$, and consequently to a family of dynamically consistent sequences of sets of probability measures (see Appendix B for definitions and related results.) In what follows, we fix such a family of DCRMs $(\rho^\gamma)_{\{\gamma>0\}}$, and denote by $\mathcal{Q} = \left(\left(\mathcal{Q}_t^\gamma\right)_{t \in \mathcal{T}}\right)_{\gamma>0}$ the corresponding family of dynamically consistent sequences of sets of probability measures.

Definition 3.1.1. *A good-deal for $\mathcal{H}(t)$ at time $t \in \{0, \dots, T-1\}$ and level $\gamma > 0$ is a cash flow $H \in \mathcal{H}(t)$ such that $\rho_t^\gamma(H)(\omega) < 0$ for some $\omega \in \Omega$.*

Note that a good-deal depends on the family of DCRMs and the level γ . A cash flow that is a good-deal with respect to a family of DCRMs might not be a

good-deal with respect to another family of DCRMs. Also, note that, for a fixed family of DCRMs, a cash flow stream that is a good-deal at level γ_0 might not be a good-deal at some other level $\gamma' > \gamma_0$. Although, since ρ^γ is monotone increasing in γ , if a cash flow is a good-deal for γ_0 , then it will also be a good deal for any level $\gamma' \leq \gamma_0$.

Definition 3.1.2. *The no-good-deal condition (NGD) is satisfied for $\mathcal{H}(t)$ at time $t \in \{0, \dots, T-1\}$ and level $\gamma > 0$ if $\rho_t^\gamma(H) \geq 0$ for all $H \in \mathcal{H}(t)$.*

In view of Theorem B.0.2, if NDG is satisfied at time t and level γ , then $\alpha_t(H) < \gamma$ for all $H \in \mathcal{H}(t)$. Therefore, there are no cash flows with acceptability level larger than γ at time t whenever NGD is satisfied at time t for $\mathcal{H}(t)$.

We will make the following technical assumption on \mathcal{Q} .

Assumption (B): We assume that, for each $\gamma > 0$ and $t \in \mathcal{T}$, any probability measure $\mathbb{Q} \in \mathcal{Q}_t^\gamma$ is equivalent to \mathbb{P} , and the set

$$\mathcal{E}_t^\gamma := \left\{ \frac{d\mathbb{Q}}{d\mathbb{P}} : \mathbb{Q} \in \mathcal{Q}_t^\gamma \right\}$$

is closed and convex.

Since it is assumed that Ω is finite and \mathbb{P} is of full support, the set \mathcal{E}_t^γ is bounded. Hence, \mathcal{E}_t^γ is compact for all $\gamma > 0$ and $t \in \mathcal{T}$. In Section 3.4.2, we show that a family of densities \mathcal{E} corresponding to the dynamic Gain-Loss Ratio satisfies this assumption.

3.2 The Fundamental Theorem of No-Good-Deal Pricing

A practitioner using a specific financial market model may want to verify whether a financial market model satisfies **NGD** for some time t and level γ . However, it is not easy to check whether $\rho_t^\gamma(H)(\omega) \geq 0$ holds for all $H \in \mathcal{H}(t)$ and $\omega \in \Omega$. Thus, it is very useful to have a condition that is equivalent to **NGD**, and that

is straightforward to verify. The Fundamental Theorem of No-Good-Deal Pricing (FTNGP) offers precisely this.

Theorem 3.2.1 (The Fundamental Theorem of No-Good-Deal Pricing). *The no-good-deal condition (NGD) is satisfied for $\mathcal{H}(t)$ at time $t \in \{0, \dots, T-1\}$ and level $\gamma > 0$ if and only if $\mathcal{R}(\mathcal{H}(t)) \cap \mathcal{Q}_t^\gamma \neq \emptyset$.*

Proof. Let us fix $t \in \{0, \dots, T-1\}$ and $\gamma > 0$ throughout the proof.

Suppose that $\mathbb{Q} \in \mathcal{R}(\mathcal{H}(t)) \cap \mathcal{Q}_t^\gamma$. By definition, we have $\mathbb{E}_{\mathbb{Q}}[\sum_{s=t+1}^T H_s | \mathcal{F}_t] \leq 0$ for all $H \in \mathcal{H}(t)$. Due to Theorem B.0.3 (Robust Representation of DCRM), we have that

$$-\rho_t^\gamma(H) = \inf_{\mathbb{Q} \in \mathcal{Q}_t^\gamma} \mathbb{E}_{\mathbb{Q}} \left[\sum_{s=t+1}^T H_s | \mathcal{F}_t \right] \leq \mathbb{E}_{\mathbb{Q}} \left[\sum_{s=t+1}^T H_s | \mathcal{F}_t \right] \leq 0,$$

for all $H \in \mathcal{H}(t)$. Thus, $\rho_t^\gamma(H) \geq 0$ for any $H \in \mathcal{H}(t)$, and hence **NGD** is satisfied for $\mathcal{H}(t)$ at time t and level γ .

Let us now prove the converse. Fix $M \in \mathbb{N}$ and $H := (H^1, H^2, \dots, H^M) \in \mathcal{H}(t) \times \mathcal{H}(t) \times \dots \times \mathcal{H}(t)$. Let \mathcal{E}_t^γ be the set defined in Assumption (B), and let us consider the set of matrices

$$\mathcal{Z}_t(H) := \left\{ \left[\mathbb{E}_{\mathbb{P}} \left[\eta \sum_{s=t+1}^T H_s^i | \mathcal{F}_t \right] (\omega_j) \right]_{j=1, \dots, N; i=1, \dots, M} : \eta \in \mathcal{E}_t^\gamma \right\} \subset \mathbb{R}^{N \times M}.$$

Since \mathcal{E}_t^γ is compact, by continuity of the mapping

$$\mathcal{E}_t^\gamma \ni \eta \mapsto \mathbb{E}_{\mathbb{P}} \left[\eta \sum_{s=t+1}^T H_s^i | \mathcal{F}_t \right] (\omega_j), \quad i = 1, 2, \dots, M; j = 1, 2, \dots, N,$$

we conclude that $\mathcal{Z}_t(H)$ is compact in $\mathbb{R}^{N \times M}$. Also note that, by convexity of \mathcal{E}_t^γ and linearity of conditional expectations above w.r.t. η , the set $\mathcal{Z}_t(H)$ is convex.

Next, we prove by contradiction that the closed, convex set $\mathcal{C} := (-\infty, 0]^{N \times M} \subseteq \mathbb{R}^{N \times M}$ satisfies $\mathcal{Z}_t(H) \cap \mathcal{C} \neq \emptyset$. Towards this end let us assume

that $\mathcal{Z}_t(H) \cap \mathcal{C} = \emptyset$. By a separation theorem (see Theorem A.0.1), there exists a linear functional $\varphi^{t,H} \in \mathbb{R}^{N \times M}$, and $\epsilon_{t,H} > 0$ such that

$$\epsilon_{t,H} \leq \varphi^{t,H}(x), \quad (3.5)$$

$$\varphi^{t,H}(z) \leq 0, \quad (3.6)$$

for all $x \in \mathcal{Z}_t(H)$, $z \in \mathcal{C}$. From the Riesz representation theorem, there exists $h^{t,H} \in \mathbb{R}^{N \times M}$ such that $\varphi^{t,H}(x) = \langle h^{t,H}, x \rangle$ for all $x \in \mathbb{R}^{N \times M}$, where $\langle x, y \rangle := \sum_{j=1}^N \sum_{i=1}^M x_{ij} y_{ij}$ for all $x \in \mathbb{R}^{N \times M}$, $y \in \mathbb{R}^{N \times M}$ denotes the Frobenius inner product in $\mathbb{R}^{N \times M}$. From (3.6), we have that $\langle h^{t,H}, z \rangle \leq 0$ for all $z \in \mathcal{C}$, and therefore, $h_{ij}^{t,H} \geq 0$ for $i = 1, \dots, M$ and $j = 1, \dots, N$. Since, in view of (3.5) we have that $h^{t,H} \neq 0$, we may assume without loss of generality that $\sum_{i=1}^M h_{ij}^{t,H} = 1$. Also in view of (3.5), we deduce that

$$0 < \epsilon_{t,H} \leq \sum_{j=1}^N \sum_{i=1}^M h_{ij}^{t,H} \mathbb{E}_{\mathbb{P}} \left[\eta \sum_{s=t+1}^T H_s \middle| \mathcal{F}_t \right] (\omega_j) = \sum_{j=1}^N \mathbb{E}_{\mathbb{P}} \left[\eta \sum_{s=t+1}^T \tilde{H}_s(j) \middle| \mathcal{F}_t \right] (\omega_j)$$

for all $\eta \in \mathcal{E}_t^\gamma$, where $\tilde{H}(j) := \sum_{i=1}^M h_{ij}^{t,H} H^i$ for $j = 1, \dots, N$. Therefore, there exists $j \in \{1, \dots, N\}$ and an $\epsilon > 0$ so that

$$0 < \epsilon < \mathbb{E}_{\mathbb{P}} \left[\eta \sum_{s=t+1}^T \tilde{H}_s(j) \middle| \mathcal{F}_t \right] (\omega_j).$$

Let us define

$$\epsilon' := \inf_{\eta \in \mathcal{E}_t^\gamma} \frac{\epsilon}{\mathbb{E}_{\mathbb{P}}[\eta | \mathcal{F}_t](\omega_j)}.$$

Since $\eta > 0$ and $\sup_{\eta \in \mathcal{E}_t^\gamma} \mathbb{E}_{\mathbb{P}}[\eta | \mathcal{F}_t](\omega_j) < \infty$, it follows that

$$0 < \epsilon' \leq \frac{\mathbb{E}_{\mathbb{P}} \left[\eta \sum_{s=t+1}^T \tilde{H}_s(j) \middle| \mathcal{F}_t \right] (\omega_j)}{\mathbb{E}_{\mathbb{P}}[\eta | \mathcal{F}_t](\omega_j)} = \mathbb{E}_{\mathbb{Q}} \left[\sum_{s=t+1}^T \tilde{H}_s(j) \middle| \mathcal{F}_t \right] (\omega_j)$$

for all $\mathbb{Q} \in \mathcal{Q}_t^\gamma$. Consequently, taking infimum with respect to $\mathbb{Q} \in \mathcal{Q}_t^\gamma$ and applying Theorem B.0.3, we get

$$0 < \epsilon' \leq -\rho_t^\gamma(\tilde{H}(j))(\omega_j). \quad (3.7)$$

By Lemma 3.0.3, the set $\mathcal{H}(t)$ is a convex cone, hence $\tilde{H}(j) \in \mathcal{H}(t)$. Thus, in view of (3.7), the cash flow $\tilde{H}(j) \in \mathcal{H}(t)$ violates **NGD** for $\mathcal{H}(t)$ at time t and level γ , which is a contradiction. Hence, $\mathcal{Z}_t(H) \cap \mathcal{C} \neq \emptyset$ for all $t \in \{0, \dots, T-1\}$ and $H \in \mathcal{H}(t) \times \dots \times \mathcal{H}(t)$. Consequently, for each $t \in \mathcal{T}$, $H \in \mathcal{H}(t) \times \dots \times \mathcal{H}(t)$, the set

$$\Gamma_t(H) := \left\{ \eta \in \mathcal{E}_t^\gamma : \mathbb{E}_{\mathbb{P}} \left[\eta \sum_{s=t+1}^T H_s^i \middle| \mathcal{F}_t \right] (\omega_j) \leq 0, i = 1, 2, \dots, M, j = 1, 2, \dots, N \right\}$$

is nonempty.

Next, we construct a risk-neutral measure for $\mathcal{H}(t)$ that is in \mathcal{Q}_t^γ . Let us define the following mapping

$$\Psi_{t,H}(\zeta) := \left[\mathbb{E}_{\mathbb{P}} \left[\zeta \sum_{s=t+1}^T H_s^i \middle| \mathcal{F}_t \right] (\omega_j) \right]_{j=1, \dots, N; i=1, \dots, M},$$

for any random variable $\zeta : \Omega \rightarrow \mathbb{R}$. Since

$$\begin{aligned} \mathcal{Z}_t(H) \cap \mathcal{C} = & \left\{ \left[\mathbb{E}_{\mathbb{P}} \left[\eta \sum_{s=t+1}^T H_s^i \middle| \mathcal{F}_t \right] (\omega_j) \right]_{j=1, \dots, N; i=1, \dots, M} \right. \\ & \left. : \eta \in \mathcal{E}_t^\gamma, \mathbb{E}_{\mathbb{P}} \left[\eta \sum_{s=t+1}^T H_s^i \middle| \mathcal{F}_t \right] (\omega_j) \leq 0, i = 1, 2, \dots, M, j = 1, 2, \dots, N \right\}, \end{aligned}$$

we have that $\Psi_{t,H}^{-1}(\mathcal{Z}_t(H) \cap \mathcal{C}) = \Gamma_t(H)$. Recall that $\mathcal{Z}_t(H)$ is compact and hence $\mathcal{Z}_t(H) \cap \mathcal{C}$ is closed, and since $\Psi_{t,H}$ is continuous, we conclude that $\Gamma_t(H)$ is closed.

Now, note that

$$\Gamma_t(H) = \bigcap_{i=1}^M \left\{ \eta \in \mathcal{E}_t^\gamma : \mathbb{E}_{\mathbb{P}} \left[\eta \sum_{s=t+1}^T H_s^i \middle| \mathcal{F}_t \right] (\omega_j) \leq 0 \text{ for all } j = 1, 2, \dots, N \right\} \neq \emptyset.$$

Therefore, the family of subsets

$$\left\{ \eta \in \mathcal{E}_t^\gamma : \mathbb{E}_{\mathbb{P}} \left[\eta \sum_{s=t+1}^T H_s \middle| \mathcal{F}_t \right] (\omega_j) \leq 0 \text{ for all } j = 1, 2, \dots, N \right\}_{H \in \mathcal{H}(t)} \subseteq \mathcal{E}_t^\gamma$$

satisfies the finite intersection property¹⁴. Since \mathcal{E}_t^γ is compact, we have by

¹⁴The family of sets $\{Y_i\}_{i \in \mathcal{I}}$ has finite intersection property if $\bigcap_{i \in \mathcal{I}'} Y_i$ is non-empty for any finite $\mathcal{I}' \subset \mathcal{I}$.

Lemma A.0.9 that the set

$$U_t := \bigcap_{H \in \mathcal{H}(t)} \left\{ \eta \in \mathcal{E}_t^\gamma : \mathbb{E}_\mathbb{P} \left[\eta \sum_{s=t+1}^T H_s \middle| \mathcal{F}_t \right] \leq 0 \right\} \quad (3.8)$$

is nonempty. Therefore, there exists an $\hat{\eta} \in \mathcal{E}_t^\gamma$ so that $\mathbb{E}_\mathbb{P}[\hat{\eta} \sum_{s=t+1}^T H_s | \mathcal{F}_t](\omega) \leq 0$ for all $\omega \in \Omega$ and $H \in \mathcal{H}(t)$. Now, let $\hat{\mathbb{Q}}$ be a measure corresponding to $\hat{\eta}$, so that $\hat{\mathbb{Q}} \in \mathcal{Q}_t^\gamma$. Using the abstract version of Bayes rule applied to $\hat{\mathbb{Q}}$ we get

$$\mathbb{E}_{\hat{\mathbb{Q}}} \left[\sum_{s=t+1}^T H_s \middle| \mathcal{F}_t \right] = \frac{\mathbb{E}_\mathbb{P}[\hat{\eta} \sum_{s=t+1}^T H_s | \mathcal{F}_t]}{\mathbb{E}_\mathbb{P}[\hat{\eta} | \mathcal{F}_t]} \leq 0$$

for all $H \in \mathcal{H}(t)$. So we see that $\hat{\mathbb{Q}} \in \mathcal{R}(\mathcal{H}(t))$. Thus, $\mathcal{R}(\mathcal{H}(t)) \cap \mathcal{Q}_t^\gamma \neq \emptyset$. This proves the theorem. \square

Since $\mathcal{R}(\mathcal{H}(t)) \cap \mathcal{Q}_t^\gamma \neq \emptyset$ implies $\mathcal{R}(\mathcal{H}(t)) \neq \emptyset$, it is immediate from Proposition 3.0.1 and Theorem 3.2.1 that if **NGD** holds, then the no-arbitrage for $\mathcal{H}(t)$ also holds.

