

Arbitrage-Free Pricing Of Derivatives In Nonlinear Market Models

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ABSTRACT: The main objective is to study no-arbitrage pricing of financial derivatives in the presence of funding costs, the counterparty credit risk and market frictions affecting the trading mechanism, such as collateralization and capital requirements. To achieve our goals, we extend in several respects the nonlinear pricing approach developed in El Karoui and Quenez [EKQ97] and El Karoui et al. [EKPQ97].

KEYWORDS: hedging, fair price, funding cost, margin agreement, market friction, BSDE
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1 Introduction

The present work contributes to the nonlinear pricing theory, which arises in a natural way when accounting for salient features of real-world trades such as: trading constraints, differential funding costs, collateralization, counterparty credit risk and capital requirements. To be more specific, the aim of our study is to extend in several respects the hedging and pricing approach in nonlinear market models developed in El Karoui and Quenez [EKQ97] and El Karoui et al. [EKPQ97] (see also [CK93, DQS14, DQS15, KK96, KK98] for hedging and pricing with constrained portfolios) by accounting for the complexity of over-the-counter financial derivatives and specific features of the trading environment after the global financial crisis. Let us briefly summarize the main contributions of this work:

- The first goal is to discuss the trading strategies in the presence of differential funding rates and adjustment processes. We stress that the need for a more general approach arises due to the fact that we study general contracts with cash flow streams, rather than simple contingent claims with a single payoff at maturity (or upon exercise).
- Second, we examine in detail the issues of the existence of arbitrage opportunities for the hedger and for the trading desk in a nonlinear trading framework and with respect to a predetermined class of contracts. We introduce the concept of no-arbitrage with respect to the null contract and a stronger notion of no-arbitrage for the trading desk. We then proceed to the issue of unilateral fair valuation of a given contract by the hedger endowed with an initial capital. We examine the link between the concept of no-arbitrage for the trading desk and the financial viability of prices computed by the hedger.
- Third, we propose and analyze the concept of a *regular* market model, which extends the concept of a nonlinear pricing system introduced in El Karoui and Quenez [EKQ97]. The goal is to identify a class of nonlinear market models, which are arbitrage-free for the trading desk and, in addition, enjoy the desirable property that for contracts that can be replicated, the cost of replication is also the fair price for the hedger.
- Next, we focus on replication of a contract in a regular market model and we discuss the BSDE approach to the valuation and hedging of contracts in a model with differential funding rates, the counterparty credit risk and trading adjustments. We propose two main definitions of no-arbitrage prices, namely, the *gained value* and the *ex-dividend price*, and we show that they in fact coincide, under suitable technical conditions, when the pricing problem under consideration is *local*. It is worth stressing that in the case of a *global* pricing problem the two above-mentioned definitions will typically yield different pricing results for the hedger. To complete our study, we also examine the marked-to-market valuation of a contract and the problem of unwinding and offsetting for an existing contract.
- We conclude the paper by briefly addressing the issue of fair valuation and hedging of a counterparty risky contract in a nonlinear market model.

It should be acknowledged that we focus on fair unilateral pricing from the perspective of the hedger, although it is clear that the same definitions and pricing methods are applicable to his counterparty as well. Hence, in principle, it is also possible to use our results in order to examine the interval of fair bilateral prices in a regular market model. Particular instances of bilateral pricing problems were studied in [NR15, NR16a, NR16c] where it was shown that a non-empty interval of fair bilateral prices (or bilaterally profitable prices) can be obtained in some nonlinear models for

contracts with either an exogenous or an endogenous collateralization. For more practical studies of pricing and hedging subject to differential funding costs and the counterparty credit risk to which our general theory can be applied, the reader is referred to [BCS15, BP14, BK11, BK13, Cré15a, Cré15b, PPB12a, PPB12b, Pit10].

2 Nonlinear Market Model

In this section, we extend a generic market model introduced in [BR15]. Throughout the paper, we fix a finite trading horizon date $T > 0$ for our model of the financial market. Let $(\Omega, \mathcal{G}, \mathbb{G}, \mathbb{P})$ be a filtered probability space satisfying the usual conditions of right-continuity and completeness, where the filtration $\mathbb{G} = (\mathcal{G}_t)_{t \in [0, T]}$ models the flow of information available to all traders. For convenience, we assume that the initial σ -field \mathcal{G}_0 is trivial. Moreover, all processes introduced in what follows are implicitly assumed to be \mathbb{G} -adapted and, as usual, any semimartingale is assumed to be càdlàg. Also, we will assume that any process Y satisfies $\Delta Y_0 := Y_0 - Y_{0-} = 0$. Let us introduce the notation for the prices of all traded assets in our model.

Risky assets. We denote by $\mathcal{S} = (S^1, \dots, S^d)$ the collection of the *ex-dividend prices* of a family of d risky assets with the corresponding *cumulative dividend streams* $\mathcal{D} = (D^1, \dots, D^d)$. The process S^i is aimed to represent the ex-dividend price of any traded security, such as, stock, sovereign or corporate bond, stock option, interest rate swap, currency option or swap, CDS, CDO, etc.

Cash accounts. The *lending cash account* $B^{0,l}$ and the *borrowing cash account* $B^{0,b}$ are used for unsecured lending and borrowing of cash, respectively. For brevity, we will sometimes write B^l and B^b instead of $B^{0,l}$ and $B^{0,b}$. Also, when the borrowing and lending cash rates are equal, the single cash account is denoted by B^0 or simply B .

Funding accounts. We denote by $B^{i,l}$ (resp. $B^{i,b}$) the *lending* (resp. *borrowing*) *funding account* associated with the i th risky asset, for $i = 1, 2, \dots, d$. The financial interpretation of these accounts varies from case to case. For an overview of trading mechanisms for risky assets, we refer to Section 2.6. In the special case when $B^{i,l} = B^{i,b}$, we will use the notation B^i and we call it the *funding account* for the i th risky asset.

For brevity, denote by $\mathcal{B} = ((B^{i,l}, B^{i,b}), i = 0, 1, \dots, d)$ the collection of all cash and funding accounts.

2.1 Contracts with Trading Adjustments

We will consider financial contracts between two parties, called the *hedger* and the *counterparty*. In what follows, all the cash flows will be viewed from the perspective of the hedger. A *bilateral financial contract* (or simply a *contract*) is given as a pair $\mathcal{C} = (A, \mathcal{X})$ where the meaning of each term is explained below.

A stochastic processes A represents the *cumulative cash flows* from time 0 till maturity date T . The process A is assumed to model the (cumulative) cash flows of a given contract, which are either paid out from the hedger's wealth or added to the his wealth via the value process of his portfolio of traded assets (including cash). Note that the price of the contract exchanged at its initiation will not be included in A . For example, if a contract stipulates that the hedger will 'receive' the (possibly random) cash flows a_1, a_2, \dots, a_m at times $t_1, t_2, \dots, t_m \in (0, T]$, then

$$A_t = \sum_{l=1}^m \mathbb{1}_{[t_l, \infty)}(t) a_l.$$

Let $(A^t, 0)$ denote a contract initiated at time t with $\mathcal{X} = 0$. Then the only cash flow exchanged between parties at time t is the price of the contract, denoted as p_t , and thus the remaining cash flows of $(A^t, 0)$ are given as $A_u^t := A_u - A_t$ for $u \in [t, T]$. In particular, the equality $A_t^t = 0$ is valid for any contract $(A, 0)$ and any date $t \in [0, T]$. All future cash flows a_l for l such that $t_l > t$ are predetermined, in the sense that they are explicitly specified by the contract covenants, but the price p_t needs to be first properly defined and next computed using some particular market model.

As an example, consider the situation where the hedger sells at time t a European call option on the risky asset S^i . Then $m = 1$, $t_1 = T$, and the terminal payoff from the perspective of the issuer equals $a_1 = -(S_T^i - K)^+$. Consequently, for every $t \in [0, T)$, the process A^t satisfies $A_u^t = -(S_T^i - K)^+ \mathbb{1}_{[T, \infty)}(u)$ for all $u \in [t, T]$. Obviously, the price p_t of the option depends both on a market model and a pricing method.

To account for market frictions, we postulate that the cash flows A (resp. A^t) of a contract are complemented by *trading adjustments*, which are formally represented by the process \mathcal{X} (resp. \mathcal{X}^t) given as $\mathcal{X} = (X^1, \dots, X^n; \alpha^1, \dots, \alpha^n; \beta^1, \dots, \beta^n)$. Its role is to describe additional trading covenants associated with a given contract, such as collateralization and regulatory capital, as well as the adjustments due to the counterparty credit risk. For each *adjustment process* X^k , the auxiliary process $\alpha^k X^k$ represents additional incoming or outgoing cash flows for the hedger, which are either stipulated in the clauses of a contract (e.g., the credit support annex) or imposed by a third party (for instance, the regulator). In addition, each process X^k , $k = 1, 2, \dots, n$ is complemented by the corresponding *remuneration process* β^k , which is used to determine net interest payments (if any) associated with the process X^k . It should be noted that the processes X^1, \dots, X^n and the associated remuneration processes β^1, \dots, β^n do not represent traded assets. It is rather clear that the processes α and β may depend on the respective adjustment process. Therefore, when the adjustment process is \mathcal{Y} , rather than \mathcal{X} , one should write $\alpha(\mathcal{Y})$ and $\beta(\mathcal{Y})$ in order to avoid confusion. However, for brevity, we will keep the shorthand notation α and β when the adjustment process is denoted as \mathcal{X} .

Valuation or pricing of a given contract means, in particular, to find the range of *fair prices* p_t at any date t from the viewpoint of either the hedger or the counterparty. Although it will be postulated that both parties adopt the same valuation paradigm, due to the asymmetry of cash flows, differential trading costs and possibly also different trading opportunities, they will typically obtain different ranges for fair unilateral prices for a given bilateral contract. The discrepancy in pricing for the two parties is a consequence of the nonlinearity of the wealth dynamics in trading strategies, so that it may occur even within the framework a complete market model where the perfect replication of any contract can be achieved by both parties.

2.2 Self-financing Trading Strategies

The concept of a portfolio refers to the family of *primary traded assets*, that is, risky assets, cash accounts, and funding accounts for risky assets. Formally, by a *portfolio* on the time interval $[t, T]$, we mean an arbitrary \mathbb{R}^{3d+2} -valued, \mathbb{G} -adapted process $(\varphi_u^t)_{u \in [t, T]}$ denoted as

$$\varphi^t = (\xi^1, \dots, \xi^d; \psi^{0,l}, \psi^{0,b}, \psi^{1,l}, \psi^{1,b}, \dots, \psi^{d,l}, \psi^{d,b}) \quad (2.1)$$

where the components represent positions in risky assets (S^i, D^i) , $i = 1, 2, \dots, d$, cash accounts $B^{0,l}, B^{0,b}$, and funding accounts $B^{i,l}, B^{i,b}$, $i = 1, 2, \dots, d$ for risky assets. It is postulated throughout that $\psi_u^{j,l} \geq 0$, $\psi_u^{j,b} \leq 0$ and $\psi_u^{j,l} \psi_u^{j,b} = 0$ for all $j = 0, 1, \dots, d$ and $u \in [t, T]$. If the borrowing and lending rates are equal, then we denote by B^j , for $j = 0, 1, \dots, d$, the corresponding cash or funding account and we denote $\psi^j = \psi^{j,l} + \psi^{j,b}$. It is also assumed throughout that the processes ξ^1, \dots, ξ^d are \mathbb{G} -predictable.

We say that a portfolio φ is *constrained* if at least one of the components of the process φ is assumed to satisfy some additional constraints. For instance, we will need to impose conditions ensuring that the funding of each risky asset is done using the corresponding funding account. Another example of an explicit constraint is obtained when we set $\psi_u^{0,b} = 0$ for all $u \in [t, T]$, meaning that an outright borrowing of cash from the account $B^{0,b}$ is prohibited. We are now in a position to state some fairly standard technical assumptions underpinning our further developments.

Assumption 2.1. We work throughout under the following standing assumptions:

- (i) for every $i = 1, 2, \dots, d$, the price S^i of the i th risky asset is a semimartingale and the cumulative dividend stream D^i is a process of finite variation with $D_0^i = 0$;
- (ii) the cash and funding accounts $B^{j,l}$ and $B^{j,b}$ are strictly positive and continuous processes of finite variation with $B_0^{j,l} = B_0^{j,b} = 1$ for $j = 0, 1, \dots, d$;
- (iii) the cumulative cash flow process A of any contract is a process of finite variation;
- (iv) the adjustment processes X^k , $k = 1, 2, \dots, n$ and the auxiliary processes α^k , $k = 1, 2, \dots, n$ are semimartingales;
- (v) the remuneration processes β^k , $k = 1, 2, \dots, n$ are strictly positive and continuous processes of finite variation with $\beta_0^k = 1$ for every k .

In the next definition, the \mathcal{G}_t -measurable random variable x_t represents the *endowment* of the hedger at time $t \in [0, T)$ whereas p_t , which at this stage is an arbitrary \mathcal{G}_t -measurable random variable, stands for the price at time t of $\mathcal{C}^t = (A^t, \mathcal{X}^t)$, as seen by the hedger. Recall that A^t denotes the cumulative cash flows of the contract A that occur after time t , that is, $A_u^t := A_u - A_t$ for all $u \in [t, T]$. Hence A^t can be seen as a contract with the same remaining cash flows as the original contract A , except that A^t is initiated and traded at time t . By the same token, we denote by \mathcal{X}^t the adjustment process related to the contract A^t . Let \mathcal{C} be a predetermined class of contracts. As expected, it is assumed throughout that the null contract $\mathcal{N} = (0, 0)$ is traded in any market model, that is, $\mathcal{N} \in \mathcal{C}$ (see Assumption 3.1).

It should be noted that the prices p_t for contracts belonging to the class \mathcal{C} are yet unspecified and thus there is a certain degree of freedom in the foregoing definitions.

Note also that we use the convention that $\int_t^u := \int_{(t,u]}$ for any $t \leq u$.