3.3 No-Good-Deal Ask and Bid Prices

In this section, we recall from Bielecki et al. [BCIR12] one of the main objects of dynamic conic finance: *the no-good-deal* (NGD) ask and bid prices corresponding to a given DCAI α . In what follows, we will denote by $\mathcal{L}^0 := L^0(\Omega, \mathcal{F}_T, (\mathcal{F}_t)_{t \in T}, \mathbb{P})$ the set of all \mathbb{F} -adapted processes. Recall from Chapter 2 that a derivative contract is any adapted process $D \in \mathcal{L}^0$.

Definition 3.3.1. *The discounted no-good-deal (NGD) ask and bid prices of a derivative contract $D \in \mathcal{L}^0$, at level $\gamma > 0$, at time $t \in \{1, \dots, T-1\}$ are defined as*

$$\Pi_t^{ask, \gamma}(D)(\omega) := \inf \{ v \in \mathbb{R} : \exists H \in \mathcal{H}(t) \text{ s.t. } \alpha_t(\delta_t(\mathbf{1}v) + H - \delta_t^+(D^*))(\omega) \geq \gamma \},$$

$$\Pi_t^{bid, \gamma}(D)(\omega) := \sup \{ v \in \mathbb{R} : \exists H \in \mathcal{H}(t) \text{ s.t. } \alpha_t(\delta_t^+(D^*) + H - \delta_t(\mathbf{1}v))(\omega) \geq \gamma \},$$

for all $\omega \in \Omega$.

The discounted NGD ask price $\Pi_t^{ask,\gamma}(D)$ can be interpreted as the minimum amount of cash v such that v plus the resulting hedging error acceptable (in the sense of acceptability index α) at least at level γ . Similarly, the discounted NGD bid price $\Pi_t^{bid,\gamma}(D)$ can be viewed as the maximum amount of cash v such that $-v$ plus the resulting hedging error is α -acceptable at least at level γ .

A natural question is then: how should γ be chosen to find the NGD prices of a derivative contract? As in Cherny and Madan [CM10] and Madan and Schoutens [MS11a, MS11b], for a given α , the level γ can be calibrated from similar securities. Then, using this γ , the NGD prices are computed for the derivative contract.

Remark 3.3.1. *We note that the NGD prices depend on the choice of DCAI α , level γ , and the set of hedging cash flows $\mathcal{H}(t)$. First, we see that the NGD ask (bid) price is non-decreasing (non-increasing) in γ from the monotonicity property of DCAIs (D3). Secondly, the NGD ask (bid) price is non-increasing (non-decreasing) in $\mathcal{H}(t)$ since*

$$\begin{aligned}\Pi_t^{ask,\gamma}(D)(\omega) &= \inf \bigcup_{H \in \mathcal{H}(t)} \{v \in \mathbb{R} : \alpha_t(\delta_t(\mathbf{1}v) + H - \delta_t^+(D^*))(\omega) \geq \gamma\}, \\ \Pi_t^{bid,\gamma}(D)(\omega) &= \sup \bigcup_{H \in \mathcal{H}(t)} \{v \in \mathbb{R} : \alpha_t(\delta_t^+(D^*) + H - \delta_t(\mathbf{1}v))(\omega) \geq \gamma\}\end{aligned}$$

for all $\omega \in \Omega$.

Remark 3.3.2. *By Theorem B.0.4, we have that*

$$\begin{aligned}&\alpha_t(\delta_t(\mathbf{1}v) + H - \delta_t^+(D^*))(\omega) \\ &= \sup \left\{ \gamma \in (0, \infty) : v + \inf_{\mathbb{Q} \in \mathcal{Q}_t^\gamma} \mathbb{E}_{\mathbb{Q}} \left[\sum_{s=t+1}^T H_s - D_s^* | \mathcal{F}_t \right] (\omega) \geq 0 \right\},\end{aligned}$$

for all $\omega \in \Omega$, $t \in \{1, \dots, T-1\}$, and $D \in \mathcal{L}^0$. Since the cash flows D^* and $H \in \mathcal{H}(t)$ are discounted, the prices $\Pi^{ask,\gamma}(D)$ and $\Pi^{bid,\gamma}(D)$ are also discounted. We took the liberty to denote them by $\Pi^{ask,\gamma}(D)$ and $\Pi^{bid,\gamma}(D)$ rather than $\Pi^{ask,\gamma,*}(D)$ and $\Pi^{bid,\gamma,*}(D)$ (which would agree with earlier notation) to ease exposition.

The following proposition guarantees that the infimum and supremum appearing in Definition 3.3.1 are taken over non-empty sets (although $\Pi_t^{ask,\gamma}$, $\Pi_t^{bid,\gamma}$ can still be $-\infty, \infty$, respectively).

Proposition 3.3.1. *For any fixed $t \in \{1, \dots, T-1\}$, $D \in \mathcal{L}^0$, and $\gamma > 0$, the sets*

$$\{v \in \mathbb{R} : \exists H \in \mathcal{H}(t) \text{ such that } \alpha_t(\delta_t(\mathbf{1}v) + H - \delta_t^+(D^*))(\omega) \geq \gamma\},$$

$$\{v \in \mathbb{R} : \exists H \in \mathcal{H}(t) \text{ such that } \alpha_t(\delta_t^+(D^*) + H - \delta_t(\mathbf{1}v))(\omega) \geq \gamma\}$$

are nonempty for all $\omega \in \Omega$.

Proof. Let us fix $t \in \{1, \dots, T-1\}$, $D \in \mathcal{L}^0$, and $\gamma > 0$. We prove by contradiction.

Suppose that

$$\alpha_t(\delta_t(\mathbf{1}v) + H - \delta_t^+(D^*)) < \gamma$$

for all $v \in \mathbb{R}$ and $H \in \mathcal{H}(t)$. By Theorem B.0.4, we have that

$$\alpha_t(\delta_t(\mathbf{1}v) + H - \delta_t^+(D^*))(\omega) = \sup \left\{ \beta \in (0, +\infty) : \right. \\ \left. v + \inf_{\mathbb{Q} \in \mathcal{Q}_t^\beta} \mathbb{E}_{\mathbb{Q}} \left[\sum_{s=t+1}^T H_s - D_s \middle| \mathcal{F}_t \right] (\omega) \geq 0 \right\} < \gamma$$

for all $v \in \mathbb{R}$ and $H \in \mathcal{H}(t)$. Since α is normalized, there exists $D' \in \mathcal{L}^0$ such that $\alpha_t(D') = +\infty$. Let us define v^* as the scalar

$$v^* := \sup_{\omega \in \Omega} \sup_{H \in \mathcal{H}(t)} \left\{ \sup_{\mathbb{Q} \in \mathcal{Q}_t^\beta} \mathbb{E}_{\mathbb{Q}} \left[\sum_{s=t+1}^T D'_s \middle| \mathcal{F}_t \right] (\omega) - \inf_{\mathbb{Q} \in \mathcal{Q}_t^\beta} \mathbb{E}_{\mathbb{Q}} \left[\sum_{s=t+1}^T H_s - D_s \middle| \mathcal{F}_t \right] (\omega) \right\}.$$

Then, we see that

$$v^* + \mathbb{E}_{\mathbb{Q}} \left[\sum_{s=t+1}^T H_s - D_s \middle| \mathcal{F}_t \right] (\omega) \geq \mathbb{E}_{\mathbb{Q}} \left[\sum_{s=t+1}^T D'_s \middle| \mathcal{F}_t \right] (\omega),$$

for all $\mathbb{Q} \in \mathcal{Q}_t^\gamma$, $\omega \in \Omega$, and $H \in \mathcal{H}(t)$. From the monotonicity property of α , we have that

$$\alpha_t(\delta_t(\mathbf{1}v^*) + H - \delta_t^+(D^*)) \geq \alpha_t(D') = +\infty,$$

which contradicts $\alpha_t(\delta_t(\mathbf{1}v) + H - \delta_t^+(D^*))(\omega) < \gamma$ for all $v \in \mathbb{R}$. \square

We now turn our attention to a representation theorem for the discounted NGD ask and bid prices. First, let us make a third standing assumption, which will make use of in the proof. In Proposition 3.4.3, we will show that the dynamic Gain-Loss Ratio satisfies this assumption.

Assumption (C): The mapping $\gamma \mapsto \rho^\gamma$ is continuous.

The next result shows that the prices $\Pi_t^{ask}(D)$ and $\Pi_t^{bid}(D)$ have useful representations in terms of the sets $\mathcal{R}(\mathcal{H}(t))$ and $\mathcal{Q}_t^\gamma(\mathcal{H}(t))$. The proof can be found in Bielecki et al. [BCIR12].

Theorem 3.3.1. *The discounted NGD ask and bid prices of a derivative contract $D \in \mathcal{L}^0$, at level $\gamma > 0$, at time $t \in \{1, \dots, T-1\}$ satisfy*

$$\begin{aligned}\Pi_t^{ask,\gamma}(D) &= \sup_{\mathbb{Q} \in \mathcal{Q}_t^\gamma \cap \mathcal{R}(\mathcal{H}(t))} \mathbb{E}_{\mathbb{Q}} \left[\sum_{s=t+1}^T D_s^* \middle| \mathcal{F}_t \right], \\ \Pi_t^{bid,\gamma}(D) &= \inf_{\mathbb{Q} \in \mathcal{Q}_t^\gamma \cap \mathcal{R}(\mathcal{H}(t))} \mathbb{E}_{\mathbb{Q}} \left[\sum_{s=t+1}^T D_s^* \middle| \mathcal{F}_t \right].\end{aligned}$$

Let us now make a few remarks regarding Theorem 3.3.1.

Remark 3.3.3. *If NGD is not satisfied for $\mathcal{H}(t)$, at time $t \in \{1, \dots, T-1\}$, at level γ , then*

$$\begin{aligned}\Pi_t^{ask,\gamma}(D)(\omega) &= -\infty, \\ \Pi_t^{bid,\gamma}(D)(\omega) &= \infty,\end{aligned}$$

for all $\omega \in \Omega$ and $D \in \mathcal{L}^0$.

In the next remark, we treat the case in which the markets are frictionless and complete.

Remark 3.3.4. *If, for $t \in \{1, \dots, T-1\}$, the set of hedging cash flows $\mathcal{H}(t)$ satisfies the no-arbitrage condition, and $\mathcal{H}(T-1)$ is complete (for any $D \in \mathcal{L}^0$, there exists*

$H \in \mathcal{H}(T-1)$ so that $H_T = D_T$), then it follows from the Fundamental Theorems of Asset Pricing that $\mathcal{R}(\mathcal{H}(t)) \neq \emptyset$, for $t = 1, 2, \dots, T-2$, and $\mathcal{R}(\mathcal{H}(T-1)) = \{\mathbb{Q}^*\}$. Since $\mathcal{R}(\mathcal{H}(0)) \subseteq \dots \subseteq \mathcal{R}(\mathcal{H}(T-1))$, we have that $\mathcal{R}(\mathcal{H}(t)) = \{\mathbb{Q}^*\} \neq \emptyset$ for $t = 0, 1, \dots, T-2$. By Theorems 3.2.1 and 3.3.1, if **NGD** holds then the NGD ask and bid prices of a derivative contract $D \in \mathcal{L}^0$, at time $t \in \mathcal{T}$ and level $\gamma >$, satisfy

$$\Pi_t^{ask, \gamma}(D) = \Pi_t^{bid, \gamma}(D) = \mathbb{E}_{\mathbb{Q}^*} \left[\sum_{s=t+1}^T D_s^* \middle| \mathcal{F}_t \right].$$

Notice that, naturally, the NGD prices no longer depend on the acceptance level γ .

The next remark treats the case in which there are no nonzero hedging cash flows.

Remark 3.3.5. *If for some $t \in \{1, \dots, T-1\}$, we have that $\mathcal{Q}_t^\gamma \neq \emptyset$ and $\mathcal{H}(t) = \{0\}$, then we have $\mathcal{R}(\mathcal{H}(t)) = \{\mathbb{Q} : \mathbb{Q} \sim \mathbb{P}\}$, so $\mathcal{Q}_t^\gamma \subseteq \mathcal{R}(\mathcal{H}(t))$. In this case the NGD ask and bid prices of a derivative contract $D \in \mathcal{L}^0$, at time $t \in \mathcal{T}$ and level $\gamma > 0$, satisfy*

$$\begin{aligned} \Pi_t^{ask, \gamma}(D) &= \sup_{\mathbb{Q} \in \mathcal{Q}_t^\gamma} \mathbb{E}_{\mathbb{Q}} \left[\sum_{s=t+1}^T D_s^* \middle| \mathcal{F}_t \right], \\ \Pi_t^{bid, \gamma}(D) &= \inf_{\mathbb{Q} \in \mathcal{Q}_t^\gamma} \mathbb{E}_{\mathbb{Q}} \left[\sum_{s=t+1}^T D_s^* \middle| \mathcal{F}_t \right]. \end{aligned}$$

We continue by explaining one of the main reasons the topic of this thesis is called dynamic conic finance.

Remark 3.3.6. *Let us consider the sets of extended cash flows associated with NGD prices $\Pi_t^{ask, \gamma}(D)$ and $\Pi_t^{bid, \gamma}(D)$:*

$$\begin{aligned} \widehat{\mathcal{H}}(t) &= \left\{ \left(0, \dots, 0, \xi_t \Pi_t^{ask, \gamma}(D), H_{t+1} - \xi_t D_{t+1}^*, \dots, H_T - \xi_t D_T^* \right) \right. \\ &\quad \left. : H \in \mathcal{H}(t), \xi_t \text{ is } \mathcal{F}_t\text{-measurable, } \xi_t \geq 0 \right\}, \\ \overline{\mathcal{H}}(t) &= \left\{ \left(0, \dots, 0, -\xi_t \Pi_t^{bid, \gamma}(D), H_{t+1} + \xi_t D_{t+1}^*, \dots, H_T + \xi_t D_T^* \right) \right. \\ &\quad \left. : H \in \mathcal{H}(t), \xi_t \text{ is } \mathcal{F}_t\text{-measurable, } \xi_t \geq 0 \right\}. \end{aligned}$$

If $\mathcal{H}(t)$ is frictionless and complete (and therefore linear), and **NGD** is satisfied, then as in Remark 3.3.4, we have that $\Pi(D) := \Pi_t^{ask,\gamma}(D) = \Pi_t^{bid,\gamma}(D)$. In this case, the set

$$\begin{aligned} \widehat{\mathcal{H}}(t) + \overline{\mathcal{H}}(t) = & \left\{ \left(0, \dots, 0, \xi_t \Pi_t(D), H_{t+1} - \xi_t D_{t+1}^*, \dots, H_T - \xi_t D_T^* \right) \right. \\ & \left. : H \in \mathcal{H}(t), \xi_t \text{ is } \mathcal{F}_t\text{-measurable} \right\} \end{aligned}$$

is a linear space. Whenever $\Pi_t^{ask,\gamma}(D) > \Pi_t^{bid,\gamma}(D)$, as in our general case, we have that

$$\begin{aligned} \widehat{\mathcal{H}}(t) + \overline{\mathcal{H}}(t) = & \left\{ \left(0, \dots, 0, \xi_t \Pi_t^{ask,\gamma}(D) - \phi_t \Pi_t^{bid,\gamma}(D), H_{t+1} - (\xi_t - \phi_t) D_{t+1}^*, \right. \right. \\ & \left. \left. \dots, H_T - (\xi_t - \phi_t) D_T^* \right) : H \in \mathcal{H}(t), \xi_t, \phi_t \text{ is } \mathcal{F}_t\text{-measurable}, \xi_t, \phi_t \geq 0 \right\} \end{aligned}$$

is only a convex cone. This is one of the main reasons why we call our approach dynamic conic finance.