Definition 2.2. A quadruplet $(x_t, p_t, \varphi^t, \mathcal{C}^t)$ is a *self-financing trading strategy* on $[t, T]$ associated with the contract $\mathcal{C} = (A, \mathcal{X})$ if the *portfolio value* $V^p(x_t, p_t, \varphi^t, \mathcal{C}^t)$, which is given by

$$V_u^p(x_t, p_t, \varphi^t, \mathcal{C}^t) := \sum_{i=1}^d \xi_u^i S_u^i + \sum_{j=0}^d \left(\psi_u^{j,l} B_u^{j,l} + \psi_u^{j,b} B_u^{j,b} \right) \quad (2.2)$$

satisfies for all $u \in [t, T]$

$$V_u^p(x_t, p_t, \varphi^t, \mathcal{C}^t) = x_t + p_t + G_u(x_t, p_t, \varphi^t, \mathcal{C}^t), \quad (2.3)$$

where the *adjusted gains process* $G(x_t, p_t, \varphi^t, \mathcal{C}^t)$ is given by

$$\begin{aligned} G_u(x_t, p_t, \varphi^t, \mathcal{C}^t) := & \sum_{i=1}^d \int_t^u \xi_v^i (dS_v^i + dD_v^i) + \sum_{j=0}^d \int_t^u \left(\psi_v^{j,l} dB_v^{j,l} + \psi_v^{j,b} dB_v^{j,b} \right) \\ & + \sum_{k=1}^n \alpha_u^k X_u^k - \sum_{k=1}^n \int_t^u X_v^k (\beta_v^k)^{-1} d\beta_v^k + A_u^t. \end{aligned} \quad (2.4)$$

For a given pair (x_t, p_t) , we denote by $\Phi^{t, x_t}(p_t, \mathcal{C}^t)$ the set of all self-financing trading strategies on $[t, T]$ associated with a contract \mathcal{C} .

When studying valuation of a contract \mathcal{C}^t for a fixed t , we will typically assume that the hedger's endowment x_t is given and we will search for the range of prices p_t for \mathcal{C}^t . For instance, when dealing with the hedger with a fixed initial endowment x_t at time t , we will consider the following set of self-financing trading strategies $\Phi^{t,x_t}(\mathcal{C}) = \cup_{\mathcal{C} \in \mathcal{C}} \cup_{p_t \in \mathcal{G}_t} \Phi^{t,x_t}(p_t, \mathcal{C}^t)$. Note, however, that in the definition of the market model we do not assume that the quantity x_t is predetermined.

Definition 2.3. The *market model* is the quintuplet $\mathcal{M} = (\mathcal{S}, \mathcal{D}, \mathcal{B}, \mathcal{C}, \Phi(\mathcal{C}))$ where $\Phi(\mathcal{C})$ stands for the set of all self-financing trading strategies associated with the class \mathcal{C} of contracts, that is, $\Phi(\mathcal{C}) = \cup_{t \in [0, T]} \cup_{x_t \in \mathcal{G}_t} \Phi^{t,x_t}(\mathcal{C})$.

Note that, in principle, the market model defined above is nonlinear, in the sense that either the portfolio value process $V^p(x_t, p_t, \varphi^t, \mathcal{C}^t)$ is not linear in $(x_t, p_t, \varphi^t, \mathcal{C}^t)$ or the class of all self-financing strategies is not a vector space (or both). Therefore, we refer to this model as to a generic *nonlinear market model*. In contrast, by a *linear market model* we will understand in this paper the version of the model defined above in which all adjustments are null (i.e., $X^k = 0$ for all $k = 1, 2, \dots, n$), there are no differential funding rates (i.e., $B^{j,b} = B^{j,l}$ for all $j = 0, 1, \dots, d$) and there are no trading constraints. In particular, in the linear market model the class of all self-financing trading strategies is a vector space and the value process $V^p(x_t, p_t, \varphi^t, \mathcal{C}^t)$ is a linear mapping in $(x_t, p_t, \varphi^t, \mathcal{C}^t)$.

To alleviate notation, we will frequently write $(x, p, \varphi, \mathcal{C})$ instead of $(x_0, p_0, \varphi^0, \mathcal{C}^0)$ when working on the interval $[0, T]$. Note that (2.3) yields the following equalities for any trading strategy $(x, p, \varphi, \mathcal{C}) \in \Phi^{0,x}(\mathcal{C})$

$$V_0^p(x, p, \varphi, \mathcal{C}) = \sum_{i=1}^d \xi_0^i S_0^i + \sum_{j=0}^d \left(\psi_0^{j,l} B_0^{j,l} + \psi_0^{j,b} B_0^{j,b} \right) = x + p + \sum_{k=1}^n \alpha_0^k X_0^k. \quad (2.5)$$

Recall that in the classical case of a frictionless market, it is common to assume that the initial endowments of traders are null. Moreover, the price of a derivative has no impact on the dynamics of the gains process. In contrast, when portfolio's value is driven by nonlinear dynamics, the initial endowment x at time 0, the initial price p and the cash flows of a contract may all affect the dynamics of the gains process and thus the classical approach is no longer valid.

2.3 Funding Adjustment

The concept of the funding adjustment refers to the spreads of funding rates with regard to some benchmark cash rate. In the present setup, it can be defined relative to either B^l or B^b . If the lending and borrowing rates are not equal, then (2.3) can be written as follows

$$\begin{aligned} V_t^p(x, p, \varphi, \mathcal{C}) &= x + p + \sum_{i=1}^d \int_0^t \xi_u^i (dS_u^i + dD_u^i) + \sum_{k=1}^n \alpha_t^k X_t^k + A_t \\ &+ \sum_{j=0}^d \int_0^t \left(\psi_u^{j,l} dB_u^{0,l} + \psi_u^{j,b} dB_u^{0,b} \right) - \sum_{k=1}^n \int_0^t \left((X_u^k)^+ (B_u^{0,l})^{-1} dB_u^{0,l} - (X_u^k)^- (B_u^{0,b})^{-1} dB_u^{0,b} \right) \\ &+ \sum_{i=1}^d \int_0^t \left(\psi_u^{i,l} ((\widehat{B}_u^{i,l} - 1) dB_u^{0,l} + B_u^{0,l} d\widehat{B}_u^{i,l}) + \psi_u^{i,b} ((\widehat{B}_u^{i,b} - 1) dB_u^{0,b} + B_u^{0,b} d\widehat{B}_u^{i,b}) \right) \\ &- \sum_{k=1}^n \int_0^t \left((X_u^k)^+ (\beta_u^{k,l})^{-1} d\widehat{\beta}_u^{k,l} - (X_u^k)^- (\beta_u^{k,b})^{-1} d\widehat{\beta}_u^{k,b} \right), \end{aligned}$$

where $\widehat{B}^{j,l/b} := B^{j,l/b}(B^{0,l/b})^{-1}$ and $\widehat{\beta}^{k,l/b} := \beta^k(B^{0,l/b})^{-1}$. The quantity

$$\begin{aligned} & \sum_{i=1}^d \int_0^t \left(\psi_u^{i,l} ((\widehat{B}_u^{i,l} - 1) dB_u^{0,l} + B_u^{0,l} d\widehat{B}_u^{i,l}) + \psi_u^{i,b} ((\widehat{B}_u^{i,b} - 1) dB_u^{0,b} + B_u^{0,b} d\widehat{B}_u^{i,b}) \right) \\ & - \sum_{k=1}^n \int_0^t \left((X_u^k)^+ (\beta_u^{k,l})^{-1} d\widehat{\beta}_u^{k,l} - (X_u^k)^- (\beta_u^{k,b})^{-1} d\widehat{\beta}_u^{k,b} \right) \end{aligned}$$

is called the *funding adjustment*. If the borrowing and lending rates are equal, then the expression for the funding adjustment simplifies to

$$\sum_{i=1}^d \int_0^t \psi_u^i ((\widehat{B}_u^i - 1) dB_u^0 + B_u^0 d\widehat{B}_u^i) - \sum_{k=1}^n \int_0^t X_u^k (\widehat{\beta}_u^k)^{-1} d\widehat{\beta}_u^k.$$

When the cash account B^0 is used for funding and remuneration for adjustment processes, that is, when $B^i = B^0$ for $i = 1, 2, \dots, d$ and $\beta^k = B^0$ for $k = 1, 2, \dots, n$, then the funding adjustment vanishes, as was expected.

2.4 Financial Interpretation of Trading Adjustments

In this study, we will devote significant attention to terms appearing in the dynamics of $V^p(x, \varphi, A, \mathcal{X})$, which correspond to the trading adjustment process \mathcal{X} .

Definition 2.4. The stochastic process $\varpi = \sum_{k=1}^n \varpi^k$, where for $k = 1, \dots, n$,

$$\varpi_t^k := \alpha_t^k X_t^k - \int_0^t X_u^k (\beta_u^k)^{-1} d\beta_u^k \quad (2.6)$$

is called the *cash adjustment*.

In general, the financial interpretation of the cash adjustment term ϖ^k is as follows: the term $\alpha_t^k X_t^k$ represents the part of the k th adjustment that the hedger can either use for his trading purposes when $\alpha_t^k X_t^k > 0$ or has to put aside (for instance, pledge as a collateral or hold as a regulatory capital) when $\alpha_t^k X_t^k < 0$. Let us illustrate alternative interpretations of cash adjustments given by (2.6). We denote $\widehat{X}^k = (\beta^k)^{-1} X^k$.

- Let us first assume that $\alpha_t^k = 1$, for all t . The term $X_t^k - \int_0^t \widehat{X}_u^k d\beta_u^k$ indicates that the cash adjustment ϖ^k is affected by both the current value X_t^k of the adjustment process and by the cost of funding of this adjustment given by the integral $\int_0^t \widehat{X}_u^k d\beta_u^k$. Such a situation occurs, for example, when X^k represents the capital charge or the rehypotecated collateral. The integration by parts formula gives

$$\varpi_t^k = X_t^k - \int_0^t \widehat{X}_u^k d\beta_u^k = X_0^k + \int_0^t \beta_u^k d\widehat{X}_u^k, \quad (2.7)$$

where the integral $\int_0^t \beta_u^k d\widehat{X}_u^k$ has the following financial interpretation: \widehat{X}_u^k is the number of units of the funding account β_u^k that are needed to fund the amount X_u^k of the adjustment process. Hence $d\widehat{X}_u^k$ is the infinitesimal change of this number and $\beta_u^k d\widehat{X}_u^k$ is the cost of this change, which has to be absorbed by the change in the value of the trading strategy. Observe that the term $\beta_u^k d\widehat{X}_u^k$ may be negative, meaning that a cash relieve situation is taking place.

In the special case when $\alpha_t^k = 1$ and $\beta_t^k = 1$ for all t , we obtain $\varpi_t^k = X_t^k$ for all t . We deal here with the cash adjustment X^k on which there is no remuneration since manifestly $\int_0^t \widehat{X}_u^k d\beta_u^k = 0$. This situation may arise, for example, if the bank does not use any external funding for financing this adjustment, but relies on its own cash reserves, which are assumed to be kept idle and neither yield interest nor require interest payouts.

- Let us now assume that $\alpha_t^k = 0$ for all t . Then the term $\varpi_t^k = -\int_0^t \widehat{X}_u^k d\beta_u^k$ indicates that the cash value of the adjustment X^k does not contribute to the portfolio value. Only the remuneration of the adjustment process X^k , which is given by the integral $\int_0^t \widehat{X}_u^k d\beta_u^k$, contributes to the portfolio's value. This happens, for example, when the adjustment process represents the collateral posted by the counterparty and kept in the segregated account.

As was argued above, in most practical applications, the cash adjustment process can be represented as follows

$$\varpi_t = \sum_{k=1}^{n_1} X_0^k + \sum_{k=1}^{n_1} \int_0^t \beta_u^k d\widehat{X}_u^k - \sum_{k=n_1+1}^{n_1+n_2} \int_0^t \widehat{X}_u^k d\beta_u^k + \sum_{k=n_1+n_2+1}^{n_1+n_2+n_3} X_t^k, \quad (2.8)$$

where the non-negative integers n_1, n_2, n_3 are assumed to satisfy $n_1 + n_2 + n_3 = n$.

2.5 Wealth Process

Let $(x, p, \varphi, \mathcal{C})$ be an arbitrary self-financing trading strategy. Then the following natural question arises: what is the wealth of a hedger at time t , say $V_t(x, p, \varphi, \mathcal{C})$? It is clear that if the hedger's initial endowment equals x , then his initial wealth equals $x + p$ when he sells a contract \mathcal{C} at the price p at time 0. By contrast, the initial value of the hedger's portfolio, that is, the total amount of cash he invests at time 0 in his portfolio of traded assets, is given by (2.5) meaning that the trading adjustments at time 0 need to be accounted for in the initial portfolio's value. However, according to the financial interpretation of trading adjustments, they have no bearing on the hedger's initial wealth and thus the relationship between the hedger's initial wealth and the initial portfolio's value reads

$$V_0(x, p, \varphi, \mathcal{C}) = V_0^p(x, p, \varphi, \mathcal{C}) - \sum_{k=1}^n \alpha_0^k X_0^k.$$

Analogous arguments can be used at any time $t \in [0, T]$, since the hedger's wealth at time t should represent the value of his portfolio of traded assets net of the value of all trading adjustments (see (2.10)). Furthermore, one needs to focus on the actual ownership (as opposed to the legal ownership) of each of the adjustment processes X^1, \dots, X^n , of course, provided that they do not vanish at time t . Although this general rule is cumbersome to formalize, it will not present any difficulties when applied to a particular contract at hand.

For instance, in the case of the rehypothecated cash collateral (see Section 2.7.1), the hedger's wealth at time t should be computed by subtracting the collateral amount C_t from the portfolio's value. This is consistent with the actual ownership of the cash amount delivered by either the hedger or the counterparty at time t . For example, if $C_t^+ > 0$ then the *legal owner* of the amount C_t^+ at time t could be either the holder or the counterparty (depending on the legal covenants of the collateral agreement) but the hedger, as a collateral taker, is allowed to use the collateral amount for his trading purposes. If there is no default before T , the collateral taker returns the collateral amount to the collateral provider. Hence the amount C_t^+ should be accounted for when dealing with the hedger's portfolio, but should be excluded from his wealth. In general, we have the following definition of the wealth process.

Definition 2.5. The *wealth process* of a self-financing trading strategy $(x_t, p_t, \varphi^t, \mathcal{C}^t)$ defined, for every $u \in [t, T]$, by

$$V_u(x_t, p_t, \varphi^t, \mathcal{C}^t) := V_u^p(x_t, p_t, \varphi^t, \mathcal{C}^t) - \sum_{k=1}^n \alpha_u^k X_u^k \quad (2.9)$$

or, more explicitly,

$$V_u(x_t, p_t, \varphi^t, \mathcal{C}^t) = \sum_{i=1}^d \xi_u^i S_u^i + \sum_{j=0}^d \left(\psi_u^{j,l} B_u^{j,l} + \psi_u^{j,b} B_u^{j,b} \right) - \sum_{k=1}^n \alpha_u^k X_u^k. \quad (2.10)$$

Let us observe that there is a lot of flexibility in the choice of the adjustment processes X^k s and corresponding processes α^k s. However, we will always assume that these processes are specified such that the above arguments of interpreting the actual ownership of the capital and thus also of the wealth process $V(x, p, \varphi, A, \mathcal{X})$ hold true.

As an immediate consequence of Definitions 2.2 and 2.5, it follows that the wealth process V of any self-financing trading strategy $(x_t, p_t, \varphi^t, \mathcal{C}^t)$ admits the dynamics, for $u \in [t, T]$,

$$\begin{aligned} V_u(x_t, p_t, \varphi^t, \mathcal{C}^t) &= x_t + p_t + \sum_{i=1}^d \int_t^u \xi_v^i d(S_v^i + D_v^i) + \sum_{j=0}^d \int_t^u \left(\psi_v^{j,l} dB_v^{j,l} + \psi_v^{j,b} dB_v^{j,b} \right) \\ &\quad - \sum_{k=1}^n \int_t^u X_v^k (\beta_v^k)^{-1} d\beta_v^k + A_u^t. \end{aligned} \quad (2.11)$$

One could argue that it would be possible to take equations (2.10) and (2.11) as the definition of a self-financing trading strategy and subsequently deduce that equality (2.3) holds for the portfolio's value $V^p(x, p, \varphi, \mathcal{C})$, which is then given by (2.9). We contend this alternative approach would not be optimal, since conditions in Definition 2.2 are obtained through a straightforward analysis of the trading mechanism and physical cash flows, whereas the financial justification of equations (2.10)–(2.11) is less appealing.

Clearly, the wealth processes of a self-financing trading strategy is characterized in terms of two equations (2.10) and (2.11). Observe that, using (2.10), it is possible to eliminate one of the processes $\psi^{j,l}$ or $\psi^{j,b}$ from (2.11) and thus to characterize the wealth process in terms of a single equation. We obtain in this way a (typically nonlinear) BSDE, which can be used to formulate various valuation problems for a given contract.

2.6 Trading in Risky Assets

Note that we do not postulate that processes S^i , $i = 1, 2, \dots, d$ are positive, unless it is explicitly stated that the process S^i models the price of a stock. Hence by the *long cash position* (resp. *short cash position*), we mean the situation when $\xi_t^i S_t^i \leq 0$ (resp., $\xi_t^i S_t^i \geq 0$), where ξ_t^i is the number of hedger's positions in the risky asset S^i at time t .

2.6.1 Cash Market Trading

For simplicity of presentation, we assume that $S_t^i \geq 0$ for all $t \in [0, T]$. Assume first that the purchase of $\xi_t^i > 0$ shares of the i th risky asset is funded using cash. Then we set $\psi_t^{i,b} = 0$ for all $t \in [0, T]$ and thus the process $B^{i,b}$ becomes irrelevant. Let us now consider the case when $\xi_t^i < 0$. If we assume that the proceeds from short selling of the risky asset S^i can be used by the hedger (this is typically not true in practice), we also set $\psi_t^{i,l} = 0$ for all $t \in [0, T]$, and thus the process

$B^{i,l}$ becomes irrelevant as well. Hence, under these *stylized* cash trading conventions, there is no need to introduce the funding accounts $B^{i,l}$ and $B^{i,b}$ for the i th risky asset. Since dividends D^i are passed over to the lender of the asset, they do not appear in the term representing the gains/losses from the short position in the risky asset. In the simplest case of no market frictions and trading adjustments, and with the single risky asset S^1 , under the present short selling convention, (2.3) becomes

$$V_t^p(x, p_0, \varphi, \mathcal{C}) = x + p + \int_0^t \xi_u^1 (dS_u^1 + dD_u^1) + \int_0^t \left(\psi_u^{0,l} dB_u^{0,l} + \psi_u^{0,b} dB_u^{0,b} \right) + A_t. \quad (2.12)$$

More practical short selling conventions are discussed in the foregoing subsections.

2.6.2 Short Selling of Risky Assets

Let us now consider the classical way of short selling of a risky asset borrowed from a broker. In that case, the hedger does not receive the proceeds from the sale of the borrowed shares of a risky asset, which are held instead by the broker as the cash collateral. The hedger may also be requested to post additional cash collateral to the broker and, in some cases, he may be paid interest on his margin account with the broker.¹ To represent these trading arrangements for the i th risky asset, we set $\psi_t^{i,l} = 0$, $\alpha_t^i = \alpha_t^{i+d} = 0$ and

$$X_t^i = -(1 + \delta_t^i)(\xi_t^i)^- S_t^i, \quad X^{i+d} = \delta_t^i(\xi_t^i)^- S_t^i,$$

where β_t^i specifies the interest (if any) on the hedger's margin account with the broker, $\delta_t^i \geq 0$ represents an additional cash collateral, and β^{i+d} specifies the interest rate paid by the hedger for financing the additional collateral.

For example, if we assume that the risky asset is purchased using cash as in Section 2.6.1, we get the following equality, which is a slight extension of equality (2.2),

$$V_t^p(x, p, \varphi, \mathcal{C}) = \sum_{i=1}^d (\xi_t^1)^+ S_t^i + \psi_t^{0,l} B_t^{0,l} + \psi_t^{0,b} B_t^{0,b} \quad (2.13)$$

whereas equation (2.3) becomes

$$\begin{aligned} V_t^p(x, p, \varphi, \mathcal{C}) = & x + p + \int_0^t \xi_u^1 (dS_u^1 + dD_u^1) + \int_0^t \left(\psi_u^{0,l} dB_u^{0,l} + \psi_u^{0,b} dB_u^{0,b} \right) + A_t \\ & + \int_0^t (\beta_u^1)^{-1} (1 + \delta_u^1)(\xi_u^1)^- S_u^1 d\beta_u^1 - \int_0^t (\beta_u^2)^{-1} \delta_u^1 (\xi_u^1)^- S_u^1 d\beta_u^2. \end{aligned} \quad (2.14)$$

If, however, a specific interest rate for remuneration of an additional collateral is not specified, then we set $X^{i+d} = 0$ and thus the last term in (2.14) should be omitted.

2.6.3 Repo Market Trading

Let us first consider the *cash-driven repo transaction*, the situation when shares of the i th risky asset owned by the hedger are used as collateral to raise cash.² To represent this transaction, we set

$$\psi_t^{i,b} = -(B_t^{i,b})^{-1} (1 - h^{i,b})(\xi_t^i)^+ S_t^i, \quad (2.15)$$

¹The interested reader may consult the web pages <http://www.investopedia.com/terms/s/shortsale.asp> and <https://www.sec.gov/investor/pubs/regsho.htm> for more details on the mechanics of short-sales.

²We refer to https://www.newyorkfed.org/medialibrary/media/research/staff_reports/sr529.pdf for a detailed description of mechanics of repo trading.

where $B^{i,b}$ specifies the interest paid to the lender by the hedger who borrows cash and pledges the risky asset S^i as collateral, and the constant $h^{i,b}$ represents the *haircut* for the i th asset pledged.

A synthetic short-selling of the risky asset S^i using the repo market is obtained through the *security-driven repo transaction*, that is, when shares of the risky asset are posted as collateral by the borrower of cash and they are immediately sold by the hedger who lends the cash. Formally, this situation corresponds to the equality

$$\psi_t^{i,l} = (B_t^{i,l})^{-1}(1 - h^{i,l})(\xi_t^i)^- S_t^i \quad (2.16)$$

where $B^{i,l}$ specifies the interest amount paid to the hedger by the borrower of the cash amount $(1 - h^{i,l})(\xi_t^i)^- S_t^i$ and $h^{i,l}$ is the corresponding haircut.

If only one risky asset is traded and transactions are exclusively in repo market, then we obtain

$$\begin{aligned} V_t^p(x, p, \varphi, C) &= x + p + \int_0^t \xi_u^1 (dS_u^1 + dD_u^1) + \int_0^t \left(\psi_u^{0,l} dB_u^{0,l} + \psi_u^{0,b} dB_u^{0,b} \right) \\ &+ \int_0^t \left((B_u^{1,l})^{-1}(1 - h^{1,l})(\xi_u^1)^- S_u^1 dB_u^{1,l} - (B_u^{1,b})^{-1}(1 - h^{1,b})(\xi_u^1)^+ S_u^1 dB_u^{1,b} \right) + A_t. \end{aligned} \quad (2.17)$$

In practice, it is reasonable to assume that the long and short repo rates for a given risky asset are identical, that is, $B^{i,l} = B^{i,b}$. In that case, we may and do set $B^i := B^{i,l} = B^{i,b}$ and $\psi_t^i = (1 - h^i)(B_t^i)^{-1} \xi_t^i S_t^i$, so that equations (2.15) and (2.16) reduce to just one equation

$$(1 - h^i) \xi_t^i S_t^i + \psi_t^i B_t^i = 0. \quad (2.18)$$

According to this interpretation of B^i , equality (2.18) means that trading in the i th risky asset is done using the (symmetric) repo market and ξ_t^i shares of a risky asset are pledged as collateral at time t , meaning that the collateral rate equals 1. Under (2.18), equation (2.17) reduces to

$$\begin{aligned} V_t^p(x, p, \varphi, C) &= x + p + \int_0^t \xi_u^1 (dS_u^1 + dD_u^1) + \int_0^t \left(\psi_u^{0,l} dB_u^{0,l} + \psi_u^{0,b} dB_u^{0,b} \right) \\ &- \int_0^t (B_u^1)^{-1}(1 - h^1) \xi_u^1 S_u^1 dB_u^1 + A_t. \end{aligned} \quad (2.19)$$

2.7 Collateralization

We consider the situation when the hedger and the counterparty enter a contract and either receive or pledge *collateral* with value denoted by C , which is assumed to be a semimartingale. Generally speaking, the process C represents the value of the *margin account*. We let

$$C_t = X_t^1 + X_t^2, \quad (2.20)$$

where $X_t^1 := C_t^+ = C_t \mathbf{1}_{\{C_t \geq 0\}}$, and $X_t^2 := -C_t^- = C_t \mathbf{1}_{\{C_t < 0\}}$. By convention, the amount C_t^+ is the cash value of collateral received at time t by the hedger from the counterparty, whereas C_t^- represents the cash value of collateral pledged by him and thus transferred to his counterparty. For simplicity of presentation and consistently with the prevailing market practice, it is postulated throughout that only cash collateral may be delivered or received (for other collateral conventions, see, e.g., [BR15]). According to ISDA Margin Survey 2014, about 75% of non-cleared OTC collateral agreements are settled in cash and about 15% in government securities. We also make the following natural assumption regarding the value of the margin account at the contract's maturity date.

Assumption 2.6. The \mathbb{G} -adapted collateral amount process C satisfies $C_T = 0$.

Typically this means that the collateral process C will have a jump at time T from C_{T-} to 0. The postulated equality $C_T = 0$ is simply a convenient way of ensuring that any collateral amount posted is returned in full to the pledger when the contract matures, provided that the default events have not occurred prior to or at maturity date T . As soon as the default events are also modeled, we will need to specify the closeout payoff (see Section 2.8.1).

Let us first make some comments from the hedger's perspective regarding the crucial features of the margin account. The financial practice may require to hold the collateral amounts in *segregated* margin accounts, so that the hedger, when he is a collateral taker, cannot make use of the collateral amount for trading. Another collateral convention mostly encountered in practice is *rehypothecation* (around 90% of cash collateral of OTC contracts are rehypothecated), which refers to the situation where the hedger may use the collateral pledged by his counterparties as collateral for his contracts with other counterparties. Obviously, if the hedger is a collateral provider, then a particular convention regarding segregation or rehypothecation is immaterial for the dynamics of the value process of his portfolio. We refer the reader to [BR15] and [CBB14] for a detailed analysis of various conventions on collateral agreements. Here we will examine some basic aspects of *collateralization* (sometimes also called *margining*) in our context.

In general, we have

$$\varpi_t = \alpha_t^1 C_t^+ - \alpha_t^2 C_t^- - \int_0^t (\beta_u^1)^{-1} C_u^+ d\beta_u^1 + \int_0^t (\beta_u^2)^{-1} C_u^- d\beta_u^2, \quad (2.21)$$

where the remuneration processes β^1 and β^2 determine the interest rates paid or received by the hedger on collateral amounts C^+ and C^- , respectively. The auxiliary processes α^1 and α^2 introduced in (2.21) are used to cover alternative conventions regarding rehypothecation and segregation of margin accounts. Note that we always set $\alpha_t^2 = 1$ for all $t \in [0, T]$ when considering the portfolio of the hedger, since a particular convention regarding rehypothecation or segregation is manifestly irrelevant for the pledger of collateral.