Next, we remark on the relationship between the NGD prices and the superhedging prices.

Remark 3.3.7. (i) In view of Lemma 3.0.4 and Theorem 3.3.1, if **NGD** is satisfied then $\Pi_t^{bid,\gamma}(D)$ and $\Pi_t^{ask,\gamma}(D)$ satisfy

$$\inf_{\mathbb{Q} \in \mathcal{R}(\mathcal{H}(t))} \mathbb{E}_t^{\mathbb{Q}} \left[\sum_{s=t+1}^T D_s^* \right] \leq \Pi_t^{bid,\gamma}(D) \leq \Pi_t^{ask,\gamma}(D) \leq \sup_{\mathbb{Q} \in \mathcal{R}(\mathcal{H}(t))} \mathbb{E}_t^{\mathbb{Q}} \left[\sum_{s=t+1}^T D_s^* \right].$$

(ii) According to Theorem 2.6.1, if **NGD** under the efficient friction assumption **(EF)** is satisfied, then the NDG prices and the superhedging and subhedging prices $\pi_t^{ask}(D)$ and $\pi_t^{bid}(D)$ satisfy

$$\pi_t^{bid}(D) \leq \Pi_t^{bid,\gamma}(D) \leq \Pi_t^{ask,\gamma}(D) \leq \pi_t^{ask}(D) \quad (3.9)$$

for all $\gamma \in (0, \infty)$ and $t \in \mathcal{T}$. This is intuitively true.

Indeed, by definition, $\pi_t^{ask}(D)$ is the least random variable such that there exists a cash flow $H \in \mathcal{H}(t)$ so that

$$\pi_t^{ask}(D)(\omega) + \sum_{s=t+1}^T (H_s(\omega) - D_s^*(\omega)) \geq 0, \quad \omega \in \Omega.$$

Now, by the definition of the NGD ask price at level γ , and in view of the Theorem B.0.4, $\Pi_t^{ask,\gamma}(D)$ is the least random variable such that there¹⁵ a cash flow $H \in \mathcal{H}(t)$ so that

$$\Pi_t^{ask,\gamma}(D)(\omega) + \inf_{\mathbb{Q} \in \mathcal{Q}_t^\gamma} \mathbb{E}_{\mathbb{Q}} \left[\sum_{s=t+1}^T (H_s - D_s^*) \middle| \mathcal{F}_t \right](\omega) \geq 0, \quad \omega \in \Omega,$$

where \mathcal{Q} is the corresponding family of dynamically consistent sequences of sets of probability measures.

Because the condition

$$\sum_{s=t+1}^T (H_s(\omega) - D_s^*(\omega)) \geq 0, \quad \omega \in \Omega,$$

is clearly stronger than

$$\inf_{\mathbb{Q} \in \mathcal{Q}_t^\gamma} \mathbb{E}_{\mathbb{Q}} \left[\sum_{s=t+1}^T (H_s - D_s^*) \middle| \mathcal{F}_t \right](\omega) \geq 0, \quad \omega \in \Omega, \quad \gamma \in (0, \infty),$$

the inequality $\Pi_t^{ask,\gamma}(D) \leq \pi_t^{ask}(D)$ holds for all $\gamma \in (0, \infty)$. Using a similar argument, we also see that $\pi_t^{bid}(D) \leq \Pi_t^{bid,\gamma}(D)$.

We proceed by introducing the NGD forward ask and bid prices. We suppose that the risk-free interest rate r is deterministic.

Definition 3.3.2. *The no-good-deal (NGD) ask and bid forward prices, with delivery at time T , written at time $t \in \{1, \dots, T-1\}$, of a derivative contract $D \in \mathcal{L}^0$, at*

¹⁵Assuming that the infimum in the definition of $\Pi_t^{ask,\gamma}(D)$ is attained.

level $\gamma > 0$ are defined as

$$\begin{aligned}
F_t^{\text{ask},\gamma,T}(D)(\omega) &:= \inf\{f \in \mathbb{R} : \exists H \in \mathcal{H}(t) \text{ s.t.} \\
&\quad \alpha_t(\delta_T(\mathbf{1}B_T^{-1}f) + H - \delta_t^+(D^*))(\omega) \geq \gamma\}, \\
F_t^{\text{bid},\gamma,T}(D)(\omega) &:= \sup\{f \in \mathbb{R} : \exists H \in \mathcal{H}(t) \text{ s.t.} \\
&\quad \alpha_t(-\delta_T(\mathbf{1}B_T^{-1}f) + H + \delta_t^+(D^*))(\omega) \geq \gamma\}
\end{aligned}$$

for all $\omega \in \Omega$.

Note that the cash flow $\delta_T(\mathbf{1}B_T^{-1}f) + H - \delta_t^+(D^*)$ represents an exchange of a cash payment f at time T for a discounted cash flow D that is hedged with H . The NGD forward ask price at level γ is the minimum amount of cash f at time T so that $\delta_T(\mathbf{1}B_T^{-1}f) + H - \delta_t^+(D^*)$ is acceptable at level γ at time t .

The following result shows that the NGD forward ask and bid prices have a useful representation. See Bielecki, et al. [BCIR12] for the proof.

Theorem 3.3.2. *The no-good-deal (NGD) ask and bid forward prices of a derivative contract $D \in \mathcal{L}^0$, with delivery at time T , written at time $t \in \{1, \dots, T-1\}$ and level $\gamma > 0$, satisfy*

$$\begin{aligned}
F_t^{\text{ask},\gamma,T}(D)(\omega) &= B_T \Pi_t^{\text{ask},\gamma}(D), \\
F_t^{\text{bid},\gamma,T}(D)(\omega) &= B_T \Pi_t^{\text{bid},\gamma}(D).
\end{aligned}$$

Remark 3.3.8. *If r is deterministic and the set of hedging cash flows $\mathcal{H}(t)$ forms a market that is frictionless, complete, and arbitrage-free, then $\mathcal{R}(\mathcal{H}(t))$ is a singleton, say $\{\mathbb{Q}^*\}$, and so by Theorem 3.3.2 we have that $F_t^{\text{ask},\gamma,T}(D) = F_t^{\text{bid},\gamma,T}(D) = B_T \mathbb{E}_{\mathbb{Q}^*} \left[\sum_{u=t+1}^T D_u^* | \mathcal{F}_t \right]$. This is compatible with the classic result that states that in a frictionless, complete, and arbitrage-free market the discounted forward price $f_t^T(D)$ of a derivative contract D , with delivery at time T , written at time $t \in \{1, \dots, T-1\}$,*

is given as

$$f_t^T(D) = B_T S_t(D),$$

where $S(D)$ is the discounted risk-neutral spot price given by

$$S_t(D) = \mathbb{E}_{\mathbb{Q}^*} \left[\sum_{u=t+1}^T D_u^* \middle| \mathcal{F}_t \right].$$

Remark 3.3.9. From Theorem 3.3.2, we see that the relationship between the good-deal ask and bid forward prices is classic, in the sense that

$$\frac{F_t^{ask,\gamma,T}(D)}{\Pi_t^{ask,\gamma}(D)} = \frac{F_t^{bid,\gamma,T}(D)}{\Pi_t^{bid,\gamma}(D)} = \frac{f_t^T(D)}{S_t(D)}, \quad \gamma \in (0, \infty), \quad D \in \mathcal{L}^0,$$

where $f_t^T(D)$ and $S_t(D)$ are the forward and spot prices, respectively, corresponding to a frictionless, complete, and arbitrage-free market.

3.4 The Dynamic Gain-Loss Ratio

In this section, we first prove some auxiliary results that hold for general DCAIs. Then, we particularize these results to the very important special case of DCAI, namely to the dynamic Gain-Loss Ratio (dGLR). In this section, without a loss of generality, we assume that $r = 0$.

3.4.1 Characterization of DCAIs. In this section, we will prove an auxiliary result for DCAIs. For basic facts and notions regarding DCAIs, we refer to Appendix B.

From [BCZ11], we recall that for every normalized and right-continuous DCAI α there exist family $\mathcal{Q} = \left((\mathcal{Q}_t^\gamma)_{t \in \mathcal{T}} \right)_{\gamma \in (0, \infty)}$ of dynamically consistent sequences of sets of probability measures that is increasing (in γ), such that the following robust representation holds true

$$\alpha_t(D)(\omega) = \sup \left\{ \gamma \in (0, \infty) : \inf_{\mathbb{Q} \in \mathcal{Q}_t^\gamma} \mathbb{E}_{\mathbb{Q}} \left[\sum_{s=t}^T D_s \middle| \mathcal{F}_t \right] (\omega) \geq 0 \right\} \quad (3.10)$$

for all $\omega \in \Omega$, $t \in \mathcal{T}$, and $D \in \mathcal{L}^0$. We say that a family \mathcal{Q} of dynamically consistent sequences of sets of probability measures that is increasing (in γ) *corresponds* to a given normalized and right-continuous DCAI α if \mathcal{Q} satisfies (3.10). Now, we will establish a characterization of families \mathcal{Q} that correspond to a given normalized and right-continuous DCAI α .

Lemma 3.4.1. *Suppose that α is a normalized and right-continuous DCAI. A family \mathcal{Q} corresponds to α if and only if $\mathcal{Q} \in \mathfrak{Q}^\alpha$, where¹⁶*

$$\mathfrak{Q}^\alpha := \left\{ \mathcal{U} : \alpha_t(D)(\omega) \geq \gamma \text{ if and only if} \right. \\ \left. \inf_{\mathbb{Q} \in \mathcal{U}_t^\gamma} \mathbb{E}_{\mathbb{Q}} \left[\sum_{s=t}^T D_s \middle| \mathcal{F}_t \right] (\omega) \geq 0, \omega \in \Omega, \gamma \in (0, \infty), t \in \mathcal{T}, D \in \mathcal{L}^0 \right\}.$$

Proof. (\Leftarrow) Let $\mathcal{U} \in \mathfrak{Q}^\alpha$. We fix $t \in \mathcal{T}$, $D \in \mathcal{L}^0$, and $\omega \in \Omega$. Define the set

$$\Gamma(\mathcal{U}) := \left\{ \beta \in (0, \infty) : \inf_{\mathbb{Q} \in \mathcal{U}_t^\beta} \mathbb{E}_{\mathbb{Q}} \left[\sum_{s=t}^T D_s \middle| \mathcal{F}_t \right] (\omega) \geq 0 \right\}.$$

If $\alpha_t(D)(\omega) = \infty$, then

$$\inf_{\mathbb{Q} \in \mathcal{U}_t^\beta} \mathbb{E}_{\mathbb{Q}} \left[\sum_{s=t}^T D_s \middle| \mathcal{F}_t \right] (\omega) \geq 0, \quad \beta \in (0, \infty).$$

Therefore, $\Gamma(\mathcal{U}) = (0, \infty)$, and thus $\sup \Gamma(\mathcal{U}) = \infty$. Hence, (3.10) holds true.

If $\Gamma(\mathcal{U}) = \emptyset$, then

$$\inf_{\mathbb{Q} \in \mathcal{U}_t^\beta} \mathbb{E}_{\mathbb{Q}} \left[\sum_{s=t}^T D_s \middle| \mathcal{F}_t \right] (\omega) < 0, \quad \beta \in (0, \infty).$$

Since $\mathcal{U} \in \mathfrak{Q}^\alpha$, it is true that $\alpha_t(D)(\omega) < \beta$ for all $\beta \in (0, \infty)$. However, α is nonnegative by definition, thus $\alpha_t(D)(\omega) = 0$. By convention, we are taking $\sup \emptyset = 0$, so we also have that $\sup \Gamma(\mathcal{U}) = 0$. Hence, (3.10) holds true.

¹⁶We will generically denote by $\mathcal{U} = \left((\mathcal{U}_t^\gamma)_{t \in \mathcal{T}} \right)_{\gamma \in (0, \infty)}$ a family of dynamically consistent sequences of sets of probability measures that is increasing (in γ).

If $\alpha_t(D)(\omega) = 0$, then, since $\mathcal{U} \in \mathfrak{Q}^\alpha$, we have that

$$\inf_{\mathbb{Q} \in \mathcal{U}_t^\beta} \mathbb{E}_{\mathbb{Q}} \left[\sum_{s=t}^T D_s \middle| \mathcal{F}_t \right] (\omega) < 0, \quad \beta \in (0, \infty).$$

It follows that $\Gamma(\mathcal{U}) = \emptyset$, and so (3.10) holds true.

Suppose $\Gamma(\mathcal{U}) \neq \emptyset$. Assume that $\alpha_t(D)(\omega) < \infty$. We first show that $\alpha_t(D)(\omega)$ is an upper bound of $\Gamma(\mathcal{U})$. Observe that if $\gamma \in \Gamma(\mathcal{U})$, then

$$\inf_{\mathbb{Q} \in \mathcal{U}_t^\gamma} \mathbb{E}_{\mathbb{Q}} \left[\sum_{s=t}^T D_s \middle| \mathcal{F}_t \right] (\omega) \geq 0.$$

Now, since $\mathcal{U} \in \mathfrak{Q}^\alpha$, we have that $\alpha_t(D)(\omega) \geq \gamma$. So $\alpha_t(D)(\omega)$ is an upper bound of $\Gamma(\mathcal{U})$. If we let $\beta' := \alpha_t(D)(\omega)$, then, because $\mathcal{U} \in \mathfrak{Q}^\alpha$, we have that

$$\inf_{\mathbb{Q} \in \mathcal{U}_t^{\beta'}} \mathbb{E}_{\mathbb{Q}} \left[\sum_{s=t}^T D_s \middle| \mathcal{F}_t \right] (\omega) \geq 0.$$

Thus, $\beta' \in \Gamma(\mathcal{U})$. It follows that (3.10) holds.

(\implies) Now, suppose \mathcal{U} satisfies (3.10), and let $\gamma \in (0, \infty)$. If

$$\inf_{\mathbb{Q} \in \mathcal{U}_t^\gamma} \mathbb{E}_{\mathbb{Q}} \left[\sum_{s=t}^T D_s \middle| \mathcal{F}_t \right] (\omega) \geq 0,$$

then $\gamma \in \Gamma(\mathcal{U})$. By (3.10), we have that $\alpha_t(D)(\omega) \geq \gamma$.