2.7.1 Rehypothecated Collateral

As it is customary in the existing literature, we assume that rehypothecation of cash collateral means that it can be used by the hedger for his trading purposes without any restrictions. To cover this stylized version of a *rehypothecated collateral* for the hedger, it suffices to set $\alpha_t^1 = \alpha_t^2 = 1$ for all $t \in [0, T]$, so that for the hedger we obtain $\alpha_t^1 X_t^1 + \alpha_t^2 X_t^2 = C_t$. Consequently, the cash adjustment corresponding to the margin account equals

$$\varpi_t = \varpi_t^1 + \varpi_t^2 = \sum_{k=1}^2 \left(X_0^k + \int_0^t \beta_u^k d\widehat{X}_u^k \right). \quad (2.22)$$

2.7.2 Segregated Collateral

Under segregation, the collateral received by the hedger is kept by the third party, so that it cannot be used by the hedger for his trading activities. In that case, we set $\alpha_t^1 = 0$ and $\alpha_t^2 = 1$ for all $t \in [0, T]$ and thus $\alpha_t^1 X_t^1 + \alpha_t^2 X_t^2 = -C_t^-$. Hence the corresponding cash adjustment term ϖ equals

$$\varpi_t = \varpi_t^1 + \varpi_t^2 = X_0^2 - \int_0^t \widehat{X}_u^1 d\beta_u^1 + \int_0^t \beta_u^2 d\widehat{X}_u^2. \quad (2.23)$$

2.7.3 Initial and Variation Margins

In market practice, the total collateral amount is usually represented by two components, which are termed the *initial margin* (also known as the *independent amount*) and the *variation margin*. In the context of self-financing trading strategies, this can be easily dealt with by introducing two (or more) collateral processes for a given contract A . It is worth mentioning that each of the collateral processes specified in the clauses of a contract is usually subject to a different convention regarding segregation and/or remuneration.

2.8 Counterparty Credit Risk

The *counterparty credit risk* in a financial contract arises from the possibility that at least one of the parties in the contract may default prior to or at the contract's maturity, which may result in failure of this party to fulfil all their contractual obligations leading to financial loss suffered by either one of the two parties in the contract. We will model defaultability of the two parties to the contract in terms of their default times. We denote by τ^h and τ^c the default times of the hedger and his counterparty, respectively. We require that τ^h and τ^c are non-negative random variables defined on $(\Omega, \mathcal{G}, \mathbb{G}, \mathbb{P})$. If $\tau^h > T$ holds a.s. (resp., $\tau^c > T$, a.s.) then the hedger (resp., the counterparty) is considered to be default-free, at least with respect to the contract under study. Hence the counterparty risk is a relevant aspect of our model provided that $\mathbb{P}(\tau \leq T) > 0$, where $\tau := \tau^h \wedge \tau^c$ is the moment of the first default.

From now on, we postulate that the process A models all promised (or nominal) cash flows of the contract, as seen from the perspective of the trading desk without taking into account the possibility of defaults of trading parties. In other words, A represents cash flows that would be realized in case none of the two parties defaulted prior to or at the contract's maturity. We will sometimes refer to A as to the *counterparty risk-free cash flows* or *counterparty clean cash flows* and we will call the contract with cash flows A the *counterparty risk-free contract* or the *counterparty clean contract*. The key concept in the context of counterparty risk is the *counterparty risky contract*.

2.8.1 Closeout Payoff

On the event $\{\tau < \infty\}$, we define the random variable Υ as

$$\Upsilon = Q_\tau + \Delta A_\tau - C_\tau, \quad (2.24)$$

where Q is the Credit Support Annex (CSA) closeout valuation process of the contract A , $\Delta A_\tau = A_\tau - A_{\tau-}$ is the jump of A at τ corresponding to a (possibly null) promised bullet dividend at τ , and C_τ is the value of the collateral process C at time τ . In the financial interpretation, Υ^+ is the amount the counterparty owes to the hedger at time τ , whereas Υ^- is the amount the hedger owes to the counterparty at time τ . It accounts for the legal value Q_τ of the contract, plus the bullet dividend ΔA_τ to be received/paid at time τ , less the collateral amount C_τ since it is already held by the hedger (if $C_\tau > 0$) or by the counterparty (if $C_\tau < 0$). We refer the reader to [CBB14, Section 3.1.3] for more details regarding the financial interpretation of Υ .

One of the key financial aspects of the counterparty risky contract is the closeout payoff, which occurs if at least one of the parties defaults either before or at the maturity of the contract. It represents the cash flow exchanged between the two parties at first-party-default time. The following definition of the closeout payoff, as seen from the perspective of the hedger, is taken from [CBB14]. The random variables R_c and R_h , which take values in $[0, 1]$, represent the recovery rates of the counterparty and the hedger, respectively.

Definition 2.7. The *CSA closeout payoff* \mathfrak{K} is defined as

$$\mathfrak{K} := C_\tau + \mathbb{1}_{\{\tau^c < \tau^h\}}(R_c \Upsilon^+ - \Upsilon^-) + \mathbb{1}_{\{\tau^h < \tau^c\}}(\Upsilon^+ - R_h \Upsilon^-) + \mathbb{1}_{\{\tau^h = \tau^c\}}(R_c \Upsilon^+ - R_h \Upsilon^-). \quad (2.25)$$

The *counterparty risky cumulative cash flows* process A^\sharp is given by

$$A_t^\sharp = \mathbb{1}_{\{t < \tau\}} A_t + \mathbb{1}_{\{t \geq \tau\}}(A_{\tau-} + \mathfrak{K}), \quad t \in [0, T]. \quad (2.26)$$

Let us make some comments on the form of the closeout payoff \mathfrak{K} . First, the term C_τ is due to the fact that legal title to the collateral amount comes into force only at the time of the first default. The following three terms correspond to the CSA convention that, in principle, the nominal cash flow at the first default from the perspective of the hedger is given as $Q_\tau + \Delta A_\tau$. Let us consider, for instance, the event $\{\tau_c < \tau_h\}$. If $\Upsilon^+ > 0$, then we obtain

$$\mathfrak{K} = C_\tau + R_c(Q_\tau + \Delta A_\tau - C_\tau) \leq Q_\tau + \Delta A_\tau,$$

where the equality holds whenever $R_c = 1$. If $\Upsilon^- > 0$, then we get

$$\mathfrak{K} = C_\tau - (-Q_\tau - \Delta A_\tau + C_\tau) = Q_\tau + \Delta A_\tau.$$

Finally, if $\Upsilon = 0$, then $\mathfrak{K} = C_\tau = Q_\tau + \Delta A_\tau$. Similar analysis can be done on the remaining two events in (2.25).

Remark 2.8. Of course, there is no counterparty credit risk present under the assumption that $\mathbb{P}(\tau > T) = 1$. Let us consider the case where $\mathbb{P}(\tau > T) < 1$. We denote by p_t^e the *clean* (that is, counterparty risk-free) ex-dividend price of the contract at time t . If we set $R_c = R_h = 1$, then we obtain

$$A_\tau^\sharp = A_\tau + Q_\tau.$$

Hence the counterparty credit risk is still present, despite the postulate of the full recovery, unless the legal value Q_τ perfectly matches the clean ex-dividend price p_τ^e . Obviously, the counterparty credit risk vanishes when $R_c = R_h = 1$ and $Q_\tau = p_\tau^e$, since in that case the so-called *exposure at default* (see [CBB14, Section 3.2.3]) is null.

2.8.2 Counterparty Credit Risk Decomposition

To effectively deal with the closeout payoff in our general framework, we now define the counterparty credit risk (CCR) cash flows, which are sometimes called CCR exposures. Note that the events $\{\tau = \tau^h\} = \{\tau^h \leq \tau^c\}$ and $\{\tau = \tau^c\} = \{\tau^c \leq \tau^h\}$ may overlap.

Definition 2.9. By the *CCR processes*, we mean the processes CL , CG and RP where the *credit loss* CL equals

$$CL_t = -\mathbb{1}_{\{t \geq \tau\}} \mathbb{1}_{\{\tau = \tau^c\}}(1 - R_c) \Upsilon^+,$$

the *credit gain* CG equals

$$CG_t = \mathbb{1}_{\{t \geq \tau\}} \mathbb{1}_{\{\tau = \tau^h\}}(1 - R_h) \Upsilon^-,$$

and the *replacement process* is given by

$$CR_t = \mathbb{1}_{\{t \geq \tau\}}(A_\tau - A_t + Q_\tau).$$

The *CCR cash flow* is given by $A^{\text{CCR}} = CL + CG + CR$.

It is worth noting that the CCR cash flows depend on processes A, C and Q . The next proposition shows that we may interpret the counterparty risky contract as the clean contract A , which is complemented by the collateral adjustment process $\mathcal{X} = (X^1, X^2) = (C^+, -C^-)$ and the CCR cash flow A^{CCR} . In view of this result, the counterparty risky contract (A^\sharp, \mathcal{X}) admits the following formal decompositions $(A^\sharp, \mathcal{X}) = (A, \mathcal{X}) + (A^{\text{CCR}}, 0)$ and $(A^\sharp, \mathcal{X}) = (A, 0) + (A^{\text{CCR}}, \mathcal{X})$.

Proposition 2.10. *The equality $A_t^\sharp = A_t + A_t^{\text{CCR}}$ holds for all $t \in [0, T]$.*

Proof. We first note that

$$\begin{aligned} \mathfrak{R} &= C_\tau + \mathbb{1}_{\{\tau^c < \tau^h\}}(R_c \Upsilon^+ - \Upsilon^-) + \mathbb{1}_{\{\tau^h < \tau^c\}}(\Upsilon^+ - R_h \Upsilon^-) + \mathbb{1}_{\{\tau^h = \tau^c\}}(R_c \Upsilon^+ - R_h \Upsilon^-) \\ &= C_\tau - \mathbb{1}_{\{\tau^c \leq \tau^h\}}(1 - R_c) \Upsilon^+ + \mathbb{1}_{\{\tau^h \leq \tau^c\}}(1 - R_h) \Upsilon^- + \Upsilon \\ &= Q_\tau + \Delta A_\tau - \mathbb{1}_{\{\tau^c \leq \tau^h\}}(1 - R_c) \Upsilon^+ + \mathbb{1}_{\{\tau^h \leq \tau^c\}}(1 - R_h) \Upsilon^- \end{aligned}$$

where we used (2.24) in the last equality. Therefore, from (2.26) we obtain

$$\begin{aligned} A_t^\sharp &= \mathbb{1}_{\{t < \tau\}} A_t + \mathbb{1}_{\{t \geq \tau\}} (A_{\tau-} + \mathfrak{R}) = \mathbb{1}_{\{t < \tau\}} A_t + \mathbb{1}_{\{t \geq \tau\}} (A_\tau - \Delta A_\tau + \mathfrak{R}) \\ &= A_{t \wedge \tau} + \mathbb{1}_{\{t \geq \tau\}} (\mathfrak{R} - \Delta A_\tau) = A_t + (A_{t \wedge \tau} - A_t) + \mathbb{1}_{\{t \geq \tau\}} (\mathfrak{R} - \Delta A_\tau) \\ &= A_t + \mathbb{1}_{\{t \geq \tau\}} (A_\tau - A_t + Q_\tau - \mathbb{1}_{\{\tau^c \leq \tau^h\}}(1 - R_c) \Upsilon^+ + \mathbb{1}_{\{\tau^h \leq \tau^c\}}(1 - R_h) \Upsilon^-), \end{aligned}$$

which is the desired equality in view of Definition 2.9. \square

More generally, given two contracts, say (A^i, \mathcal{X}^i) for $i = 1, 2$, we are interested in pricing and hedging issues for a *compound contract* (A, \mathcal{X}) where $A = f(A^1, A^2)$ in relation to pricing and hedging of its components (A^1, \mathcal{X}^1) and (A^2, \mathcal{X}^2) . To be more specific, we wish to find out whether individual no-arbitrage pricing for (A^1, \mathcal{X}^1) and (A^2, \mathcal{X}^2) leads to, at least approximate, fair valuation for the contract (A, \mathcal{X}) . We need to stress that there may be a feedback effect involved between the compound contract and its components. For instance, it follows from Proposition 2.10 that the counterparty risky contract can be decomposed into the clean component $(A^1, \mathcal{X}^1) = (A, \mathcal{X})$ and the CCR component $(A^2, \mathcal{X}^2) = (A^{\text{CCR}}, 0)$. In bank's practice, the exit price for a counterparty risky contract is the combination of the clean price of the contract and the price of the counterparty credit risk, which is referred to as the *CCR price* in what follows. The clean price and the corresponding hedge are established by the trading desk, whereas the price and hedge for the counterparty credit risk are dealt with by the dedicated CVA desk. To sum up, the typical procedure used in industry to derive the exit price of the contract (A^\sharp, \mathcal{X}) is based on the following additive decomposition

$$\text{price}(A^\sharp, \mathcal{X}) = \text{price}(A, \mathcal{X}) + \text{price}(A^{\text{CCR}}, 0) = \text{clean price} + \text{CCR price}. \quad (2.27)$$

It is unclear under which conditions this procedure results in an overall arbitrage-free valuation and hedging of the counterparty risky contract, in general, since the implicitly assumed additivity of pricing does not necessarily hold under market frictions.

In the existing literature, the counterpart of the above relationship is usually represented by the equality

$$\text{counterparty risky price} = \text{clean price} - \text{TVA}$$

where TVA stands for the *total valuation adjustment*. This requires two comments. First, the TVA term accounts for several adjustments, and not only the counterparty credit risk, typically represented by the CVA and DVA. In particular, it may account for the funding valuation adjustment (FVA). In our approach, the funding adjustment results from the funding costs attributed to

hedging the two components of $(A^\#, \mathcal{X})$. Second, in our formula we have the sum, rather than the difference, in the right-hand side of (2.27). This discrepancy is simply due to our definition of the adjustments CL, CG and CR, which are set to be negatives of their counterparts encountered in other papers.

2.9 Regulatory Capital

The case of the *regulatory capital* can be formally covered by adding the process $X^k = -K$, where K is a non-negative stochastic process and by setting $\alpha_t^k = 1$ for all $t \in [0, T]$. If the regulatory capital is remunerated, then we also need to specify the corresponding remuneration process β^k . Of course, an important issue of explicit specifications of these processes will arise when the general theory is implemented. A practical approach to the *capital valuation adjustment* associated with the regulatory capital was developed in a recent work by Albanese et al. [ACC16].

3 Arbitrage-Free Trading Models

The analysis of the self-financing property of a trading strategy should be complemented by the study of some kind of a no-arbitrage property for the adopted market model. Due to the nonlinearity of a market model with differential funding rates, the concept of no-arbitrage is nontrivial, even when no trading adjustments are present. We will argue that it can be effectively dealt with using a reasonably general definition of an *arbitrage opportunity* associated with trading. Let us stress that we only consider here a nonlinear extension of the classical concept of an arbitrage opportunity (as opposed to other related concepts, such as: NFLVR, NUPBR, etc.).

3.1 No-arbitrage Pricing Principles

Let us first describe very briefly the commonly adopted pricing paradigm for financial derivatives. In essence, a general approach to no-arbitrage pricing hinges on the following steps:

Step (L.1). One first checks whether a market model with predetermined trading rules and primary traded assets is arbitrage-free, where the definition of an arbitrage opportunity is a mathematical formalization of the real-world concept of a risk-free profitable trading opportunity.

Step (L.2). Given a financial derivative for which the price is yet unspecified, one proposes a price (not necessarily unique) and checks whether the extended model (that is, the model where the financial derivative is postulated to be an additional traded asset) is also arbitrage-free in the sense made precise in Step (L.1).

The valuation procedure outlined above can be referred to as the *no-arbitrage pricing paradigm*. In any linear market model (see the comments after Definition 2.3), the unique price given by replication (or the range of no-arbitrage prices obtained using the concept of superhedging strategies in the case of an incomplete market) is consistent with the no-arbitrage pricing paradigm (L.1)–(L.2), although to establish this property in a continuous-time framework, one needs also to introduce the concept of *admissibility* of a trading strategy. This is feasible since the strict comparison property of linear BSDEs can be employed to show that replication (or superhedging) will indeed yield prices for derivatives that are consistent with the no-arbitrage pricing paradigm. Alternatively, the fundamental theorem of asset pricing (FTAP) can be used to show that the discounted prices defined through admissible trading strategies are local martingales (in fact, supermartingales) under an equivalent local martingale measure. The latter property is a well known fundamental feature of stochastic integration, so it covers all linear market models.

Let us now comment on the existing approaches to nonlinear pricing of derivatives. The most common approach to the pricing problem in a nonlinear framework seems to hinge, at least implicitly, on the following steps in which it is usually assumed that the hedger's initial endowment is null. In fact, Step (N.1) was explicitly addressed only in some works, whereas most papers in the existing literature were concerned with finding a replicating strategy mentioned in Step (N.2). Also, to the best of our knowledge, the important issue outlined in Step (N.3) was up to now completely ignored.

Step (N.1). The strict comparison argument for the BSDE associated with the wealth dynamics is used to show that it is not possible to construct an admissible trading strategy with the null initial wealth and the terminal wealth, which is non-negative almost surely and strictly positive with a positive probability.

Step (N.2). The price for a European contingent claim is defined using either the cost of replication or the minimal cost of superhedging. A suitable version of the strict comparison property for the wealth dynamics can be used to show that in some market models (referred to as regular models in this work) the two pricing approaches yield the same value for any European claim that can be replicated.

Step (N.3). It remains to check if the prices given by the cost of replication (or selected to be below the upper bound given by the minimal cost of superhedging) comply with some form of the no-arbitrage principle.

The problem whether the extended nonlinear market model is still arbitrage-free in some sense is much harder to tackle than it was the case for the linear framework, since trading in derivatives may essentially change the properties of the market. However, in the case of a *regular* model, this step is relatively easy to handle due to the postulated regularity conditions (see, in particular, Definition 4.6 and Proposition 4.8).

3.2 Discounted Wealth and Admissible Strategies

To deal with the issue of no-arbitrage, we need to introduce the discounted wealth process and properly define the concept of admissibility of trading strategies. Let us denote

$$\mathcal{B}_t(x) := \mathbf{1}_{\{x \geq 0\}} B_t^{0,l} + \mathbf{1}_{\{x < 0\}} B_t^{0,b}. \quad (3.1)$$

Note that if $B^{0,l} = B^{0,b}$, then $\mathcal{B}(x) = B^0 = B$. Furthermore, if $x = 0$, then $x B_T^{0,b} = x B_T^{0,l} = 0$ and thus the choice of either $B^{0,l}$ or $B^{0,b}$ in the right-hand side of (3.1) is immaterial. It is natural to postulate that the initial endowment $x \geq 0$ (resp. $x < 0$) has the future value $x B_t^{0,l}$ (resp. $x B_t^{0,b}$) at time $t \in [0, T]$ when invested in the cash account $B^{0,l}$ (resp. $B^{0,b}$). We thus henceforth work under the following assumption.

Assumption 3.1. We postulate that:

- (i) for any initial endowment $x \in \mathbb{R}$ of the hedger, the *null contract* $\mathcal{N} = (0, 0)$ belongs to \mathcal{C} ,
- (ii) for any $x \in \mathbb{R}$, the trading strategy $(x, 0, \widehat{\varphi}, \mathcal{N})$ where $\widehat{\varphi}$ has all components null except for either $\psi^{0,l}$ (if $x \geq 0$) or $\psi^{0,b}$ (if $x < 0$) belongs to $\Phi^{0,x}(\mathcal{C})$ and $V_t^p(x, 0, \widehat{\varphi}, \mathcal{N}) = V_t(x, 0, \widehat{\varphi}, \mathcal{N}) = x \mathcal{B}_t(x)$ for all $t \in [0, T]$.

Assumption 3.1 may look redundant at the first glance, but it is nevertheless needed and useful in derivation of basic properties of fair prices. Condition (i) is a rather obvious requirement. Note that condition (ii) cannot be deduced directly from the self-financing condition, since it hinges on the additional postulate that there are no trading adjustments (such as: taxes, transactions costs,

margin account, etc.) when the initial endowment is invested in the cash account. Formally, it states that the null contract $\mathcal{N} = (0, 0)$ can be entered into by the hedger and it will be used to show that the null contract has zero fair price at any date $t \in [0, T]$. Also, the trading strategy introduced in condition (ii) will serve as a natural benchmark for assessment of profits or losses incurred by the hedger.

We also follow the standard approach of introducing the concept of admissibility for the discounted wealth. Towards this end, for any fixed $t \in [0, T]$ we consider a hedger who starts trading at time t with the initial endowment x_t and who uses a self-financing trading strategy $(x_t, p_t, \varphi^t, \mathcal{C}^t)$, where the price $p_t \in \mathcal{G}_t$ at which the contract \mathcal{C}^t is traded at time t is arbitrary. We also consider the discounting process $\mathcal{B}^t(x_t)$ on $[t, T]$, which is given by

$$\mathcal{B}_u^t(x_t) := \mathbf{1}_{\{x_t \geq 0\}} B_u^{0,l} (B_t^{0,l})^{-1} + \mathbf{1}_{\{x_t < 0\}} B_u^{0,b} (B_t^{0,b})^{-1}, \quad (3.2)$$

so that, in particular, $\mathcal{B}_t^t(x_t) = x_t$. Then the wealth process discounted back to time t satisfies

$$\tilde{V}_u(x_t, p_t, \varphi^t, \mathcal{C}^t) := (\mathcal{B}_u^t(x_t))^{-1} V_u(x_t, p_t, \varphi^t, \mathcal{C}^t), \quad u \in [t, T], \quad (3.3)$$

and we have the following natural concept of admissibility of a trading strategy on $[t, T]$.

Definition 3.2. Let $t \in [0, T]$. We say that a trading strategy $(x_t, p_t, \varphi^t, \mathcal{C}^t) \in \Phi^{t,x_t}(\mathcal{C})$ is *admissible* if the discounted wealth $\tilde{V}_u(x_t, p_t, \varphi^t, \mathcal{C}^t)$ is bounded from below by a constant. We denote by $\Psi^{t,x_t}(p_t, \mathcal{C}^t)$ the class of admissible strategies corresponding to $(x_t, p_t, \varphi^t, \mathcal{C}^t)$, and we let

$$\Psi^{t,x_t}(\mathcal{C}) := \cup_{\mathcal{C} \in \mathcal{C}} \cup_{p_t \in \mathcal{G}_t} \Psi^{t,x_t}(p_t, \mathcal{C}^t)$$

to denote the class of all admissible trading strategies on $[t, T]$ relative to the class \mathcal{C} of contracts for the hedger with the initial endowment x_t at time t . In particular, $\Psi^{0,x}(\mathcal{C})$ represents the class of all trading strategies admissible at time $t = 0$ for the hedger with the initial endowment x .

3.3 No-arbitrage with Respect to the Null Contract

A minimal no-arbitrage requirement for an underlying market model is that it should be arbitrage-free with respect to the null contract. Note that, consistently with Assumption 3.1 and the concept of replication (for the general formulation of replication of a non-null contract, see Definition 4.4), it is implicitly assumed in Definition 3.3 that the price at which the null contract is traded at time zero equals zero. Needless to say, this is a rather obvious postulate in any trading model.

Definition 3.3. Consider an underlying market model $\mathcal{M} = (\mathcal{S}, \mathcal{D}, \mathcal{B}, \mathcal{C}, \Psi^{0,x}(\mathcal{C}))$. An *arbitrage opportunity with respect to the null contract* (or a *primary arbitrage opportunity*) for the hedger with an initial endowment x is a strategy $(x, 0, \varphi, \mathcal{N}) \in \Psi^{0,x}(0, \mathcal{N})$ such that

$$\mathbb{P}(\tilde{V}_T(x, 0, \varphi, \mathcal{N}) \geq x) = 1, \quad \mathbb{P}(\tilde{V}_T(x, 0, \varphi, \mathcal{N}) > x) > 0. \quad (3.4)$$

If no primary arbitrage opportunity exists in the market model \mathcal{M} then we say that \mathcal{M} is *arbitrage-free* with respect to the null contract for the hedger with an initial endowment x .

For any linear market model, Definition 3.3 reduces to the classical definition of an arbitrage opportunity. It is well known that the no-arbitrage property introduced in this definition is a sufficiently strong tool for the development of no-arbitrage pricing for financial derivatives in the linear framework. This does not mean, however, that Definition 3.3 is sufficiently strong to allow us to develop nonlinear no-arbitrage pricing theory, which would enjoy the properties, which are desirable from either mathematical or financial perspective. On the one hand, a natural definition

of a *hedger's fair value* (see Definition 4.1) is consistent with the concept of no-arbitrage with respect to the null contract and thus it seems to be theoretically sound. On the other hand, as we argue below, Definition 3.3 is manifestly not sufficient to ensure an efficient pricing and hedging approach in a general nonlinear market.

First, it may occur that the replication cost of a contract does not satisfy the definition of a fair price, since selling of a contract at its replication cost may generate an arbitrage opportunity for the hedger. Explicit examples of market models, which are arbitrage-free in the sense of Definition 3.3, but suffer from this deficiency, are given in Section 7.1.

Second, and more importantly, not well established method of finding a fair price in a general nonlinear market satisfying Definition 3.3 is available. We contend that the drawback of the definition of an arbitrage-free model with respect to the null contract is that it does not make an explicit reference to a class \mathcal{C} of contracts under study. Indeed, it hinges on the specification of the class $\Psi^{0,x}(0, \mathcal{N})$ of trading strategies, but it does not take into account the larger class $\Psi^{0,x}(\mathcal{C})$. To amend this drawback of Definition 3.3, it was proposed in [BR15] to consider the concept of the no-arbitrage for the trading desk with respect to a predetermined family \mathcal{C} of contracts.

3.4 No-arbitrage for the Trading Desk

Following [BR15], we will now examine a stronger no-arbitrage property of a model, which is directly related to a predetermined family \mathcal{C} of contracts. The goal to impose a more stringent no-arbitrage condition, which not only accounts for the nonlinearity of the market, but also explicitly refers to a family of contracts under consideration. Regrettably, the class of models that are arbitrage-free in the sense of Definition 3.7 could still be too encompassing, and thus it is unclear whether the pricing irregularities mentioned in the preceding section will be eliminated (for an example, see Section 7.2).

For simplicity of notation, we consider here the case of $t = 0$, but all definitions can be extended to the case of any date t . The symbols $\mathcal{X} = \mathcal{X}(A)$ and $\mathcal{Y} = \mathcal{Y}(-A)$ are used to emphasize that there is no reason to expect that the trading adjustments will satisfy the equality $\mathcal{X}(-A) = -\mathcal{X}(A)$, in general. Therefore, we denote by $\mathcal{Y} = (Y^1, \dots, Y^n; \alpha^1(\mathcal{Y}), \dots, \alpha^n(\mathcal{Y}); \beta^1(\mathcal{Y}), \dots, \beta^n(\mathcal{Y}))$ the trading adjustments associated with the cumulative cash flows process $-A$. In order to avoid confusion, we will use the full notation for the wealth process, for instance, $V(x, p, \varphi, \mathcal{C}) = V(x, p, \varphi, A, \mathcal{X})$, etc.

Definition 3.4. For a contract $\mathcal{C} = (A, \mathcal{X})$ and an initial endowment x , the *combined wealth* is defined as

$$V^{\text{com}}(x_1, x_2, \varphi, \bar{\varphi}, A, \mathcal{X}, \mathcal{Y}) := V(x_1, 0, \varphi, A, \mathcal{X}) + V(x_2, 0, \bar{\varphi}, -A, \mathcal{Y}), \quad (3.5)$$

where x_1, x_2 are any real numbers such that $x = x_1 + x_2$, $\varphi \in \Psi^{0,x_1}(0, A, \mathcal{X})$, $\bar{\varphi} \in \Psi^{0,x_2}(0, -A, \mathcal{Y})$. In particular, $V_0^{\text{com}}(x_1, x_2, \varphi, \bar{\varphi}, A, \mathcal{X}, \mathcal{Y}) = x_1 + x_2 = x$.