Assume $\alpha_t(D)(\omega) \geq \gamma$. We consider the cases $\alpha_t(D)(\omega) > \gamma$ and $\alpha_t(D)(\omega) = \gamma$ separately. If $\alpha_t(D)(\omega) > \gamma$, then, since \mathcal{U}^γ is increasing in γ , we have that

$$\inf_{\mathbb{Q} \in \mathcal{U}_t^\gamma} \mathbb{E}_{\mathbb{Q}} \left[\sum_{s=t}^T D_s \middle| \mathcal{F}_t \right] (\omega) \geq 0.$$

Next, suppose that $\alpha_t(D)(\omega) = \gamma$ and

$$\inf_{\mathbb{Q} \in \mathcal{U}_t^\gamma} \mathbb{E}_{\mathbb{Q}} \left[\sum_{s=t}^T D_s \middle| \mathcal{F}_t \right] (\omega) < 0.$$

By Theorem B.0.3, the mapping

$$\gamma \longmapsto \inf_{\mathbb{Q} \in \mathcal{U}_t^\gamma} \mathbb{E}_{\mathbb{Q}} \left[\sum_{s=t}^T D_s \middle| \mathcal{F}_t \right] (\omega)$$

is left-continuous and monotone decreasing. Thus, by left-continuity there exists $\epsilon > 0$ so that

$$\inf_{\mathbb{Q} \in \mathcal{U}_t^{\gamma-\epsilon}} \mathbb{E}_{\mathbb{Q}} \left[\sum_{s=t}^T D_s \middle| \mathcal{F}_t \right] (\omega) < 0,$$

and by monotonicity and (3.10), we deduce that $\alpha_t(D)(\omega) \leq \gamma - \epsilon$. This implies that $\epsilon \leq 0$, which is a contradiction. Hence, we have that

$$\inf_{\mathbb{Q} \in \mathcal{U}_t^{\gamma}} \mathbb{E}_{\mathbb{Q}} \left[\sum_{s=t}^T D_s \middle| \mathcal{F}_t \right] (\omega) \geq 0,$$

which concludes the proof. \square

3.4.2 Characterization of the dGLR. A performance measure that is very popular among practitioners is the Sharpe Ratio, which was introduced in Sharpe [Sha64]. However, the Sharpe ratio is not monotone. Bernardo and Ledoit [BL00] proposed the *static* Gain-Loss Ratio, which is a monotone performance measure that is unbounded for arbitrage opportunities, and, as proved in Cherny and Madan [CM09], is also a static coherent acceptability index. Recently, Bielecki et al. [BCZ11] introduced the *dynamic* Gain-Loss Ratio, which is defined¹⁷ as

$$\text{dGLR}_t(D)(\omega) := \begin{cases} \frac{\mathbb{E}_t^{\mathbb{P}}[\sum_{s=t}^T D_s](\omega)}{\mathbb{E}_t^{\mathbb{P}}[(\sum_{s=t}^T D_s)^-](\omega)}, & \text{if } \mathbb{E}_t^{\mathbb{P}}\left[\sum_{s=t}^T D_s\right](\omega) > 0, \\ 0, & \text{otherwise.} \end{cases} \quad (3.11)$$

It is shown in [BCZ11] that the dGLR satisfies conditions (D1)–(D7), and therefore it is a dynamic coherent acceptability index (see Definition B.0.1).

Remark 3.4.1. *It is worth to remark on the interpretation of the dGLR in the context of arbitrage, which was first noticed in Bernardo and Ledoit [BL00] for the*

¹⁷By convention, $\text{dGLR}(0) = \infty$.

static Gain-Loss Ratio. Towards this end we note that

$$\sum_{s=t}^T H_s(\omega) \geq 0 \text{ for all } \omega \in \Omega, \quad \mathbb{E}_{\mathbb{P}} \left[\sum_{s=t}^T H_s \middle| \mathcal{F}_t \right] (\omega) > 0 \text{ for some } \omega \in \Omega,$$

$$\text{if and only if } \mathbb{E}_{\mathbb{P}} \left[\left(\sum_{s=t}^T H_s \right)^- \middle| \mathcal{F}_t \right] (\omega) = 0 \text{ for all } \omega \in \Omega,$$

$$\mathbb{E}_{\mathbb{P}} \left[\sum_{s=t}^T H_s \middle| \mathcal{F}_t \right] (\omega) > 0 \text{ for some } \omega \in \Omega$$

$$\text{if and only if } \text{dGLR}_t(H)(\omega) = \infty \text{ for some } \omega \in \Omega.$$

Therefore, in view of Definition 3.0.2, a cash flow $H \in \mathcal{H}(t)$ is an arbitrage opportunity at time $t \in \mathcal{T}$ if and only if $\text{dGLR}_t(H)(\omega) = \infty$ for some $\omega \in \Omega$. Equivalently, the no-arbitrage condition holds true at time $t \in \mathcal{T}$ if and only if $\text{dGLR}_t(H)$ is bounded for all $H \in \mathcal{H}(t)$. This equivalence gives an intuitive interpretation of the dGLR in terms of the no-arbitrage condition. \square

Next, we will show that $\widehat{\mathcal{Q}} := \{\widehat{\mathcal{Q}}^\gamma, \gamma > 0\}$ defined in (3.12) below is an increasing family of dynamically consistent sets of probability measures corresponding to the dGLR. We define a family $\widehat{\mathcal{Q}}$ as

$$\widehat{\mathcal{Q}}^\gamma := \left\{ \mathbb{Q} : \text{d}\mathbb{Q}/\text{d}\mathbb{P} = c(1 + \Lambda), c > 0, \Lambda \in \mathfrak{L}^\gamma, c\mathbb{E}^{\mathbb{P}}[1 + \Lambda] = 1 \right\}, \quad (3.12)$$

for all $\gamma \in (0, \infty)$, where

$$\mathfrak{L}^\gamma := \{\Lambda : \Lambda \text{ is an } \mathcal{F}_T\text{-measurable r.v., } 0 \leq \Lambda \leq \gamma\}.$$

Proposition 3.4.1. *The family $\widehat{\mathcal{Q}}$ is an increasing family of dynamically consistent sets of probability measures. In addition, this family corresponds to the dGLR.*

Proof. We start by observing that, for each $\gamma > 0$, the set $\widehat{\mathcal{Q}}^\gamma$ is nonempty since, in particular, we may take $\Lambda = 0$ in the definition of $\widehat{\mathcal{Q}}^\gamma$. Also, we note that $\widehat{\mathcal{Q}}^\gamma$ is increasing in γ .

For the rest of the proof we fix $\gamma > 0$. We denote by $\Upsilon^t = \{P_1^t, P_2^t, \dots, P_{n_t}^t\}$ the unique partition of Ω at time t that generates \mathcal{F}_t . In order to prove our result it suffices to show that \widehat{Q}^γ is weakly consistent (see Corollary 4.1.1 in [Zha11]), which is

$$\mathbb{I}_{P_i^t} \inf_{\mathbb{Q} \in \mathcal{Q}^\gamma} \mathbb{E}_{\mathbb{Q}}[X|\mathcal{F}_t] \leq \mathbb{I}_{P_i^t} \max_{\omega \in P_i^t} \left\{ \inf_{\mathbb{Q} \in \mathcal{Q}^\gamma} \mathbb{E}_{\mathbb{Q}}[X|\mathcal{F}_{t+1}](\omega) \right\}, \quad (3.13)$$

for every $t \in \{0, \dots, T-1\}$, $P_i^t \in \Upsilon^t$, and $X \in \mathcal{F}_T$. Next, take $0 \leq \Lambda \leq \gamma$ and suppose that

$$\max_{\omega \in P_i^t} \frac{\mathbb{E}_{\mathbb{P}}[(1+\Lambda)X|\mathcal{F}_{t+1}](\omega)}{\mathbb{E}_{\mathbb{P}}[1+\Lambda|\mathcal{F}_{t+1}](\omega)} \leq a,$$

for some $a \in \mathbb{R}$. Hence,

$$\mathbb{E}_{\mathbb{P}}[(1+\Lambda)X|\mathcal{F}_{t+1}](\omega) \leq a \mathbb{E}_{\mathbb{P}}[1+\Lambda|\mathcal{F}_t](\omega),$$

for all $\omega \in P_i^t$. Therefore, using the tower property of conditional expectations, we have that

$$\mathbb{E}_{\mathbb{P}}[(1+\Lambda)X|\mathcal{F}_t](\omega) \leq a \mathbb{E}_{\mathbb{P}}[1+\Lambda|\mathcal{F}_t](\omega),$$

for all $\omega \in P_i^t$, and, consequently

$$\max_{\omega \in P_i^t} \frac{\mathbb{E}_{\mathbb{P}}[(1+\Lambda)X|\mathcal{F}_t](\omega)}{\mathbb{E}_{\mathbb{P}}[1+\Lambda|\mathcal{F}_t](\omega)} \leq a.$$

Thus, we showed that for any $a \in \mathbb{R}$ the following implication holds,

$$\max_{\omega \in P_i^t} \frac{\mathbb{E}_{\mathbb{P}}[(1+\Lambda)X|\mathcal{F}_{t+1}](\omega)}{\mathbb{E}_{\mathbb{P}}[1+\Lambda|\mathcal{F}_{t+1}](\omega)} \leq a \implies \max_{\omega \in P_i^t} \frac{\mathbb{E}_{\mathbb{P}}[(1+\Lambda)X|\mathcal{F}_t](\omega)}{\mathbb{E}_{\mathbb{P}}[1+\Lambda|\mathcal{F}_t](\omega)} \leq a,$$

so that

$$\max_{\omega \in P_i^t} \frac{\mathbb{E}_{\mathbb{P}}[(1+\Lambda)X|\mathcal{F}_t](\omega)}{\mathbb{E}_{\mathbb{P}}[1+\Lambda|\mathcal{F}_t](\omega)} \leq \max_{\omega \in P_i^t} \frac{\mathbb{E}_{\mathbb{P}}[(1+\Lambda)X|\mathcal{F}_{t+1}](\omega)}{\mathbb{E}_{\mathbb{P}}[1+\Lambda|\mathcal{F}_{t+1}](\omega)}.$$

Hence, we have

$$\begin{aligned} \mathbb{I}_{P_i^t} \frac{\mathbb{E}_{\mathbb{P}}[(1+\Lambda)X|\mathcal{F}_t](\omega)}{\mathbb{E}_{\mathbb{P}}[1+\Lambda|\mathcal{F}_t](\omega)} &\leq \mathbb{I}_{P_i^t} \max_{\omega \in P_i^t} \frac{\mathbb{E}_{\mathbb{P}}[(1+\Lambda)X|\mathcal{F}_t](\omega)}{\mathbb{E}_{\mathbb{P}}[1+\Lambda|\mathcal{F}_t](\omega)} \\ &\leq \mathbb{I}_{P_i^t} \max_{\omega \in P_i^t} \frac{\mathbb{E}_{\mathbb{P}}[(1+\Lambda)X|\mathcal{F}_{t+1}](\omega)}{\mathbb{E}_{\mathbb{P}}[1+\Lambda|\mathcal{F}_{t+1}](\omega)} \end{aligned}$$

for all $\omega \in \Omega$. Thus, for $\mathbb{Q} = c(1 + \Lambda)\mathbb{P}$, we have that

$$\mathbb{I}_{P_i^t} \mathbb{E}_{\mathbb{Q}}[X|\mathcal{F}_t](\omega) \leq \mathbb{I}_{P_i^t} \max_{\omega \in P_i^t} \mathbb{E}_{\mathbb{Q}}[X|\mathcal{F}_{t+1}](\omega),$$

for all $\omega \in \Omega$. Therefore,

$$\begin{aligned} \mathbb{I}_{P_i^t} \inf_{\mathbb{Q} \in \mathcal{Q}^\gamma} \mathbb{E}_{\mathbb{Q}}[X|\mathcal{F}_t] &\leq \mathbb{I}_{P_i^t} \inf_{\mathbb{Q} \in \mathcal{Q}^\gamma} \left\{ \max_{\omega \in P_i^t} \mathbb{E}_{\mathbb{Q}}[X|\mathcal{F}_{t+1}](\omega) \right\} \\ &\leq \mathbb{I}_{P_i^t} \max_{\omega \in P_i^t} \left\{ \inf_{\mathbb{Q} \in \mathcal{Q}^\gamma} \mathbb{E}_{\mathbb{Q}}[X|\mathcal{F}_{t+1}](\omega) \right\}, \end{aligned}$$

which proves the weak consistency of $\widehat{\mathcal{Q}}^\gamma$.

We now show that the family $\widehat{\mathcal{Q}}$ corresponds to the dGLR. By Lemma 3.4.1, this is equivalent to show that

$$\text{dGLR}_t(D)(\omega) \geq \gamma \iff \inf_{\mathbb{Q} \in \widehat{\mathcal{Q}}^\gamma} \mathbb{E}_{\mathbb{Q}}[X_t^T|\mathcal{F}_t](\omega) \geq 0, \quad (3.14)$$

for all $\omega \in \Omega$, $t \in \mathcal{T}$ and $D \in \mathcal{L}^0$, where for convenience we denoted $X_t^T = \sum_{u=T}^t D_u$.

In the rest of the proof we fix $\omega \in \Omega$, $t \in \mathcal{T}$ and $D \in \mathcal{L}^0$.

In order to show (3.14) we first observe that since any $\eta \in \mathcal{E}^\gamma$ is strictly positive, we may apply the abstract Bayes formula to write

$$\begin{aligned} \inf_{\mathbb{Q} \in \widehat{\mathcal{Q}}^\gamma} \mathbb{E}_{\mathbb{Q}}[X_t^T|\mathcal{F}_t](\omega) \geq 0 &\iff \inf_{\eta \in \mathcal{E}^\gamma} \frac{\mathbb{E}_{\mathbb{P}}[\eta X_t^T|\mathcal{F}_t](\omega)}{\mathbb{E}_{\mathbb{P}}[\eta|\mathcal{F}_t](\omega)} \geq 0 \\ &\iff \frac{\mathbb{E}_{\mathbb{P}}[\eta X_t^T|\mathcal{F}_t](\omega)}{\mathbb{E}_{\mathbb{P}}[\eta|\mathcal{F}_t](\omega)} \geq 0, \quad \eta \in \mathcal{E}^\gamma \\ &\iff \mathbb{E}_{\mathbb{P}}[\eta X_t^T|\mathcal{F}_t](\omega) \geq 0, \quad \eta \in \mathcal{E}^\gamma \\ &\iff \inf_{\eta \in \mathcal{E}^\gamma} \mathbb{E}_{\mathbb{P}}[\eta X_t^T|\mathcal{F}_t](\omega) \geq 0. \end{aligned} \quad (3.15)$$

Next, recall that by definition of \mathcal{E}^γ we have that

$$\inf_{\eta \in \mathcal{E}^\gamma} \mathbb{E}_{\mathbb{P}}[\eta X_t^T|\mathcal{F}_t](\omega) = \inf_{\Lambda \in \mathcal{E}^\gamma} \mathbb{E}_{\mathbb{P}}[(1 + \Lambda)X_t^T|\mathcal{F}_t](\omega). \quad (3.16)$$

Observing that

$$\begin{aligned}
\mathbb{E}_{\mathbb{P}}[(1 + \Lambda)X_t^T | \mathcal{F}_t](\omega) &= \mathbb{E}_{\mathbb{P}}[X_t^T + \Lambda \mathbb{I}_{\{X_t^T \leq 0\}} X_t^T + \Lambda \mathbb{I}_{\{X_t^T > 0\}} X_t^T | \mathcal{F}_t](\omega) \\
&\geq \mathbb{E}_{\mathbb{P}}[X_t^T + \Lambda \mathbb{I}_{\{X_t^T \leq 0\}} X_t^T | \mathcal{F}_t](\omega) \\
&\geq \mathbb{E}_{\mathbb{P}}[X_t^T + \gamma \mathbb{I}_{\{X_t^T \leq 0\}} X_t^T | \mathcal{F}_t](\omega) \\
&= \mathbb{E}_{\mathbb{P}}[(1 + \Lambda^*)X_t^T | \mathcal{F}_t](\omega),
\end{aligned}$$

where $\Lambda^* := \gamma \mathbb{I}_{\{X_t^T \leq 0\}} \in \mathcal{L}^\gamma$.