The rationale for the term *combined wealth* comes from the financial interpretation of the process given by the right-hand side in (3.5). We argue that it represents the aggregated wealth of two traders, which are members of the same trading desk, who are supposed to act as follows:

- The first trader takes the long position in a contract (A, \mathcal{X}) , whereas the second one takes the short position in the same contract, which is thus formally represented by $(-A, \mathcal{Y})$. Since we assume that the long and short positions have exactly opposite prices, the corresponding cash flows p and $-p$ coming to the trading desk (and not to individual traders) offset each other and thus the initial endowment x of the trading desk remains unchanged.

- In addition, it is assumed that after the cash flows p and $-p$ have been netted so they are no longer present, the initial endowment x is split into arbitrary amounts x_1 and x_2 meaning that $x = x_1 + x_2$. Then each trader is allocated the respective amount x_1 and x_2 as an initial endowment and each of them hedges his position. It is now clear that the level of the initial price p at which the contract is traded at time zero is immaterial for both hedging strategies and the combined wealth of the two traders is given by the right-hand side in (3.5).

Alternatively, the combined wealth may be used to describe the situation where a single trader takes long and short positions with two external counterparties and hedges them independently using his initial endowment x split into x_1 and x_2 . Of course, in that case it is even more clear that the initial price p has no impact on his trading strategies.

Remark 3.5. One can also observe that the following equality is manifestly satisfied for any real number p

$$V(x_1, 0, \varphi, A, \mathcal{X}) + V(x_2, 0, \bar{\varphi}, -A, \mathcal{Y}) = V(\tilde{x}_1, p, \varphi, A, \mathcal{X}) + V(\tilde{x}_2, -p, \bar{\varphi}, -A, \mathcal{Y}),$$

where $\tilde{x}_1 = x_1 - p$ and $\tilde{x}_2 = x_2 + p$ is another decomposition of x such that $x = \tilde{x}_1 + \tilde{x}_2$. However, equation (3.5) much better reflects the actual trading arrangements and it has a clear advantage that an unknown number p does not appear in the formula for the combined wealth. It thus emphasizes the crucial feature that the combined wealth is independent of p . In fact, one can remark that the issue whether the trading desk has been informed about the actual level of the price p does not matter at all.

Definition 3.6. A pair $(x_1, \varphi; x_2, \bar{\varphi})$ of trading strategies introduced in Definition 3.4 is *admissible for the trading desk* if the discounted combined wealth process

$$\tilde{V}^{\text{com}}(x_1, x_2, \varphi, \bar{\varphi}, A, \mathcal{X}, \mathcal{Y}) := (\mathcal{B}(x))^{-1} V^{\text{com}}(x_1, x_2, \varphi, \bar{\varphi}, A, \mathcal{X}, \mathcal{Y}) \tag{3.6}$$

is bounded from below by a constant. The class of such strategies is denoted by $\Psi^{0, x_1, x_2}(A, \mathcal{X}, \mathcal{Y})$.

We are in a position to formalize the concept of an arbitrage-free model for the trading desk with respect to a particular family of contracts.

Definition 3.7. We say that a pair $(x_1, \varphi; x_2, \bar{\varphi}) \in \Psi^{0, x_1, x_2}(A, \mathcal{X}, \mathcal{Y})$ is an *arbitrage opportunity for the trading desk* with respect to a contract (A, \mathcal{X}) if the following conditions are satisfied

$$\mathbb{P}(\tilde{V}_T^{\text{com}}(x_1, x_2, \varphi, \bar{\varphi}, A, \mathcal{X}, \mathcal{Y}) \geq x) = 1, \quad \mathbb{P}(\tilde{V}_T^{\text{com}}(x_1, x_2, \varphi, \bar{\varphi}, A, \mathcal{X}, \mathcal{Y}) > x) > 0.$$

We say that the market model $\mathcal{M} = (\mathcal{S}, \mathcal{D}, \mathcal{B}, \mathcal{C}, \Psi^{0, x}(\mathcal{C}))$ is *arbitrage-free for the trading desk* if no arbitrage opportunity exists for the trading desk with respect to any contract \mathcal{C} from \mathcal{C} .

Our main purpose in Sections 3.3 and 3.4 was to provide some simple criteria that would allow one to eliminate models in which some particular form of arbitrage appears. Definition 3.3 and Definition 3.7 provide such criteria for accepting or rejecting any tentative nonlinear market model. It is easy to see that a model which is rejected according to Definition 3.7 is also rejected when Definition 3.3 is applied. We do not claim, however, that these tests are sufficient to discriminate between acceptable and non-acceptable nonlinear models for pricing of derivatives. Indeed, in Definition 4.6 we formulate additional conditions that should be satisfied by a viable market model.

3.5 Dynamics of the Discounted Wealth Process

It is natural to ask whether the no-arbitrage for the trading desk can be checked for a given market model. Before we illustrate a simple verification method for this property, we need to introduce additional notation. Let us write

$$\begin{aligned}\tilde{B}^{i,l}(x) &:= (\mathcal{B}(x))^{-1} B^{i,l}, & \tilde{B}^{i,b}(x) &:= (\mathcal{B}(x))^{-1} B^{i,b}, \\ \tilde{\beta}^k(x, \mathcal{X}) &:= (\mathcal{B}(x))^{-1} \beta^k(\mathcal{X}), & \tilde{\beta}^k(x, \mathcal{Y}) &:= (\mathcal{B}(x))^{-1} \beta^k(\mathcal{Y}), \\ \hat{X}^k &:= (\beta^k(\mathcal{X}))^{-1} X^k, & \hat{Y}^k &:= (\beta^k(\mathcal{Y}))^{-1} Y^k, \\ B^{0,b,l} &:= (B^{0,l})^{-1} B^{0,b}, & B^{0,l,b} &:= (B^{0,b})^{-1} B^{0,l}.\end{aligned}$$

Lemma 3.8. *The discounted combined wealth satisfies*

$$\begin{aligned}d\tilde{V}_t^{com}(x_1, x_2, \varphi, \bar{\varphi}, A, \mathcal{X}, \mathcal{Y}) &= \sum_{i=1}^d (\xi_t^i + \bar{\xi}_t^i) d\tilde{S}_t^{i,cld}(x) + \sum_{i=1}^d (\psi_t^{i,l} + \bar{\psi}_t^{i,l}) d\tilde{B}_t^{i,l}(x) \\ &+ \sum_{i=1}^d (\psi_t^{i,b} + \bar{\psi}_t^{i,b}) d\tilde{B}_t^{i,b}(x) + \mathbb{1}_{\{x \geq 0\}} (\psi_t^{0,b} + \bar{\psi}_t^{0,b}) dB_t^{0,b,l} + \mathbb{1}_{\{x < 0\}} (\psi_t^{0,l} + \bar{\psi}_t^{0,l}) dB_t^{0,l,b} \\ &- \sum_{k=1}^n \hat{X}_t^k d\tilde{\beta}_t^k(x, \mathcal{X}) - \sum_{k=1}^n \hat{Y}_t^k d\tilde{\beta}_t^k(x, \mathcal{Y}) + \sum_{k=1}^n ((1 - \alpha_t^k(\mathcal{X}))X_t^k + (1 - \alpha_t^k(\mathcal{Y}))Y_t^k) d(\mathcal{B}_t(x))^{-1},\end{aligned}\quad (3.7)$$

where we set

$$\tilde{S}_t^{i,cld}(x) := (\mathcal{B}_t(x))^{-1} S_t^i + \int_0^t (\mathcal{B}_u(x))^{-1} dD_u^i. \quad (3.8)$$

Proof. For an arbitrary decomposition $x = x_1 + x_2$, we write (note that the notation introduced in Definition 3.2 is extended here, since $x \neq x_i$, in general)

$$\tilde{V}(x_1, p, \varphi, A, \mathcal{X}) := (\mathcal{B}(x))^{-1} \tilde{V}(x_1, p, \varphi, A, \mathcal{X}), \quad \tilde{V}(x_2, p, \bar{\varphi}, -A, \mathcal{Y}) := (\mathcal{B}(x))^{-1} \tilde{V}(x_2, p, \bar{\varphi}, -A, \mathcal{Y}).$$

From (2.10) and (2.11), using the Itô integration by parts formula, we obtain

$$\begin{aligned}d\tilde{V}_t(x_1, p, \varphi, A, \mathcal{X}) &= \sum_{i=1}^d \xi_t^i d\tilde{S}_t^{i,cld}(x) + \sum_{i=1}^d \left(\psi_t^{i,l} d\tilde{B}_t^{i,l}(x) + \psi_t^{i,b} d\tilde{B}_t^{i,b}(x) \right) \\ &+ \mathbb{1}_{\{x \geq 0\}} \psi_t^{0,b} dB_t^{0,b,l} + \mathbb{1}_{\{x < 0\}} \psi_t^{0,l} dB_t^{0,l,b} + (\mathcal{B}_t(x))^{-1} dA_t - \sum_{k=1}^n \hat{X}_t^k d\tilde{\beta}_t^k(x, \mathcal{X}) \\ &+ \sum_{k=1}^n (1 - \alpha_t^k(\mathcal{X})) X_t^k d(\mathcal{B}_t(x))^{-1},\end{aligned}\quad (3.9)$$

and an analogous equality holds for $\tilde{V}(x_2, p, \bar{\varphi}, -A, \mathcal{Y})$. Hence (3.7) follows from (3.5) and (3.6). \square

It is worth noting that it follows from (3.9) that condition (ii) in Assumption 3.1 is satisfied provided that no additional constraints on trading strategies are imposed (recall that condition (i) in Assumption 3.1 is always postulated to hold).

Assume now, in addition, that $B^{i,l} = B^{i,b} = B^i$ for $i = 1, 2, \dots, d$. We define the processes $S^{i,cld}$ and $\hat{S}^{i,cld}$

$$S_t^{i,cld} := S_t^i + B_t^i \int_0^t (B_u^i)^{-1} dD_u^i, \quad \hat{S}_t^{i,cld} := (B_t^i)^{-1} S_t^{i,cld} = \hat{S}_t^i + \int_0^t (B_u^i)^{-1} dD_u^i,$$

where in turn $\widehat{S}^i := (B^i)^{-1}S^i$. It is easy to check that

$$d\widetilde{S}_t^{i,cld}(x) = \widetilde{B}_t^i(x) d\widehat{S}_t^{i,cld} + \widehat{S}_t^i d\widetilde{B}_t^i(x), \quad (3.10)$$

where $\widetilde{B}^i(x) := (\mathcal{B}(x))^{-1}B^i$ and thus (3.7) becomes

$$\begin{aligned} d\widetilde{V}_t^{\text{com}}(x_1, x_2, \varphi, \bar{\varphi}, A, \mathcal{X}, \mathcal{Y}) &= \sum_{i=1}^d (\xi_t^i + \bar{\xi}_t^i) \widetilde{B}_t^i(x) d\widehat{S}_t^{i,cld} + \sum_{i=1}^d ((\xi_t^i + \bar{\xi}_t^i) \widehat{S}_t^i + (\psi_t^i + \bar{\psi}_t^i)) d\widetilde{B}_t^i(x) \\ &+ \mathbb{1}_{\{x \geq 0\}} (\psi_t^{0,b} + \bar{\psi}_t^{0,b}) dB_t^{0,b,l} + \mathbb{1}_{\{x < 0\}} (\psi_t^{0,l} + \bar{\psi}_t^{0,l}) dB_t^{0,l,b} - \sum_{k=1}^n \widehat{X}_t^k d\widetilde{\beta}_t^k(x, \mathcal{X}) \\ &- \sum_{k=1}^n \widehat{Y}_t^k d\widetilde{\beta}_t^k(x, \mathcal{Y}) + \sum_{k=1}^n ((1 - \alpha_t^k(\mathcal{X}))X_t^k + (1 - \alpha_t^k(\mathcal{Y}))Y_t^k) d(\mathcal{B}_t(x))^{-1}. \end{aligned} \quad (3.11)$$

3.6 Sufficient Conditions for the Trading Desk No-Arbitrage

The following result gives a sufficient condition for a market model to be arbitrage-free for the trading desk. The proof of Proposition 3.9 is straightforward and thus it is omitted.

Proposition 3.9. *Assume that there exists a probability measure \mathbb{Q} , equivalent to \mathbb{P} on (Ω, \mathcal{G}_T) , and such that for any decomposition $x = x_1 + x_2$ and any admissible combination of trading strategies $(x_1, \varphi, A, \mathcal{X})$ and $(x_2, \bar{\varphi}, -A, \mathcal{Y})$ for any contract (A, \mathcal{X}) belonging to \mathcal{C} the discounted combined wealth $\widetilde{V}^{\text{com}}(x_1, x_2, \varphi, \bar{\varphi}, A, \mathcal{X}, \mathcal{Y})$ is a supermartingale under \mathbb{Q} . Then the market model $\mathcal{M} = (\mathcal{S}, \mathcal{D}, \mathcal{B}, \mathcal{C}, \Psi^{0,x}(\mathcal{C}))$ is arbitrage-free for the trading desk.*

Although Proposition 3.9 is fairly abstract, the sufficient condition stated there can readily be verified as soon as a specific market model is adopted (see, for instance, [BR15, NR15, NR16a, NR16c]). To support this claim, we will examine a market model with idiosyncratic funding of risky assets and rehypothecated cash collateral.

Example 3.10. We consider the special case where $B^{0,l} = B^{0,b} = B = \mathcal{B}(x)$ and $B^{i,l} = B^{i,b} = B^i$ for all $i = 1, 2, \dots, d$. Under the assumption of no additional constraints on trading strategies, (3.11) yields (for a special case of this formula, see Corollary 2.1 in [BR15])

$$\begin{aligned} d\widetilde{V}_t^{\text{com}}(x_1, x_2, \varphi, \bar{\varphi}, A, \mathcal{X}, \mathcal{Y}) &= \sum_{i=1}^d (\xi_t^i + \bar{\xi}_t^i) \widetilde{B}_t^i(x) d\widehat{S}_t^{i,cld} + \sum_{i=1}^d (\xi_t^i S_t^i + \psi_t^i B_t^i) (B_t^i)^{-1} d\widetilde{B}_t^i(x) \\ &+ \sum_{i=1}^d (\bar{\xi}_t^i S_t^i + \bar{\psi}_t^i B_t^i) (B_t^i)^{-1} d\widetilde{B}_t^i(x) - \sum_{k=1}^n \widehat{X}_t^k d\widetilde{\beta}_t^k(x, \mathcal{X}) - \sum_{k=1}^n \widehat{Y}_t^k d\widetilde{\beta}_t^k(x, \mathcal{Y}) \\ &+ \sum_{k=1}^n ((1 - \alpha_t^k(\mathcal{X}))X_t^k + (1 - \alpha_t^k(\mathcal{Y}))Y_t^k) dB_t^{-1}. \end{aligned}$$

We postulate that the cash collateral is rehypothecated, so that $n = 2$ in Section 2.4. Then $\alpha_t^1 = \alpha_t^2 = \alpha_t^1(\mathcal{Y}) = \alpha_t^2(\mathcal{Y}) = 1$ and $X_t^1 + Y_t^1 = X_t^2 + Y_t^2 = 0$ for all $t \in [0, T]$. Let us assume, in addition, that $\xi_t^i S_t^i + \psi_t^i B_t^i = \bar{\xi}_t^i S_t^i + \bar{\psi}_t^i B_t^i = 0$ for all i and $t \in [0, T]$, meaning that the i th risky asset is fully funded from the repo account B^i (see Section 2.6.3). More generally, it suffices to assume that the following equalities are satisfied for all $t \in [0, T]$

$$\sum_{i=1}^d \int_0^t (\xi_u^i S_u^i + \psi_u^i B_u^i) (B_u^i)^{-1} d\widetilde{B}_u^i(x) = \sum_{i=1}^d \int_0^t (\bar{\xi}_u^i S_u^i + \bar{\psi}_u^i B_u^i) (B_u^i)^{-1} d\widetilde{B}_u^i(x) = 0. \quad (3.12)$$

Finally, let the remuneration processes satisfy $\beta^k(\mathcal{X}) = \beta^k(\mathcal{Y})$. Then the formula for the dynamics of the discounted combined wealth for the trading desk reduces to

$$d\widetilde{V}_t^{\text{com}}(x_1, x_2, \varphi, \bar{\varphi}, A, \mathcal{X}, \mathcal{Y}) = \sum_{i=1}^d (\xi_t^i + \bar{\xi}_t^i) \widetilde{B}_t^i(x) d\widehat{S}_t^{i,\text{cld}}$$

and thus the model is arbitrage-free for the trading desk provided that there exists a probability measure \mathbb{Q} , which is equivalent to \mathbb{P} on (Ω, \mathcal{G}_T) , and such that the processes $\widehat{S}^{i,\text{cld}}$, $i = 1, 2, \dots, d$ are \mathbb{Q} -local martingales. This property is still a sufficient condition for the trading desk no-arbitrage when the cash account $B^{0,l}$ and $B^{0,b}$ differ, but the borrowing rate dominates the lending rate.

4 Hedger's Fair Pricing

We now address the issue of a fair pricing in our nonlinear model under the assumption that the hedger has the initial endowment x_t at time t . We assume that the model is arbitrage-free either with respect to the null contract or for the trading desk and we consider the hedger who contemplates entering into the contract \mathcal{C}^t at time t . The first goal is to describe the range of the *hedger's fair prices* for the contract \mathcal{C}^t . Let $p_t \in \mathcal{G}_t$ denote a generic price of a contract at time t , as seen from the perspective of the hedger. Hence a positive value of p_t means that the hedger receives at time t the cash amount p_t from the counterparty, whereas a negative value of p_t means that he agrees to pay the cash amount $-p_t$ to the counterparty at time t . In the next definition, we fix a date $t \in [0, T)$ and we assume that the contract \mathcal{C}^t is traded at the price p_t at time t . It is natural to ask whether in this situation the hedger can make a risk-free profit by entering into the contract and hedging it with an admissible trading strategy over $[t, T]$. We propose to call it a *hedger's pricing* arbitrage opportunity. Recall that the arbitrage opportunities defined in Section 3 are related to the properties of a trading model, and they do not depend on the level of a price p_t for \mathcal{C}^t .

Definition 4.1. A trading strategy $(x_t, p_t, \varphi^t, \mathcal{C}^t) \in \Psi^{t, x_t}(\mathcal{C})$ is a *hedger's pricing arbitrage opportunity on $[t, T]$* associated with a contract \mathcal{C}^t traded at p_t at time t (or, briefly, a *secondary arbitrage opportunity*) if

$$\mathbb{P}(\widetilde{V}_T(x_t, p_t, \varphi^t, \mathcal{C}^t) \geq x_t) = 1 \quad (4.1)$$

and

$$\mathbb{P}(\widetilde{V}_T(x_t, p_t, \varphi^t, \mathcal{C}^t) > x_t) > 0. \quad (4.2)$$

It is clear that $(x_t, p_t, \varphi^t, \mathcal{C}^t) \in \Psi^{t, x_t}(\mathcal{C})$ is not a hedger's pricing arbitrage opportunity on $[t, T]$ if either

$$\mathbb{P}(\widetilde{V}_T(x_t, p_t, \varphi^t, \mathcal{C}^t) = x_t) = 1 \quad (4.3)$$

or

$$\mathbb{P}(\widetilde{V}_T(x_t, p_t, \varphi^t, \mathcal{C}^t) < x_t) > 0. \quad (4.4)$$

We will refer to condition (4.4) as the *strict subhedging condition*.

Definition 4.2. We say that $p_t^f = p_t^f(x_t, \mathcal{C}^t)$ is a *fair hedger's price* at time t for \mathcal{C}^t if there is no hedger's secondary arbitrage opportunity $(x_t, p_t^f, \varphi^t, \mathcal{C}^t) \in \Psi^{t, x_t}(\mathcal{C})$. A fair hedger's price p_t^f such that the strict subhedging condition holds for every trading strategy $(x_t, p_t^f, \varphi^t, \mathcal{C}^t) \in \Psi^{t, x_t}(\mathcal{C})$ is called a *strict subhedging price*.

It is clear from Definitions 4.2 and 4.1 that if p_t^f is a fair price, then any trading strategy $(x_t, p_t, \varphi^t, \mathcal{C}^t) \in \Psi^{t, x_t}(\mathcal{C})$ necessarily satisfies either condition (4.3) or condition (4.4). Obviously, a fair hedger's price depends both on the given endowment x_t and the contract \mathcal{C}^t and thus the notation $p_t^f(x_t, \mathcal{C}^t)$ is appropriate, but it will be frequently simplified to p_t^f when no danger of confusion may arise. It is important to note that if p_t^f is a fair hedger's price (resp. a strict subhedging price) at time t for \mathcal{C}^t , then any $p_t \leq p_t^f$ is also a fair hedger's price (resp. a strict subhedging price) at time t for that contract, provided that the amount $p_t^f - p_t \geq 0$ can be invested in $B^{0,t}$ with no trading adjustments. Recall that the latter property has been already postulated in our framework and thus it is not restrictive.

A fair price prevents the hedger from making sure profit with a positive probability and with no risk of losing money. In contrast, it does not prevent the hedger from losing money, in general. It is thus natural to search for the highest possible value of a fair price. This idea motivates the following definition of the upper bound for the hedger's fair prices

$$\underline{p}_t^f(x_t, \mathcal{C}^t) := \text{ess sup} \{p_t^f \in \mathcal{G}_t \mid p_t^f \text{ is a fair hedger's price for } \mathcal{C}^t\}. \quad (4.5)$$

We find it useful to study also the upper bound for the strict subhedging prices, which is given by

$$\underline{p}_t(x_t, \mathcal{C}^t) := \text{ess sup} \{p_t^f \in \mathcal{G}_t \mid p_t^f \text{ is a strict subhedging price for } \mathcal{C}^t\}. \quad (4.6)$$

Definition 4.3. A trading strategy $(x_t, p_t^s, \varphi^t, \mathcal{C}^t) \in \Psi^{t, x_t}(\mathcal{C})$ is said to be a *superhedging strategy* for a contract \mathcal{C}^t if (4.1) holds, whereas a *strict superhedging* means that (4.1) and (4.2) are satisfied. In the former (resp. latter) case, $p_t^s = p_t^s(x_t, \mathcal{C}^t)$ is called a *superhedging cost* (resp. *strict superhedging cost*) at time t for \mathcal{C}^t .

If a superhedging (resp. strict superhedging) strategy exists when \mathcal{C}^t is entered into at the price p_t^s , then a superhedging (resp. strict superhedging) strategy exists as well for any p_t satisfying $p_t \geq p_t^s$. The lower bound for superhedging costs is given by

$$\bar{p}_t^s(x_t, \mathcal{C}^t) := \text{ess inf} \{p_t^s \in \mathcal{G}_t \mid p_t^s \text{ is a superhedging cost for } \mathcal{C}^t\} \quad (4.7)$$

and the lower bound for strict superhedging costs equals

$$\bar{p}_t(x_t, \mathcal{C}^t) := \text{ess inf} \{p_t^s \in \mathcal{G}_t \mid p_t^s \text{ is a strict superhedging cost for } \mathcal{C}^t\}. \quad (4.8)$$

For the basic relationships between the quantities introduced above when $t = 0$, see Lemma 4.5 (analogous relationships hold in fact for any $t > 0$ as well, but they are not reported here).

In any linear (either complete or incomplete) market model, the quantities \underline{p}_t and \bar{p}_t do not depend on the hedger's endowment. Moreover, the equality $\underline{p}_t = \bar{p}_t$ can be established in some arbitrage-free incomplete models for all square-integrable European claims (see El Karoui and Quenez [EKQ95]). It was shown in [EKQ95] that the common value is not a fair hedger's price for a European claim, unless it can be replicated. Of course, in the latter case we have that $\underline{p}_t = \bar{p}_t = \hat{p}_t$ where \hat{p}_t denotes the unique cost of replication.

Obviously, this does not mean that such desirable properties of $\underline{p}_t(x_t, \mathcal{C}^t)$, $\bar{p}_t(x_t, \mathcal{C}^t)$ and the cost of replication (see Definition 4.4) will still be valid when the assumption of a model's linearity is relaxed, even when a nonlinear model is assumed to be arbitrage-free either with respect to the null contract or for the trading desk. Furthermore, Definition 4.2 is too broad, in the sense that it is not constructive and thus it does not offer any guidance how to implement fair pricing, let alone how to derive the dynamics of fair prices for a given contract. These important shortcomings are addressed and overcome in Section 4.2 where the notion of a *regular market* is introduced.

4.1 Replication on $[0, T]$ and the Gained Value

Admittedly, the most commonly used technique for valuation of derivatives hinges on the concept of replication. In the present framework, it is given by the following definition, in which we consider the hedger with the initial endowment x at time 0 and \hat{p}_0 stands for an arbitrary real number.

Definition 4.4. A trading strategy $(x, \hat{p}_0, \hat{\varphi}, \mathcal{C}) \in \Psi^{0,x}(\mathcal{C})$ is said to *replicate a contract* \mathcal{C} on $[0, T]$ whenever $V_T(x, \hat{p}_0, \hat{\varphi}, \mathcal{C}) = x\mathcal{B}_T(x)$ or, equivalently, $\tilde{V}_T(x, \hat{p}_0, \hat{\varphi}, \mathcal{C}) = x$. Then $\hat{p}_0 = \hat{p}_0(x, \mathcal{C})$ is said to be a *hedger's replication cost* for \mathcal{C} at time 0 and the process $\hat{p}(x, \mathcal{C})$ given by

$$\hat{p}_t(x, \mathcal{C}) := V_t(x, \hat{p}_0(x, \mathcal{C}), \hat{\varphi}, \mathcal{C}) - x\mathcal{B}_t(x) \quad (4.9)$$

is called a *hedger's gained value*.

Note that the uniqueness of a replication cost $\hat{p}_0(x, \mathcal{C})$ is not yet guaranteed and indeed there is no reason to expect that it will always hold in every market model satisfying either Definition 3.3 or Definition 3.7. Nevertheless, the financial interpretation of a hedger's replication cost $\hat{p}_0(x, \mathcal{C})$ for a given contract $\mathcal{C} \in \mathcal{C}$ is fairly straightforward. It represents either a positive or a negative adjustment to the initial endowment, which is required to implement a trading strategy ensuring that the hedger's wealth at time T , after the terminal payoff of the contract has been settled, perfectly matches the value at time T of his initial endowment invested in the cash account.

It is clear that the equality $\hat{p}_T(x, \mathcal{C}) = 0$ holds for any replicable contract $\mathcal{C} \in \mathcal{C}$. As expected, for the null contract $\mathcal{N} = (0, 0)$, the self-financing strategy $(x, 0, \hat{\varphi}, \mathcal{N})$, with the portfolio $\hat{\varphi}$ putting/taking all the money into/from the bank account, will be a replicating strategy such that $\hat{p}_t(x, \mathcal{N}) = 0$ for all $t \in [0, T]$. The fact that such a trading strategy is self-financing was postulated in Section 3.2 but, of course, this a priori assumption needs to be checked for any market model at hand.

As was already mentioned, it should not be taken for granted that in a nonlinear market model, which is assumed to be arbitrage-free with respect to the null contract (or even arbitrage-free for the trading desk), the cost of replication for a contract \mathcal{C} will satisfy the definition of a fair price for the hedger. In fact, it is possible to show by means of counter-examples that this highly desirable property may fail to hold (see the appendix). To amend this drawback, we introduce in the next section a class of models in which replication can be proven to be a reliable method of fair pricing for any contract from a predetermined family \mathcal{C} for which a replicating strategy exists.