Consequently, we obtain that

$$\inf_{\Lambda \in \mathcal{L}^\gamma} \mathbb{E}_{\mathbb{P}}[(1 + \Lambda)X_t^T | \mathcal{F}_t](\omega) = \mathbb{E}_{\mathbb{P}}[(1 + \Lambda^*)X_t^T | \mathcal{F}_t](\omega).$$

Thus, in view of (3.16), we get

$$\begin{aligned}
\inf_{\eta \in \mathcal{E}^\gamma} \mathbb{E}_{\mathbb{P}}[\eta X_t^T | \mathcal{F}_t](\omega) &= \mathbb{E}_{\mathbb{P}}[X_t^T | \mathcal{F}_t](\omega) + \gamma \mathbb{E}_{\mathbb{P}}[\mathbb{I}_{\{X_t^T \leq 0\}} X_t^T | \mathcal{F}_t](\omega) \\
&= \mathbb{E}_{\mathbb{P}}[X_t^T | \mathcal{F}_t](\omega) + \gamma \mathbb{E}_{\mathbb{P}}[\mathbb{I}_{\{X_t^T \leq 0\}} ((X_t^T)^+ - (X_t^T)^-) | \mathcal{F}_t](\omega) \\
&= \mathbb{E}_{\mathbb{P}}[X_t^T | \mathcal{F}_t](\omega) - \gamma \mathbb{E}_{\mathbb{P}}[(X_t^T)^- | \mathcal{F}_t](\omega).
\end{aligned}$$

From here and (3.15) we deduce that

$$\inf_{\mathbb{Q} \in \widehat{\mathcal{Q}}^\gamma} \mathbb{E}_{\mathbb{Q}}[X_t^T | \mathcal{F}_t](\omega) \geq 0 \iff \mathbb{E}_{\mathbb{P}}[X_t^T | \mathcal{F}_t](\omega) \geq \gamma \mathbb{E}_{\mathbb{P}}[(X_t^T)^- | \mathcal{F}_t](\omega). \quad (3.17)$$

To complete the proof of (3.14) we shall consider the following three cases:

$$\mathbb{E}_{\mathbb{P}}[X_t^T | \mathcal{F}_t](\omega) > 0, \quad \mathbb{E}_{\mathbb{P}}[X_t^T | \mathcal{F}_t](\omega) < 0, \quad \text{and} \quad \mathbb{E}_{\mathbb{P}}[X_t^T | \mathcal{F}_t](\omega) = 0.$$

Case 1: $\mathbb{E}_{\mathbb{P}}[X_t^T | \mathcal{F}_t](\omega) > 0$.

From the definition of the dGLR and from (3.17) we have that

$$\inf_{\mathbb{Q} \in \widehat{\mathcal{Q}}^\gamma} \mathbb{E}_{\mathbb{Q}}[X_t^T | \mathcal{F}_t](\omega) \geq 0 \iff \mathbb{E}_{\mathbb{P}}[X_t^T | \mathcal{F}_t](\omega) \geq \gamma \mathbb{E}_{\mathbb{P}}[(X_t^T)^- | \mathcal{F}_t](\omega) \quad (3.18)$$

$$\iff \text{dGLR}_t(D)(\omega) \geq \gamma. \quad (3.19)$$

Therefore, (3.14) holds true.

Case 2: $\mathbb{E}_{\mathbb{P}}[X_t^T | \mathcal{F}_t](\omega) < 0$.

Since $\mathbb{P} \in \widehat{\mathcal{Q}}^\gamma$, we have that

$$\inf_{\mathbb{Q} \in \widehat{\mathcal{Q}}^\gamma} \mathbb{E}_{\mathbb{Q}}[X_t^T | \mathcal{F}_t](\omega) \leq \mathbb{E}_{\mathbb{P}}[X_t^T | \mathcal{F}_t](\omega) < 0.$$

Also, by the definition of the dGLR, we have that $\text{dGLR}_t(D)(\omega) = 0$. As a result,

$$\text{dGLR}_t(D)(\omega) < \gamma \iff \inf_{\mathbb{Q} \in \widehat{\mathcal{Q}}^\gamma} \mathbb{E}_{\mathbb{Q}}[X_t^T | \mathcal{F}_t](\omega) < 0,$$

and so (3.14) holds true.

Case 3: $\mathbb{E}_{\mathbb{P}}[X_t^T | \mathcal{F}_t](\omega) = 0$.

Case 3a: If $\mathbb{E}_{\mathbb{P}}[(X_t^T)^- | \mathcal{F}_t](\omega) = 0$, then $\mathbb{E}_{\mathbb{P}}[(X_t^T)^+ | \mathcal{F}_t](\omega) = 0$. Since $\omega \in \Omega$ is arbitrary, we may conclude that in this case $X_t^T = 0$. Thus $\text{dGLR}_t(D) = \infty$ and $\inf_{\mathbb{Q} \in \widehat{\mathcal{Q}}^\gamma} \mathbb{E}_{\mathbb{Q}}[X_t^T | \mathcal{F}_t] = 0$, showing that (3.14) holds true.

Case 3b: Now, assume that $\gamma \mathbb{E}_{\mathbb{P}}[(X_t^T)^- | \mathcal{F}_t](\omega) > \mathbb{E}_{\mathbb{P}}[X_t^T | \mathcal{F}_t](\omega) = 0$. By (3.17), it follows that $\inf_{\mathbb{Q} \in \widehat{\mathcal{Q}}^\gamma} \mathbb{E}_{\mathbb{Q}}[X_t^T | \mathcal{F}_t](\omega) < 0$. Due to the definition of the dGLR, we thus have that $\text{dGLR}_t(D)(\omega) = 0$, and so (3.14) holds true in this case as well.

The proof of the proposition is complete. \square

The following two propositions will be needed in order to apply dGLR for pricing. In the first one we show that the family $\widehat{\mathcal{E}}$ satisfies Assumption (B). In the second one, we show that, for fixed $t \in \mathcal{T}$ and $D \in \mathcal{L}^0$, the function

$$\rho_t^\gamma(D) = \inf_{\mathbb{Q} \in \widehat{\mathcal{Q}}^\gamma} \mathbb{E}_{\mathbb{Q}} \left[\sum_{s=t}^T D_s \middle| \mathcal{F}_t \right], \quad \gamma \in (0, \infty)$$

satisfies Assumption (C).

Proposition 3.4.2. *For each $\gamma \in (0, \infty)$, the set of densities $\widehat{\mathcal{E}}^\gamma$ is closed and convex.*

Proof. Fix $\gamma \in (0, \infty)$. We first show that $\widehat{\mathcal{E}}^\gamma$ is closed (in \mathbb{R}^N).¹⁸ Let η_k be a sequence in $\widehat{\mathcal{E}}^\gamma$ converging to some η . By the definition of $\widehat{\mathcal{E}}^\gamma$, there exist sequences Λ_k and c_k so that $\eta_k = c_k(1 + \Lambda_k)$, $c_k > 0$, $c_k = 1/\mathbb{E}_{\mathbb{P}}[1 + \Lambda_k]$, and $0 \leq \Lambda_k(\omega_j) \leq \gamma$ for $j = 1, \dots, N$. For each ω_j , we have that $\Lambda_k(\omega_j)$ is bounded by γ , so Λ_k is bounded. By the Bolzano-Weierstrass Theorem, there exists a subsequence Λ_{k_m} such that Λ_{k_m} converges to some Λ . This limit must satisfy $0 \leq \Lambda(\omega_j) \leq \gamma$ for $j = 1, \dots, N$, since a sequence converges in \mathbb{R}^N if and only if it converges coordinate-wise. If Λ_{k_m} converges, then $\mathbb{E}_{\mathbb{P}}[1 + \Lambda_{k_m}]$ converges. Since $\mathbb{E}_{\mathbb{P}}[1 + \Lambda_{k_m}]$ is strictly greater than zero, we have that $1/\mathbb{E}_{\mathbb{P}}[1 + \Lambda_{k_m}]$ converges to $c := 1/\mathbb{E}_{\mathbb{P}}[1 + \Lambda]$, which means that c_{k_m} converges to c . Consequently, η_{k_m} converges to $c(1 + \Lambda)$. It follows that $\eta \in \widehat{\mathcal{E}}^\gamma$. Hence, $\widehat{\mathcal{E}}^\gamma$ is closed.

We proceed by showing that $\widehat{\mathcal{E}}^\gamma$ is convex. Let $\eta_1, \eta_2 \in \widehat{\mathcal{E}}^\gamma$ and $0 \leq \lambda \leq 1$. Let c_i and Λ_i correspond to η_i , in the sense of definition of $\widehat{\mathcal{E}}^\gamma$, that is, $\eta_i = c_i(1 + \Lambda_i)$, $i = 1, 2$.

We need to show that $\lambda c_1(1 + \Lambda_1) + (1 - \lambda)c_2(1 + \Lambda_2) \in \widehat{\mathcal{E}}^\gamma$. Define

$$\tilde{c} := \lambda c_1 + (1 - \lambda)c_2 \quad \text{and} \quad \tilde{\Lambda} := \frac{\lambda c_1 \Lambda_1 + (1 - \lambda)c_2 \Lambda_2}{\lambda c_1 + (1 - \lambda)c_2}.$$

Since

$$\lambda c_1(1 + \Lambda_1) + (1 - \lambda)c_2(1 + \Lambda_2) = \tilde{c}(1 + \tilde{\Lambda}),$$

it suffices to show that $0 \leq \tilde{\Lambda} \leq \gamma$ and $\tilde{c} = 1/\mathbb{E}_{\mathbb{P}}[1 + \tilde{\Lambda}]$. We first notice that since $0 \leq \Lambda_1, \Lambda_2 \leq \gamma$, the scalars c_1, c_2 satisfy $c_1, c_2 > 0$, and since $0 \leq \lambda \leq 1$, we have that

$$0 \leq \frac{\lambda c_1 \Lambda_1 + (1 - \lambda)c_2 \Lambda_2}{\lambda c_1 + (1 - \lambda)c_2} \leq \gamma \frac{\lambda c_1 + (1 - \lambda)c_2}{\lambda c_1 + (1 - \lambda)c_2} = \gamma.$$

¹⁸Clearly, we may consider $\widehat{\mathcal{E}}^\gamma$ as a subset of \mathbb{R}^N .

Therefore, $0 \leq \tilde{\Lambda} \leq \gamma$. Next, because $c_1 \mathbb{E}_{\mathbb{P}}[1 + \Lambda_1] = c_2 \mathbb{E}_{\mathbb{P}}[1 + \Lambda_2] = 1$, it is true that

$$\begin{aligned} \tilde{c} \mathbb{E}_{\mathbb{P}}[1 + \tilde{\Lambda}] &= (\lambda c_1 + (1 - \lambda) c_2) \mathbb{E}_{\mathbb{P}} \left[1 + \frac{\lambda c_1 \Lambda_1 + (1 - \lambda) c_2 \Lambda_2}{\lambda c_1 + (1 - \lambda) c_2} \right] \\ &= \lambda c_1 + (1 - \lambda) c_2 + \lambda c_1 \mathbb{E}_{\mathbb{P}}[1 + \Lambda_1] + (1 - \lambda) c_2 \mathbb{E}_{\mathbb{P}}[1 + \Lambda_2] - \lambda c_1 - (1 - \lambda) c_2 \\ &= 1. \end{aligned}$$

As a result, $\hat{\mathcal{E}}^\gamma$ is convex. \square

Proposition 3.4.3. *For each $t \in \mathcal{T}$, $D \in \mathcal{L}^0$ the function of $\gamma \in (0, \infty)$ defined as*

$$\rho_t^\gamma(D) := \inf_{\mathbb{Q} \in \hat{\mathcal{Q}}^\gamma} \mathbb{E}_{\mathbb{Q}} \left[\sum_{s=t}^T D_s \middle| \mathcal{F}_t \right], \quad (3.20)$$

is continuous.

Proof. Let $\omega \in \Omega$. By the abstract Bayes Theorem, we have that

$$\begin{aligned} \inf_{\mathbb{Q} \in \hat{\mathcal{Q}}^\gamma} \mathbb{E}_{\mathbb{Q}} \left[\sum_{s=t}^T D_s \middle| \mathcal{F}_t \right] (\omega) &= \inf_{\eta \in \hat{\mathcal{E}}^\gamma} \frac{\mathbb{E}_{\mathbb{P}} [\eta \sum_{s=t}^T D_s \middle| \mathcal{F}_t] (\omega)}{\mathcal{E}^{\mathbb{P}} [\eta \middle| \mathcal{F}_t] (\omega)} \\ &= \inf_{\Lambda \in \mathcal{L}^\gamma} \frac{\mathbb{E}_{\mathbb{P}} [(1 + \Lambda) \sum_{s=t}^T D_s \middle| \mathcal{F}_t] (\omega)}{\mathbb{E}_{\mathbb{P}} [1 + \Lambda \middle| \mathcal{F}_t] (\omega)}. \end{aligned}$$

The function g defined as

$$g(\Lambda)(\omega) := \frac{\mathbb{E}_{\mathbb{P}} [(1 + \Lambda) \sum_{s=t}^T D_s \middle| \mathcal{F}_t] (\omega)}{\mathbb{E}_{\mathbb{P}} [1 + \Lambda \middle| \mathcal{F}_t] (\omega)}, \quad 0 \leq \Lambda \text{ an } \mathcal{F}_T\text{-measurable r.v.}$$

is continuous in Λ . Applying Lemma A.0.4, we conclude that the proposition holds true. \square

Remark 3.4.2. *Note that the LHS of (3.20) is the value of a DCRM associated with $\hat{\mathcal{Q}}$ (see B.0.3).*

3.5 Good-Deal Prices for Asian Options

One of the main advantages of our dynamic framework is that NGD ask and bid prices, as defined in 3.3.1, can be computed for path-dependent options in a

dynamically consistent manner. In this section, using a simple model for ask and bid prices of a security, and choosing the dGLR as acceptability index, we compute the NGD ask and bid prices of European-style Asian and Barrier call options in a market with transaction costs¹⁹. We compare these NGD prices with the corresponding superhedging and subhedging prices.

According to Theorem 3.3.1, the NGD ask and bid prices of a derivative contract $D \in \mathcal{L}^0$, at level $\gamma > 0$, at time $t = 0$ satisfy

$$\begin{aligned}\Pi_0^{ask,\gamma}(D) &= \sup_{\mathbb{Q} \in \widehat{\mathcal{Q}}^\gamma \cap \mathcal{R}(\mathcal{H}(0))} \mathbb{E}^{\mathbb{Q}} \left[\sum_{s=1}^T D_s \right], \\ \Pi_0^{bid,\gamma}(D) &= \inf_{\mathbb{Q} \in \widehat{\mathcal{Q}}^\gamma \cap \mathcal{R}(\mathcal{H}(0))} \mathbb{E}^{\mathbb{Q}} \left[\sum_{s=1}^T D_s \right].\end{aligned}$$

We recall that $\widehat{\mathcal{Q}}$, defined in (3.12), is a dynamically consistent family of sets of probability measures that corresponds to the dGLR. In what follows, we will use the representations above to compute the NGD ask and bid prices of the options. To compute the superhedging and subhedging prices, we use the representation in Theorem 2.6.1:

$$\begin{aligned}\pi_0^{ask}(D) &= \sup_{\mathbb{Q} \in \mathcal{R}(\mathcal{H}(0))} \mathbb{E}^{\mathbb{Q}} \left[\sum_{s=1}^T D_s \right], \\ \pi_0^{bid}(D) &= \inf_{\mathbb{Q} \in \mathcal{R}(\mathcal{H}(0))} \mathbb{E}^{\mathbb{Q}} \left[\sum_{s=1}^T D_s \right].\end{aligned}$$

We suppose that the bid price of the security²⁰ is given in Table 3.1. The ask price process is assumed to satisfy $P^{ask} := P^{bid}(1 + \lambda)$, where $\lambda \in [0, \infty)$ is the *transaction costs coefficient* (cf. Bensaid et al. [BLPS92]; Boyle and Vorst [BV92]).

¹⁹We explain how the NGD ask and bid prices behave for different γ and λ only for the arithmetic mean Asian option only: the remaining options display similar behavior.

²⁰See Example 4.10 in Pliska [Pli97], page 134.

Table 3.1. Bid Price Process of the Security

ω	$t = 0$	$t = 1$	$t = 2$
ω_1	50	80	90
ω_2	50	80	70
ω_3	50	80	60
ω_4	50	40	60
ω_5	50	40	30

We define $P^{ask} := (1 + \lambda)P^{bid}$ as the ask price process for the security. Also, we define the *mid price process* as $P^{mid} := (P^{ask} + P^{bid})/2$.

We recall that $\widehat{\mathbb{Q}}$ is defined in terms of the reference measure \mathbb{P} , which we will now assume to be

$$(\mathbb{P}(\omega_1), \mathbb{P}(\omega_2), \mathbb{P}(\omega_3), \mathbb{P}(\omega_4), \mathbb{P}(\omega_5)) = (1/10, 1/8, 1/4, 1/4, 11/40).$$

3.5.1 Arithmetic Mean Asian option. We now compute the ask and bid price of a European-style arithmetic Asian call option with a strike of 75. The derivative contract associated with this option is

$$D^a := \left(0, 0, \left(\frac{1}{3}(P_0^{mid} + P_1^{mid} + P_2^{mid}) - 75\right)^+\right).$$

Recall that $\Pi_0^{ask, \gamma}(D^a)$ and $\Pi_0^{bid, \gamma}(D^a)$ denotes the NGD prices computed using dGLR, whereas $\pi_0^{ask}(D^a)$ and $\pi_0^{bid}(D^a)$ represents the ask superhedging price and subhedging bid price, respectively.

Our results are presented in Tables 3.2, 3.2, 3.2 for different transaction cost coefficients. In Figure 3.1 we display the “liquidity surface”, which is the plot of good-deal bid-ask spread as a function of the level γ and transaction costs coefficient λ .

Table 3.2. Prices of an Arithmetic Asian Call Option with $\lambda = 0$

γ	$\pi_0^{ask}(D^a)$	$\Pi_0^{ask,\gamma}(D^a)$	$\Pi_0^{bid,\gamma}(D^a)$	$\pi_0^{bid}(D^a)$
0.0001	1.388854	1.341775	1.341559	1.250035
0.001	–	1.342746	1.340587	–
0.005	–	1.347062	1.336288	–
0.01	–	1.352446	1.330952	–
0.05	–	1.388853	1.289754	–
0.1	–	1.388853	1.250036	–
0.25	–	1.388853	1.250036	–
0.5	–	1.388854	1.250036	–
0.75	–	1.388854	1.250036	–
1	–	1.388854	1.250035	–
1.25	–	1.388854	1.250035	–

Table 3.3. Prices of an Arithmetic Asian Call Option with $\lambda = 0.005$

γ	$\pi_0^{ask}(D^a)$	$\Pi_0^{ask,\gamma}(D^a)$	$\Pi_0^{bid,\gamma}(D^a)$	$\pi_0^{bid}(D^a)$
0.0001	1.484025	1.376819	1.376598	1.230204
0.001	–	1.377816	1.375601	–
0.005	–	1.382244	1.371189	–
0.01	–	1.387769	1.365714	–
0.05	–	1.431586	1.323440	–
0.1	–	1.484025	1.274140	–
0.25	–	1.484024	1.230207	–
0.5	–	1.484024	1.230205	–
0.75	–	1.484024	1.230205	–
1	–	1.484025	1.230205	–
1.25	–	1.484025	1.230205	–

Table 3.4. Prices of an Arithmetic Asian Call Option with $\lambda = 0.01$

γ	$\pi_0^{ask}(D^a)$	$\Pi_0^{ask,\gamma}(D^a)$	$\Pi_0^{bid,\gamma}(D^a)$	$\pi_0^{bid}(D^a)$
0.0001	1.550037	1.411864	1.411636	1.167264
0.001	–	1.412886	1.410614	–
0.005	–	1.417427	1.406090	–
0.01	–	1.423092	1.400476	–
0.05	–	1.468024	1.357126	–
0.1	–	1.523222	1.306571	–
0.25	–	1.550036	1.175234	–
0.5	–	1.550035	1.167266	–
0.75	–	1.550036	1.167265	–
1	–	1.550036	1.167264	–
1.25	–	1.550036	1.167264	–

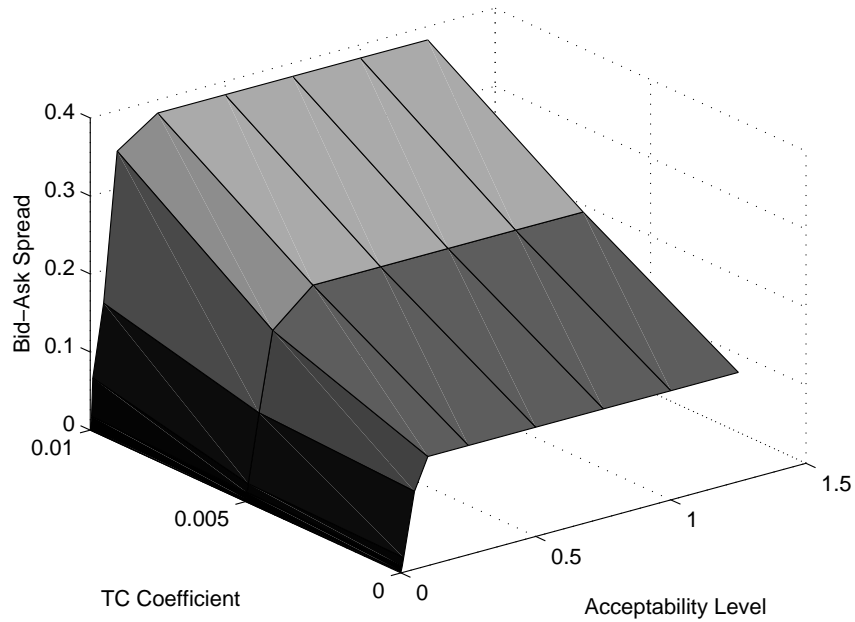


Figure 3.1. Liquidity Surface for an Arithmetic Asian call Option

In Figure 3.1, it is apparent that the good-deal bid-ask spread is increasing both in the acceptance level γ and in the transaction cost coefficient λ . The good-deal bid-ask spread increases in γ because of the representations in Theorem 3.3.1, and since \mathcal{Q}^γ is increasing in γ . On the other hand, the good-deal bid-ask spread, as well as the difference between superhedging and subhedging prices, increases in λ since hedging the derivative contract becomes more expensive as λ increases.

We also note from Tables 3.2, 3.2, 3.2 that the superhedging, subhedging, and good-deal prices increase in λ , and that the good-deal ask and bid prices converge to the no-arbitrage bounds at higher γ values. This is also due to the fact that hedging is more expensive as λ increases. For example, in case $\lambda = 0$, the prices $\Pi_0^{ask,\gamma}(D^a)$ and $\Pi_0^{bid,\gamma}(D^a)$ approximately converge to $\pi_0^{ask}(D^a)$ and $\pi_0^{bid}(D^a)$, respectively, at $\gamma = 0.1$, whereas if $\lambda = 0.005$ this happens at approximately $\gamma = 0.25$, and in the case $\lambda = 0.01$ it happens at approximately $\gamma = 0.5$.

3.5.1.1 Geometric Asian Option. We continue by computing the ask and bid prices of a geometric Asian call option with strike 75. The derivative contract associated with this option is

$$D^g := \left(0, 0, \left(\left(P_0^{mid} P_1^{mid} P_2^{mid} \right)^{\frac{1}{3}} - 75 \right)^+ \right).$$

Table 3.5. Prices of a Geometric Asian Call Option with $\lambda = 0$

γ	$\pi_0^{ask}(D^g)$	$\Pi_0^{ask,\gamma}(D^g)$	$\Pi_0^{bid,\gamma}(D^g)$	$\pi_0^{bid}(D^g)$
0.0001	1.022926	0.954010	0.953852	0.819950
0.001	–	0.954721	0.953141	–
0.005	–	0.957881	0.949995	–
0.01	–	0.961824	0.946092	–
0.05	–	0.993159	0.916010	–
0.1	–	1.022926	0.881052	–
0.25	–	1.022923	0.819953	–
0.5	–	1.022923	0.819953	–
0.75	–	1.022924	0.819951	–
1	–	1.022925	0.819951	–
1.25	–	1.022926	0.819951	–

Table 3.6. Prices of a Geometric Asian Call Option with $\lambda = 0.005$

γ	$\pi_0^{ask}(D^g)$	$\Pi_0^{ask,\gamma}(D^g)$	$\Pi_0^{bid,\gamma}(D^g)$	$\pi_0^{bid}(D^g)$
0.0001	1.100503	0.988085	0.987922	0.815067
0.001	–	0.988816	0.987191	–
0.005	–	0.992063	0.983958	–
0.01	–	0.996116	0.979947	–
0.05	–	1.028320	0.949031	–
0.1	–	1.068030	0.913103	–
0.25	–	1.100502	0.820396	–
0.5	–	1.100501	0.815071	–
0.75	–	1.100499	0.815070	–
1	–	1.100502	0.815069	–
1.25	–	1.100503	0.815069	–

Table 3.7. Prices of a Geometric Asian Call Option with $\lambda = 0.01$

γ	$\pi_0^{ask}(D^g)$	$\Pi_0^{ask,\gamma}(D^g)$	$\Pi_0^{bid,\gamma}(D^g)$	$\pi_0^{bid}(D^g)$
0.0001	1.149457	1.022159	1.021992	0.767216
0.001	–	1.022910	1.021242	–
0.005	–	1.026244	1.017921	–
0.01	–	1.030407	1.013802	–
0.05	–	1.063480	0.982052	–
0.1	–	1.104262	0.945155	–
0.25	–	1.149455	0.849945	–
0.5	–	1.149456	0.767218	–
0.75	–	1.149456	0.767217	–
1	–	1.149457	0.767216	–
1.25	–	1.149457	0.767216	–

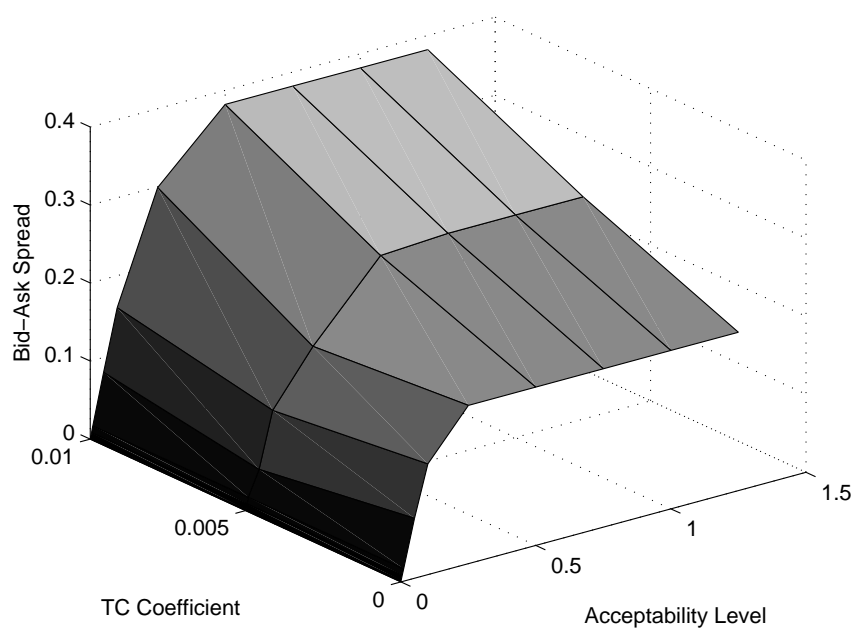


Figure 3.2. Liquidity Surface for a Geometric Asian call Option

We notice that the subhedging, superhedging, and good-deal prices for the geometric Asian call option display a similar relationship to the acceptance level γ and transaction cost coefficient λ as for the arithmetic Asian call option in the previous section.

CHAPTER 4

FUTURE WORK

The following are open questions and further research problems regarding the no-arbitrage pricing for dividend-paying securities in markets with transaction costs studied in Chapter 2.

1. In Theorem 2.5.1, we proved that no-arbitrage condition under the efficient friction assumption (**NAEF**) is satisfied if and only if there exists a consistent pricing system (CPS) for the case in which there are no transaction costs on the dividends paid by the securities. An open question is to prove or disprove, in the general, whether a CPS exists whenever **NAEF** (or versions of) holds.
2. We illustrated our no-arbitrage pricing framework with a vanilla credit default swap contract. A further research problem is to construct a financial market model for pricing and hedging CDSs in markets with transaction costs.
3. In this thesis, we studied a version of the First Fundamental Theorem of Asset Pricing (FFTAP) in the context of **NAEF**. An open research problem is to study the present version of the FFTAP in the context of *the robust no-arbitrage condition* (or versions of) defined in Schachermayer [Sch04].
4. Develop the no-arbitrage pricing for a continuous-time setting. One possible direction is to work with the *generalized arbitrage pricing model* condition introduced in Cherny [Che07b, Che07a]. Therein, simple trading strategies are considered, and the price processes are assumed to be adapted, infinite-dimensional, càdlàg, and have components that are bounded from below (or above). Another possible direction is to adopt the *robust no-free-lunch with vanishing risk condition* studied in Denis et al. [DGR11]. Both approaches allow for securities' prices to be negative (which is required for securities such as CDSs).

Next, we state open questions and further research problems regarding the dynamic conic finance framework studied in Chapter 3.

1. We studied dynamic conic finance in a discrete-time setting under the assumption that the state space is finite. An open research problem is to prove the FTNGDP in a continuous-time setting and/or for a general state space.
2. We particularized our results to the case in which the chosen dynamic coherent acceptability index is the dynamic Gain-Loss ratio. A research problem is to apply dynamic conic finance to dynamic coherent acceptability indices other than the dynamic Gain-Loss ratio. Insights into the problem of finding examples of dynamic coherent acceptability indices can be found in Bielecki et al. [BCZ11].

APPENDIX A
ANALYSIS

We begin by recalling a basic result in probability (see for instance Jacod and Protter [JP04]).

Lemma A.0.1. *Let $f : \mathbb{R}^N \rightarrow \mathbb{R}$ be a continuous function, and suppose that X^m is a sequence of \mathbb{R}^N -valued random variables that converges a.s. to X . Then $f(X^m)$ converges a.s. to $f(X)$.*

The next useful result is a consequence of the previous lemma.

Lemma A.0.2. *Let $(X^m, Y^m) \in L^0(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R}^N \times \mathbb{R}^N)$ be sequence converging a.s. to (X, Y) . Then*

(i) *The sequence $Y^m X^m$ converges a.s. to YX .*

(ii) *For any Y^a, Y^b , the sequence*

$$Z^m := 1_{\{X^m \geq 0\}} X^m Y^b + 1_{\{X^m < 0\}} X^m Y^a$$

converges a.s. to

$$Z := 1_{\{X \geq 0\}} XY^b + 1_{\{X < 0\}} XY^a.$$

Proof. (i): This claim follows directly from Lemma A.0.1 by considering the function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ defined as $f(x, y) := xy$.

(ii): In view of Lemma A.0.1, it is enough to prove that the function $g : \mathbb{R} \rightarrow \mathbb{R}$ defined as

$$g(x) := 1_{\{x \geq 0\}} xy^b + 1_{\{x < 0\}} xy^a,$$

where $y^a, y^b \in \mathbb{R}$, is continuous. It is immediate that $g(x)$ is continuous for all $x \neq 0$, so we only show it is continuous at $x = 0$. By the triangle inequality, we notice that

$$\begin{aligned} |g(x)| &= |1_{\{x \geq 0\}} xy^b + 1_{\{x < 0\}} xy^a| \leq 1_{\{x \geq 0\}} x|y^b| + 1_{\{x < 0\}} |x||y^a| \\ &\leq |x| \max\{|y^a|, |y^b|\} \leq |x| \max\{|y^a|, |y^b|, 1\}. \end{aligned}$$

Thus, for any arbitrary $\epsilon > 0$, if $|x| \leq \delta$, then choosing $\delta := \epsilon / (\max\{|y^a|, |y^b|, 1\})$ proves the claim. \square

The following result is related to a.s. convergence of bounded random variables.

Lemma A.0.3. *Suppose $X \in L^0(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R})$ and let $\Omega^n := \{\omega \in \Omega : |X(\omega)| < n\}$. Then 1_{Ω^m} converges a.s. to 1.*

Proof. First, define

$$\tilde{\Omega}^n := \{\omega \in \Omega : n - 1 \leq |X(\omega)| < n\}, \quad n \in \mathbb{N}.$$

The sets $\{\tilde{\Omega}^n\}_{n \in \mathbb{N}}$ forms a partition of Ω , and $\Omega^m = \bigcup_{n=1}^m \tilde{\Omega}^n$. Hence, $1_{\bigcup_{n=1}^m \tilde{\Omega}^n}(\omega) = \sum_{n=1}^m 1_{\tilde{\Omega}^n}(\omega)$. Also, because that the Dirac measure δ defined as $\delta(E) := 1_E$ for any $E \in \mathcal{F}$ is countably additive, we have $1_{\bigcup_{n=1}^{\infty} \tilde{\Omega}^n}(\omega) = \sum_{n=1}^{\infty} 1_{\tilde{\Omega}^n}(\omega)$. Therefore, we see that

$$\begin{aligned} \lim_{m \rightarrow \infty} 1_{\Omega^m}(\omega) &= \lim_{m \rightarrow \infty} 1_{\bigcup_{n=1}^m \tilde{\Omega}^n}(\omega) = \lim_{m \rightarrow \infty} \sum_{n=1}^m 1_{\tilde{\Omega}^n}(\omega) \\ &= \sum_{n=1}^{\infty} 1_{\tilde{\Omega}^n}(\omega) = 1_{\bigcup_{n=1}^{\infty} \tilde{\Omega}^n}(\omega) = 1_{\Omega} = 1, \quad \text{for a.e. } \omega \in \Omega. \end{aligned}$$

Therefore, $\mathbb{P}(\lim_m 1_{\Omega^m} = 1) = 1$. \square

Next, we prove a lemma concerning the continuity of the infimum of a continuous function.

Lemma A.0.4. *If $g : \mathbb{R} \rightarrow \mathbb{R}$ is continuous, then the function $f : (0, \infty) \rightarrow \mathbb{R}$ defined by $f(\gamma) := \inf_{0 \leq y \leq \gamma} g(y)$ is continuous.*

Proof. Since g is continuous, $f(\gamma) = \min_{0 \leq y \leq \gamma} g(y)$. We first show that

$$\lim_{\gamma \rightarrow \gamma_0^+} \min_{\gamma_0 \leq y \leq \gamma} g(y) = \lim_{\gamma \rightarrow \gamma_0^-} \min_{\gamma_0 \leq y \leq \gamma} g(y) = g(\gamma_0).$$

Fix $\gamma_0 \in (0, \infty)$ and suppose that $\epsilon > 0$ and $\gamma \in [\gamma_0, \infty)$. From the continuity of g , for all $\epsilon' > 0$ there exists $\delta > 0$ such that $|\gamma - \gamma_0| < \delta$ implies $|g(\gamma) - g(\gamma_0)| < \epsilon'$. We notice that

$$\begin{aligned}
 |g(\gamma_0) - \min_{\gamma_0 \leq y \leq \gamma} g(y)| &= g(\gamma_0) - \min_{\gamma_0 \leq y \leq \gamma} g(y) \\
 &= \min_{\gamma_0 \leq y \leq \gamma} \{g(\gamma_0) - g(y)\} \\
 &\leq \min_{\gamma_0 \leq y \leq \gamma} \{|g(\gamma_0) - g(y)|\} \\
 &\leq \min_{\gamma_0 \leq y \leq \gamma} \{|g(\gamma_0) - g(\gamma)| + |g(\gamma) - g(y)|\} \\
 &\leq |g(\gamma_0) - g(\gamma)| + \min_{\gamma_0 \leq y \leq \gamma} \{|g(\gamma) - g(y)|\} \\
 &\leq |g(\gamma_0) - g(\gamma)| + |g(\gamma) - g(\gamma_0)| \\
 &= 2|g(\gamma_0) - g(\gamma)| < 2\epsilon'.
 \end{aligned}$$

Taking $\epsilon = 2\epsilon'$ shows that $\lim_{\gamma \rightarrow \gamma_0^+} \min_{\gamma_0 \leq y \leq \gamma} g(y) = g(\gamma_0)$.

We now show that $\lim_{\gamma \rightarrow \gamma_0^-} \min_{\gamma \leq y \leq \gamma_0} g(y) = g(\gamma_0)$. Let $\gamma_0 \in (0, \infty)$ and suppose that $\epsilon > 0$ and $\gamma \in (0, \gamma_0]$. Since g is continuous, for any $\epsilon' > 0$ there exists $\delta > 0$ such that $|\gamma - \gamma_0| < \delta$ implies $|g(\gamma) - g(\gamma_0)| < \epsilon'$. Notice that

$$\begin{aligned}
|g(\gamma_0) - \min_{\gamma \leq y \leq \gamma_0} g(y)| &= g(\gamma_0) - \min_{\gamma \leq y \leq \gamma_0} g(y) \\
&= \min_{\gamma \leq y \leq \gamma_0} \{g(\gamma_0) - g(y)\} \\
&\leq \min_{\gamma \leq y \leq \gamma_0} \{|g(\gamma_0) - g(y)|\} \\
&= \min_{\gamma \leq y \leq \gamma_0} \{|g(\gamma_0) - g(\gamma) + g(\gamma) - g(y)|\} \\
&\leq \min_{\gamma \leq y \leq \gamma_0} \{|g(\gamma_0) - g(\gamma)| + |g(\gamma) - g(y)|\} \\
&\leq |g(\gamma_0) - g(\gamma)| + \min_{\gamma \leq y \leq \gamma_0} \{|g(\gamma) - g(y)|\} \\
&\leq |g(\gamma_0) - g(\gamma)| + |g(\gamma) - g(\gamma_0)| \\
&= 2|g(\gamma_0) - g(\gamma)| < 2\epsilon'.
\end{aligned}$$

Taking $\epsilon = 2\epsilon'$ shows that $\lim_{\gamma \rightarrow \gamma_0^-} \min_{\gamma_0 \leq y \leq \gamma} g(y) = g(\gamma_0)$.

We now show that f is continuous:

$$\lim_{\gamma \rightarrow \gamma_0^+} f(\gamma) = \lim_{\gamma \rightarrow \gamma_0^-} f(\gamma) = f(\gamma_0).$$

Since f is non-increasing and bounded, the limit exists. For any $\gamma_0 \in (0, \infty)$, let $\gamma \in [\gamma_0, \infty)$. Since $\min(\cdot, \cdot) : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function it follows that

$$\begin{aligned}
f(\gamma_0) - \lim_{\gamma \rightarrow \gamma_0^+} f(\gamma) &= \min_{0 \leq y \leq \gamma_0} g(y) - \lim_{\gamma \rightarrow \gamma_0^+} \min_{0 \leq y \leq \gamma} g(y) \\
&= \min_{0 \leq y \leq \gamma_0} g(y) - \lim_{\gamma \rightarrow \gamma_0^+} \min \left(\min_{0 \leq y \leq \gamma_0} g(y), \min_{\gamma_0 \leq y \leq \gamma} g(y) \right) \\
&= \min_{0 \leq y \leq \gamma_0} g(y) - \min \left(\min_{0 \leq y \leq \gamma_0} g(y), \lim_{\gamma \rightarrow \gamma_0^+} \min_{\gamma_0 \leq y \leq \gamma} g(y) \right) \\
&= \min_{0 \leq y \leq \gamma_0} g(y) - \min \left(\min_{0 \leq y \leq \gamma_0} g(y), g(\gamma_0) \right) \\
&= 0.
\end{aligned}$$

It follows that f is right-continuous.

Now let $\gamma \in (0, \gamma_0]$. Similarly as above, we obtain

$$\begin{aligned}
f(\gamma_0) - \lim_{\gamma \rightarrow \gamma_0^-} f(\gamma) &= \min_{0 \leq y \leq \gamma_0} g(y) - \lim_{\gamma \rightarrow \gamma_0^-} \min_{0 \leq y \leq \gamma} g(y) \\
&= \lim_{\gamma \rightarrow \gamma_0^-} \min_{0 \leq y \leq \gamma_0} g(y) - \lim_{\gamma \rightarrow \gamma_0^-} \min_{0 \leq y \leq \gamma} g(y) \\
&= \lim_{\gamma \rightarrow \gamma_0^-} \min \left(\min_{0 \leq y \leq \gamma} g(y), \min_{\gamma \leq y \leq \gamma_0} g(y) \right) - \lim_{\gamma \rightarrow \gamma_0^-} \min_{0 \leq y \leq \gamma} g(y) \\
&= \min \left(\lim_{\gamma \rightarrow \gamma_0^-} \min_{0 \leq y \leq \gamma} g(y), \lim_{\gamma \rightarrow \gamma_0^-} \min_{\gamma \leq y \leq \gamma_0} g(y) \right) - \lim_{\gamma \rightarrow \gamma_0^-} \min_{0 \leq y \leq \gamma} g(y) \\
&= \min \left(\lim_{\gamma \rightarrow \gamma_0^-} \min_{0 \leq y \leq \gamma} g(y), g(\gamma_0) \right) - \lim_{\gamma \rightarrow \gamma_0^-} \min_{0 \leq y \leq \gamma} g(y)
\end{aligned}$$

From the continuity of g and $\min(\cdot, \cdot)$, we see that

$$\begin{aligned}
&\min \left(\lim_{\gamma \rightarrow \gamma_0^-} \min_{0 \leq y \leq \gamma} g(y), g(\gamma_0) \right) - \lim_{\gamma \rightarrow \gamma_0^-} \min_{0 \leq y \leq \gamma} g(y) \\
&= \min \left(\lim_{\gamma \rightarrow \gamma_0^-} \min_{0 \leq y \leq \gamma} g(y), \lim_{\gamma \rightarrow \gamma_0^-} g(\gamma) \right) - \lim_{\gamma \rightarrow \gamma_0^-} \min_{0 \leq y \leq \gamma} g(y) \\
&= \lim_{\gamma \rightarrow \gamma_0^-} \min \left(\min_{0 \leq y \leq \gamma} g(y), g(\gamma) \right) - \lim_{\gamma \rightarrow \gamma_0^-} \min_{0 \leq y \leq \gamma} g(y) \\
&= 0
\end{aligned}$$

Thus, f is left-continuous, so we conclude that f is continuous. \square

The following lemma is an auxiliary result needed for Theorem 3.3.1.

Lemma A.0.5. *For any monotone increasing, continuous function $f : (0, \infty) \rightarrow \mathbb{R}$, we have that $f(\gamma) \leq 0$ if and only if*

$$\sup\{\beta \in (0, \infty) : f(\beta) \leq 0\} \geq \gamma,$$

for any $\gamma > 0$.

Proof. Let us define the set $\Gamma := \{\beta \in (0, \infty) : f(\beta) \leq 0\}$. Assume that $f(\gamma) \leq 0$ for some $\gamma > 0$. Then $\gamma \in \Gamma$, and therefore $\sup \Gamma \geq \gamma$.

Conversely, suppose that $\sup \Gamma \geq \gamma$ and define $\beta^* := \sup \Gamma$. If $\sup \Gamma = \infty$, then $f(x) \leq 0$, for all $x > 0$, and in particular for $x = \gamma$. Now assume that $\beta^* \in (0, \infty)$. We first argue by contradiction that $\beta^* \in \Gamma$. If $\beta^* \notin \Gamma$, then $f(\beta^*) > 0$. Now, since f is continuous, there exists $\epsilon' > 0$ so that $0 < f(\beta^* - \epsilon')$. By the definition of the supremum of a set, we have that, for all $\epsilon > 0$, there exists $\beta^\epsilon \in \Gamma$ so that $\beta^* - \epsilon < \beta^\epsilon$. Therefore, because f is monotonically increasing, $f(\beta^* - \epsilon) \leq f(\beta^\epsilon)$. Hence, $0 < f(\beta^* - \epsilon) \leq f(\beta^\epsilon)$, which contradicts $\beta^\epsilon \in \Gamma$. We proceed by showing that $f(\gamma) \leq 0$. Since $\gamma \leq \beta^*$ and f is monotonically increasing, we have that $f(\gamma) \leq f(\beta^*)$. However, $\beta^* \in \Gamma$, so $f(\gamma) \leq f(\beta^*) \leq 0$. \square

We proceed by showing an important lemma that we use throughout Chapter 2.

Lemma A.0.6. *For any $W, X, Y, Z \in L^0(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R})$ with $W \geq Z$, the following random variables R^1 and R^2 are nonnegative:*

$$\begin{aligned} R^1 &:= (1_{\{X \geq 0\}}X + 1_{\{Y \geq 0\}}Y)W + (1_{\{X < 0\}}X + 1_{\{Y < 0\}}Y)Z \\ &\quad - 1_{\{X+Y \geq 0\}}(X+Y)W - 1_{\{X+Y < 0\}}(X+Y)Z, \\ R^2 &:= -(1_{\{X \geq 0\}}X + 1_{\{Y \geq 0\}}Y)Z - (1_{\{X < 0\}}X + 1_{\{Y < 0\}}Y)W \\ &\quad + 1_{\{X+Y \geq 0\}}(X+Y)Z + 1_{\{X+Y < 0\}}(X+Y)W. \end{aligned}$$

Proof. Define the subsets

$$\begin{aligned} \Omega^1 &:= \{X \geq 0\} \cap \{Y \geq 0\}, & \Omega^2 &:= \{X < 0\} \cap \{Y < 0\}, \\ \Omega^3 &:= \{X < 0\} \cap \{Y \geq 0\}, & \Omega^4 &:= \{X \geq 0\} \cap \{Y < 0\}, \\ \Omega^5 &:= \{X + Y \geq 0\}, & \Omega^6 &:= \{X + Y < 0\}. \end{aligned}$$

Notice that the subsets $\Omega^1, \Omega^2, \Omega^3 \cap \Omega^5, \Omega^3 \cap \Omega^6, \Omega^4 \cap \Omega^5, \Omega^4 \cap \Omega^6$ form a partition of Ω , so we may argue that the R^1 is nonnegative on each of these subset separately.

We first show that $R^1 \geq 0$.

On the sets Ω^1 and Ω^2 : $R^1 = 0$. On the set $\Omega^3 \cap \Omega^5$:

$$R^1 = XZ + YW - (X + Y)W = -X(W - Z) \geq 0.$$

On the set $\Omega^3 \cap \Omega^6$:

$$R^1 = XZ + YW - (X + Y)Z = Y(W - Z) \geq 0.$$

On the set $\Omega^4 \cap \Omega^5$:

$$R^1 = XW + YZ - (X + Y)W = -Y(W - Z) \geq 0.$$

On the set $\Omega^4 \cap \Omega^6$:

$$R^1 = XW + YZ - (X + Y)Z = X(W - Z) \geq 0.$$

Therefore, $R^1 \geq 0$ on Ω .

Next, if we define $\tilde{Z} := -Z$ and $\tilde{W} := -W$, then $\tilde{Z} \geq \tilde{W}$. Applying the result for R^1 to \tilde{Z}, \tilde{W} gives us that $R^2 \geq 0$. \square

The following result is used in Chapter 2.

Lemma A.0.7. *Let $\{Y_i\}_{i=1,\dots,M} \in L^1(\Omega, \mathbb{P}, \mathcal{F}; \mathbb{R})$. If $\mathbb{E}_{\mathbb{P}}[\sum_{i=1}^M X_i Y_i] \leq 0$ for all $\{X_i\}_{i=1,\dots,M} \in L_+^{\infty}(\Omega, \mathbb{P}, \mathcal{F}; \mathbb{R})$, then $\{Y_i\}_{i=1,\dots,M}$ are non-positive.*

Proof. Let us take $X_i := 1_{\{Y_i > 0\}}$ for $i = 1, 2, \dots, M$. Then

$$0 \geq \mathbb{E}_{\mathbb{P}} \left[\sum_{i=1}^M X_i Y_i \right] = \sum_{i=1}^M \mathbb{E}_{\mathbb{P}} [1_{\{Y_i > 0\}} Y_i].$$

Hence, $\mathbb{P}(Y_i > 0) = 0$ for $j = 1, 2, \dots, M$. \square

The next lemma is the celebrated result due to Yan [Yan80] and Kreps [Kre81].

Lemma A.0.8 (Kreps-Yan). *Let \mathcal{C} be a closed convex cone in $L^1(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R})$ containing $L^1_-(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R})$ such that $\mathcal{C} \cap L^1_+(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}) = \{0\}$. Then there exists a functional $f \in L^\infty(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R})$ such that, for each $h \in L^1_+(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R})$ with $h \neq 0$, we have that $\mathbb{E}_{\mathbb{P}}[fh] > 0$ and $\mathbb{E}_{\mathbb{P}}[fg] \leq 0$ for any $g \in \mathcal{C}$.*

Next, we recall a well-known characterization of compact sets. For a proof, see Lemma I.5.6 in Dunford and Schwartz [DS58].

Lemma A.0.9. *A subset of a topological space is compact if and only if every family of closed sets with the finite intersection property has a nonempty intersection.*

The following theorem is an application of Hahn-Banach theorem, regarding the separation of hyperplanes.

Theorem A.0.1. *If \mathcal{Z} and \mathcal{C} are disjoint, closed, convex subsets of \mathbb{R}^N , and if \mathcal{Z} is compact, then there exists a constant ϵ with $\epsilon > 0$, and a continuous linear functional $\varphi \in \mathbb{R}^N$, so that*

$$\varphi(c) \leq 0 < \epsilon < \varphi(z)$$

for all $z \in \mathcal{Z}$ and $c \in \mathcal{C}$.

Proof. By Theorem V.2.10 in Dunford and Schwartz [DS58], there exists constants a and ϵ' with $\epsilon' > 0$, and a continuous linear functional $\varphi \in \mathbb{R}^N$, so that

$$\varphi(x) \leq a - \epsilon' < a \leq \varphi(z) \tag{A.1}$$

for all $z \in \mathcal{Z}$ and $x \in \mathcal{C}$. We now argue that $\varphi(x) \leq 0$ for all $x \in \mathcal{C}$. Suppose there exists $a_0 > 0$ and $x_0 \in \mathcal{C}$ so that $\varphi(x_0) = a_0$. Since \mathcal{C} is a cone, we have that $\lambda x_0 \in \mathcal{C}$ for all $\lambda > 0$. Thus,

$$\sup_{x \in \mathcal{C}} \varphi(x) \geq \sup_{\lambda > 0} \varphi(\lambda x_0) = \sup_{\lambda > 0} \lambda a_0 = +\infty,$$

which contradicts (A.1), and hence $\varphi(x) \leq 0$, $x \in \mathcal{C}$. From here, and since φ is linear and $0 \in \mathcal{C}$, it follows that $\sup_{x \in \mathcal{C}} \varphi(x) = 0$. Thus, $a - \epsilon' \geq 0$, and hence $a > 0$. Taking $\epsilon = a$ concludes the proof. \square

APPENDIX B
DYNAMIC COHERENT ACCEPTABILITY INDICES

In this section, we present some important definitions and results from the theory of Dynamic Coherent Acceptability Indices. For a more detailed discussion and proofs of results we refer to Bielecki, Cialenco, and Zhang [BCZ11].

We first recall the definition of a dynamic coherent acceptability index.

Definition B.0.1. *A dynamic coherent acceptability index (DCAI) is a function $\alpha : \mathcal{T} \times L^0 \times \Omega \rightarrow [0, \infty]$ that satisfies the following properties:*

- (D1) **Adaptiveness.** *For any $t \in \mathcal{T}$ and $D \in L^0$, $\alpha_t(D)$ is \mathcal{F}_t -measurable;*
- (D2) **Independence of the past.** *For any $t \in \mathcal{T}$ and $D, D' \in L^0$, if there exists $A \in \mathcal{F}_t$ such that $\mathbb{I}_A D_s = \mathbb{I}_A D'_s$ for all $s \geq t$, then $\mathbb{I}_A \alpha_t(D) = \mathbb{I}_A \alpha_t(D')$;*
- (D3) **Monotonicity.** *For any $t \in \mathcal{T}$ and $D, D' \in L^0$, if $D_s(\omega) \geq D'_s(\omega)$ for all $s \geq t$ and $\omega \in \Omega$, then $\alpha_t(D) \geq \alpha_t(D')$ for all $\omega \in \Omega$;*
- (D4) **Scale invariance.** *$\alpha_t(\lambda D) = \alpha_t(D)$ for all $\lambda > 0$, $D \in L^0$, $t \in \mathcal{T}$, and $\omega \in \Omega$;*
- (D5) **Quasi-concavity.** *If $\alpha_t(D) \geq x$ and $\alpha_t(D') \geq x$ for some $t \in \mathcal{T}$, $\omega \in \Omega$, $D, D' \in L^0$, and $x \in (0, \infty]$, then $\alpha_t(\lambda D + (1 - \lambda)D') \geq x$ for all $\lambda \in [0, 1]$;*
- (D6) **Translation invariance.** *$\alpha_t(D + m\mathbb{I}_{\{t\}}) = \alpha_t(D + m\mathbb{I}_{\{s\}})$ for every $t \in \mathcal{T}$, $D \in L^0$, $\omega \in \Omega$, $s \geq t$ and every \mathcal{F}_t -measurable random variable m ;*
- (D7) **Dynamic consistency.** *For any $t \in [0, \dots, T - 1]$ and $D, D' \in L^0$, if $D_t(\omega) \geq 0 \geq D'_t(\omega)$ for all $\omega \in \Omega$, and there exists a non-negative \mathcal{F}_t -measurable random variable m such that $\alpha_{t+1}(D) \geq m(\omega) \geq \alpha_{t+1}(D')$ for all $\omega \in \Omega$, then $\alpha_t(D) \geq m(\omega) \geq \alpha_t(D')$ for all $\omega \in \Omega$.*

Next, we recall the definition of a dynamic coherent risk measure.

Definition B.0.2. *Dynamic coherent risk measure (DCRM) is a function $\rho : \{0, \dots, T\} \times L^0 \times \Omega \rightarrow \mathbb{R}$ that satisfies the following properties:*

- (A1) **Adaptiveness.** $\rho_t(D)$ is \mathcal{F}_t -measurable for all $t \in \mathcal{T}$ and $D \in L^0$;
- (A2) **Independence of the past.** If $\mathbb{I}_A D_s = \mathbb{I}_A D'_s$ for some $t \in \mathcal{T}$, $D, D' \in L^0$, and $A \in \mathcal{F}_t$ and for all $s \geq t$, then $\mathbb{I}_A \rho_t(D) = \mathbb{I}_A \rho_t(D')$;
- (A3) **Monotonicity.** If $D_s(\omega) \geq D'_s(\omega)$ for some $t \in \mathcal{T}$ and $D, D' \in L^0$, and for all $s \geq t$ and $\omega \in \Omega$, then $\rho_t(D) \leq \rho_t(D')$ for all $\omega \in \Omega$;
- (A4) **Homogeneity.** $\rho_t(\lambda D) = \lambda \rho_t(D)$ for all $\lambda > 0$, $D \in L^0$, $t \in \mathcal{T}$, and $\omega \in \Omega$;
- (A5) **Subadditivity.** $\rho_t(D + D') \leq \rho_t(D) + \rho_t(D')$ for all $t \in \mathcal{T}$, $D, D' \in L^0$, and $\omega \in \Omega$;
- (A6) **Translation invariance.** $\rho_t(D + m\mathbb{I}_{\{s\}}) = \rho_t(D) - m$ for every $t \in \mathcal{T}$, $D \in L^0$, \mathcal{F}_t -measurable random variable m , and all $s \geq t$;
- (A7) **Dynamic consistency.**

$$\mathbb{I}_A(\min_{\omega \in A} \rho_{t+1}(D) - D_t) \leq \mathbb{I}_A \rho_t(D) \leq \mathbb{I}_A(\max_{\omega \in A} \rho_{t+1}(D) - D_t),$$

for every $t \in \{0, 1, \dots, T-1\}$, $D \in L^0$ and $A \in \mathcal{F}_t$.

We now recall an important result that provides the representation of a DCAI in terms of a family of DCRMs, and the representation of DCRM in terms of a DCAI. The proof the following theorem can be found in [BCZ11].

Theorem B.0.2.

- (i) If α is a normalized, right-continuous, dynamic coherent acceptability index, then there exists a left-continuous and increasing family of dynamic coherent risk measures $(\rho^\gamma)_{\gamma \in (0, \infty)}$, such that

$$\alpha_t(D)(\omega) = \sup\{\gamma \in (0, \infty) : \rho_t^\gamma(D)(\omega) \leq 0\}, \quad \omega \in \Omega, t \in \mathcal{T}, D \in L^0. \tag{B.1}$$

(ii) If $(\rho^\gamma)_{\gamma \in (0, \infty)}$ is a left-continuous and increasing family of dynamic coherent risk measures, then there exists a right-continuous and normalized dynamic coherent acceptability index α such that,

$$\rho_t^\gamma(D)(\omega) = \inf\{c \in \mathbb{R} : \alpha_t(D + \delta_t(1c))(\omega) \geq \gamma\}, \quad \omega \in \Omega, t \in \mathcal{T}, D \in L^0.$$

We take $\inf \emptyset = \infty$ and $\sup \emptyset = 0$.

Next, we recall the definitions of a dynamically consistent sequence of sets of probability measures and an increasing family of sequences of sets of probability measures.

Definition B.0.3.

(i) A sequence of sets of probability measures $(\mathcal{Q}_t)_{t=0}^T$ absolutely continuous with respect to \mathbb{P} is called dynamically consistent with respect to the filtration $(\mathcal{F}_t)_{t=0}^T$ if the sequence is of full-support and the following inequality holds true

$$\begin{aligned} \mathbb{I}_E \min_{\omega \in E} \left\{ \inf_{\mathbb{Q} \in \mathcal{Q}_{t+1}} \mathbb{E}_{\mathbb{Q}}[X | \mathcal{F}_{t+1}](\omega) \right\} &\leq \mathbb{I}_E \inf_{\mathbb{Q} \in \mathcal{Q}_t} \mathbb{E}_{\mathbb{Q}}[X | \mathcal{F}_t] \\ &\leq \mathbb{I}_E \max_{\omega \in E} \left\{ \inf_{\mathbb{Q} \in \mathcal{Q}_{t+1}} \mathbb{E}_{\mathbb{Q}}[X | \mathcal{F}_{t+1}](\omega) \right\} \end{aligned}$$

for all $t \in \{0, 1, \dots, T-1\}$, $E \in \mathcal{F}_t$, and \mathcal{F}_T -measurable random variables X .

(ii) A family of sequences of sets of probability measures $(\mathcal{Q}_t^\gamma)_{t=0}^T)_{\gamma \in (0, \infty)}$ is called increasing if $\mathcal{Q}_t^\gamma \supseteq \mathcal{Q}_t^\beta$, for all $\gamma \geq \beta > 0$ and $t \in \mathcal{T}$.

The following is representation theorem for dynamic coherent risk measures in terms of dynamically consistent set of probabilities. These results, combined with the results from Theorem B.0.2 about duality between DCAI and DCRM, gives a representation theorem for dynamic coherent acceptability indices.

Theorem B.0.3 (Robust Representation Theorem for DCRM). *For $\gamma > 0$, a function $\rho^\gamma : \{0, 1, \dots, T\} \times L^0 \times \Omega \rightarrow \mathbb{R}$ is a dynamic coherent risk measure if and only if there exists a dynamically consistent family of sets of probabilities $(\mathcal{Q}_t^\gamma)_{t=0}^T$ such that,*

$$\rho_t^\gamma(D) = - \inf_{\mathbb{Q} \in \mathcal{Q}_t^\gamma} \mathbb{E}^{\mathbb{Q}} \left[\sum_{s=t}^T D_s \middle| \mathcal{F}_t \right], \quad t \in \mathcal{T}, D \in L^0. \quad (\text{B.2})$$

The proof this theorem can be found in [BCZ11].

A direct consequence of Theorem B.0.2 and Theorem B.0.3 is the following:

Theorem B.0.4.

(i) *Assume that $(\mathcal{Q}_t^\gamma)_{t=0}^T$ for $\gamma \in (0, \infty)$ is an increasing family of dynamically consistent sequences of sets of probability measures. Then, the function $\alpha : \{0, 1, \dots, T\} \times L^0 \times \Omega \rightarrow [0, \infty]$ defined as*

$$\alpha_t(D)(\omega) = \sup \left\{ \gamma \in (0, \infty) : \inf_{\mathbb{Q} \in \mathcal{Q}_t^\gamma} \mathbb{E}^{\mathbb{Q}} \left[\sum_{s=t}^T D_s \middle| \mathcal{F}_t \right] (\omega) \geq 0 \right\},$$

for $\omega \in \Omega$, $t \in \mathcal{T}$, and $D \in L^0$ is a normalized and right-continuous dynamic coherent acceptability index.

(ii) *If α is a normalized and right-continuous dynamic coherent acceptability index, then there exists a family of dynamically consistent sequences of sets of probability measures $(\mathcal{Q}_t^\gamma)_{t=0}^T$ for $\gamma \in (0, \infty)$ such that*

$$\alpha_t(D)(\omega) = \sup \left\{ \gamma \in (0, \infty) : \inf_{\mathbb{Q} \in \mathcal{Q}_t^\gamma} \mathbb{E}_t^{\mathbb{Q}} \left[\sum_{s=t}^T D_s \middle| \mathcal{F}_t \right] (\omega) \geq 0 \right\}$$

for $\omega \in \Omega$, $t \in \mathcal{T}$, and $D \in L^0$. Here we adopt the convention that $\inf \emptyset = \infty$ and $\sup \emptyset = 0$.

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