4.2 Market Regularity on $[0, T]$

Once again, we consider the hedger with the initial endowment x at time 0. Intuitively, the concept of *regularity* with respect to a given family \mathcal{C} of contracts is motivated by our desire to ensure that, for any contract from \mathcal{C} , the cost of replication is never higher than the minimal cost of superhedging and, in addition, the cost of replication is a fair hedger's price, in the sense of Definition 4.1. The following lemma is an easy consequence of Definitions 4.2 and 4.3.

Lemma 4.5. *We have that*

$$\underline{p}_0(x, \mathcal{C}) \leq \underline{p}_0^f(x, \mathcal{C}) = \bar{p}_0^s(x, \mathcal{C}) \leq \bar{p}_0(x, \mathcal{C}).$$

Therefore, either (a) $\underline{p}_0(x, \mathcal{C}) = \bar{p}_0(x, \mathcal{C})$ or (b) $\underline{p}_0(x, \mathcal{C}) < \bar{p}_0(x, \mathcal{C})$.

Suppose first that case (a) occurs and a contract \mathcal{C} can be replicated. Then it may happen that $\underline{p}_0(x, \mathcal{C}) = \bar{p}_0(x, \mathcal{C}) = \hat{p}_0(x, \mathcal{C})$. It is not obvious, however, that $\hat{p}_0(x, \mathcal{C})$ would be in this situation a hedger's fair price, since it is still possible that a strict superhedging strategy with the same initial

cost might exist. Furthermore, it may also happen that $\bar{p}_0(x, \mathcal{C}) < \hat{p}_0(x, \mathcal{C})$ meaning that a strict superhedging is less expensive than replication.

Suppose now that case (a) occurs but a contract \mathcal{C} cannot be replicated. Then it is not clear whether $p_0 := \underline{p}_0^f(x, \mathcal{C}) = \bar{p}_0^s(x, \mathcal{C})$ is a fair price, although we have that

$$\underline{p}_0(x, \mathcal{C}) = \underline{p}_0^f(x, \mathcal{C}) = \bar{p}_0^s(x, \mathcal{C}) = \bar{p}_0(x, \mathcal{C})$$

so that p_0 is equal to the upper bound for strict subhedging prices and to the lower bound for strict superhedging costs.

Suppose now that case (b) occurs. Then, it can be checked that any number $p \in (\underline{p}_0(x, \mathcal{C}), \bar{p}_0(x, \mathcal{C}))$ is a replication cost. It is not clear, however, whether $\underline{p}_0(x, \mathcal{C})$ is a replication cost or a strict subhedging price. One can also ask whether there exists a replication cost equal to or higher than $\bar{p}_0(x, \mathcal{C})$.

To eliminate at least some of such ambiguities from our further study of nonlinear pricing techniques, we will later restrict our attention to nonlinear market models satisfying some additional regularity conditions.

Definition 4.6. We say that the market model $\mathcal{M} = (\mathcal{S}, \mathcal{D}, \mathcal{B}, \mathcal{C}, \Psi^{0,x}(\mathcal{C}))$ is *regular on* $[0, T]$ *with respect to* \mathcal{C} if for every replicable contract $\mathcal{C} \in \mathcal{C}$ and for every replicating strategy $(x, \hat{p}_0(x), \hat{\varphi}, \mathcal{X})$ the following properties hold:

- (i) if p is such that there exists $(x, p, \varphi, \mathcal{C}) \in \Psi^{0,x}(p, \mathcal{C})$ satisfying

$$\mathbb{P}(\tilde{V}_T(x, p, \varphi, \mathcal{C}) \geq x) = 1, \quad (4.10)$$

then $p \geq \hat{p}_0(x)$;

- (ii) if p is such that there exists $(x, p, \varphi, \mathcal{C}) \in \Psi^{0,x}(p, \mathcal{C})$ such that

$$\mathbb{P}(\tilde{V}_T(x, p, \varphi, \mathcal{C}) \geq x) = 1 \quad (4.11)$$

and

$$\mathbb{P}(\tilde{V}_T(x, p, \varphi, \mathcal{C}) > x) > 0, \quad (4.12)$$

then $p > \hat{p}_0(x)$.

By applying Definition 4.6 to the null contract $\mathcal{N} = (0, 0)$, we deduce that any regular market model is arbitrage-free for the hedger with respect to the null contract. It is not clear, however, whether an arbitrage opportunity for the trading desk may arise in a regular model.

It is important to note that condition (i) implies that the replication cost $\hat{p}_0(x)$ for \mathcal{C} is unique. Moreover, if condition (i) holds, then condition (ii) is equivalent to the following condition:

- (iii) if p is such that there exists $(x, p, \varphi, \mathcal{C}) \in \Psi^{0,x}(p, \mathcal{C})$ satisfying

$$\mathbb{P}(\tilde{V}_T(x, p, \varphi, \mathcal{C}) \geq x) = 1 \quad (4.13)$$

then the following implication is valid: if $p = \hat{p}_0(x)$, then

$$\mathbb{P}(\tilde{V}_T(x, p, \varphi, \mathcal{C}) = x) = 1. \quad (4.14)$$

Condition (i) in Definition 4.1 states that superhedging cannot be less expensive than replication, whereas condition (ii) states that strict superhedging is in fact always more costly than replication.

Remark 4.7. In the special case of European claims with maturity T and no trading adjustments, conditions (i) and (ii) correspond to the comparison and strict comparison properties of solutions to BSDEs satisfied by the wealth process with different terminal conditions. In fact, the same idea underpins Definition 2.7 of a nonlinear pricing system introduced by El Karoui and Quenez [EKQ97]. As in [EKQ97], we will show that regularity of a market model can be established for a large variety of models using a BSDE approach.

4.2.1 Replicable Contracts

We first focus on contracts that can be replicated. Proposition 4.8 shows that, in a regular market model, the cost of replication is the unique fair price of a contract $\mathcal{C} \in \mathcal{C}$ that can be replicated and the equalities $\widehat{p}_0(x, \mathcal{C}) = \bar{p}_0(x, \mathcal{C}) = \underline{p}_0(x, \mathcal{C})$ hold for such a contract. This means that the replication of a contract is a viable method of pricing within the framework of a regular model, although this statement is not necessarily true for any nonlinear market model.

Proposition 4.8. *Assume that a market model $\mathcal{M} = (\mathcal{S}, \mathcal{D}, \mathcal{B}, \mathcal{C}, \Psi^{0,x}(\mathcal{C}))$ is regular on $[0, T]$ with respect to \mathcal{C} . Then for every contract $\mathcal{C} \in \mathcal{C}$ that can be replicated on $[0, T]$ we have:*

- (i) *the replication cost $\widehat{p}_0(x, \mathcal{C})$ is unique,*
- (ii) *$\widehat{p}_0(x, \mathcal{C})$ is the lower bound for superhedging costs and strict superhedging costs, that is,*
 $\widehat{p}_0(x, \mathcal{C}) = \bar{p}_0^s(x, \mathcal{C}) = \bar{p}_0(x, \mathcal{C}),$
- (iii) *$\widehat{p}_0(x, \mathcal{C})$ is the maximal fair price and the upper bound for strict subhedging prices, that is,*
 $\widehat{p}_0(x, \mathcal{C}) = \underline{p}_0^f(x, \mathcal{C}) = \underline{p}_0(x, \mathcal{C}).$

Proof. As was already mentioned, the uniqueness of the replication cost $\widehat{p}_0(x, \mathcal{C})$ is an immediate consequence of condition (i) in Definition 4.6.

For part (ii), we first observe that condition (i) in Definition 4.6 means that the initial cost p of any superhedging strategy satisfies $p \geq \widehat{p}_0(x, \mathcal{C})$ and thus, since a replicating strategy is also a superhedging strategy, the equality $\widehat{p}_0(x, \mathcal{C}) = \bar{p}_0^s(x, \mathcal{C})$ is obvious. Moreover, since the initial cost p_0 of any strict superhedging strategy also satisfies $p \geq \widehat{p}_0(x, \mathcal{C})$ and, for any $p > \widehat{p}_0(x, \mathcal{C})$, there manifestly exists a strict superhedging strategy with the initial cost p , we conclude that the equality $\widehat{p}_0(x, \mathcal{C}) = \bar{p}_0(x, \mathcal{C})$ is valid as well.

For part (iii), we note that condition (ii) in Definition 4.6 implies that there is no trading strategy $(x, \widehat{p}_0(x, \mathcal{C}), \varphi, \mathcal{C}) \in \Psi^{0,x}(\mathcal{C})$ such that conditions (4.11) and (4.12) are satisfied. This means that no hedger's arbitrage opportunity arises (that is, no strict superhedging strategy for \mathcal{C} exists) if \mathcal{C} is traded at time 0 at its replication price $\widehat{p}_0(x, \mathcal{C})$, so that $\widehat{p}_0(x, \mathcal{C})$ is a fair hedger's price. Furthermore, no fair price for \mathcal{C} strictly higher than $\widehat{p}_0(x, \mathcal{C})$ may exist, since for any $p > \widehat{p}_0(x, \mathcal{C})$ there exists a superhedging strategy with cost p , which contradicts the definition of a fair price. Hence the equality $\widehat{p}_0(x, \mathcal{C}) = \underline{p}_0^f(x, \mathcal{C})$ holds.

Let us now consider any number $p < \widehat{p}_0(x, \mathcal{C})$. We claim that p is a strict subhedging price. Indeed, p is a fair price and it cannot be a replication price (since the latter is unique), so it is a strict subhedging price. We conclude that $\widehat{p}_0(x, \mathcal{C}) = \underline{p}_0(x, \mathcal{C})$. \square

4.2.2 Non-replicable Contracts

Let us make some comments on the pricing of a contract \mathcal{C} for which replication in a regular model $\mathcal{M} = (\mathcal{S}, \mathcal{D}, \mathcal{B}, \mathcal{C}, \Psi^{0,x}(\mathcal{C}))$ is not feasible. Since we already know that the equality $\underline{p}_0(x, \mathcal{C}) = \bar{p}_0(x, \mathcal{C})$ holds in any regular model and any contract $\mathcal{C} \in \mathcal{C}$, the question of great interest is whether the common value is a fair price or not. Unfortunately, the definitive answer is not available, since it may happen that the common value is a fair price, but it may also occur that it is a cost of a strict superhedging strategy.

5 Pricing in Regular Models

In this section, we assume that a market model is regular and we examine the properties of prices of a contract \mathcal{C} , which can be replicated on $[t, T]$ for every $t \in [0, T]$. Recall that for any fixed

$t \in [0, T)$ we denote by $(x_t, p_t, \varphi^t, \mathcal{C}^t)$ a hedger's trading strategy starting at time t with a \mathcal{G}_t -measurable endowment x_t when a contract \mathcal{C}^t is traded at a \mathcal{G}_t -measurable price p_t . For simplicity, we focus on contracts $\mathcal{C} = (A, \mathcal{X})$ with a constant maturity date T , which correspond to non-defaultable contracts of European style. To deal with the counterparty credit risk, it suffices to replace A by the process A^\sharp introduced in Section 2.8.1 and a fixed maturity T by the effective maturity of a contract at hand, for instance, by $T \wedge \tau$ where τ is the random time of the first default or, more generally, by the effective settlement date of a contract in the presence of the gap risk. Furthermore, in the case of contracts of American style or game contracts, the effective settlement date is also affected by the decisions of both parties to prematurely terminate a contract.

5.1 Replication and Market Regularity on $[t, T]$

The notion of the hedger's gained value $\widehat{p}_t(x, \mathcal{C})$, $t \in [0, T)$ reduces to the classical no-arbitrage price obtained through replication in the linear set-up provided that the only cash flow of A after time 0 is the terminal payoff. Unfortunately, in a general nonlinear set-up considered in this work, the financial interpretation of the hedger's gained value at time $t > 0$ is less transparent, since it depends on the hedger's initial endowment, the past cash flows of a contract and the strategy implemented by the hedger on $[0, t]$. The following definition mimics Definition 4.4, but focuses on the restriction of a contract \mathcal{C} to the interval $[t, T]$. Note that here the discounted wealth process is given by equation (3.3). It is assumed in this section that \mathcal{C}^t can be replicated on $[t, T]$ at some price \widehat{p}_t , in the sense of the following definition.

Definition 5.1. For a fixed $t \in [0, T]$, let \widehat{p}_t be a \mathcal{G}_t -measurable random variable. If there exists a trading strategy $(x_t, \widehat{p}_t, \varphi^t, \mathcal{C}^t) \in \Psi^{t, x_t}(\mathcal{C})$ such that

$$\widetilde{V}_T(x_t, \widehat{p}_t, \varphi^t, \mathcal{C}^t) = x_t, \quad (5.1)$$

then $\widehat{p}_t = \widehat{p}_t(x_t, \mathcal{C}^t)$ is called a *replication cost* at time t for the contract \mathcal{C}^t relative to the hedger's endowment x_t at time t .

As expected, Definition 4.6, and thus also Proposition 4.8, can be extended to any date $t \in [0, T)$.

Definition 5.2. We say that a market model $\mathcal{M} = (\mathcal{S}, \mathcal{D}, \mathcal{B}, \mathcal{C}, \Psi^{t, x_t}(\mathcal{C}))$ is *regular on $[t, T]$ with respect to \mathcal{C}* if the following properties hold for every contract $\mathcal{C} \in \mathcal{C}$ that can be replicated:

- (i) if $p_t \in \mathcal{G}_t$ and there exists $(x_t, p_t, \varphi^t, \mathcal{C}^t) \in \Psi^{t, x_t}(\mathcal{C})$ satisfying

$$\mathbb{P}(\widetilde{V}_T(x_t, p_t, \varphi^t, \mathcal{C}^t) \geq x_t) = 1, \quad (5.2)$$

then $p_t \geq \widehat{p}_t(x_t, \mathcal{C}^t)$;

- (ii) if $p_t \in \mathcal{G}_t$ and there exists $(x_t, p_t, \varphi^t, \mathcal{C}^t) \in \Psi^{t, x_t}(\mathcal{C})$ such that for some $D \in \mathcal{G}_t$

$$\mathbb{P}(\mathbf{1}_D \widetilde{V}_T(x_t, p_t, \varphi^t, \mathcal{C}^t) \geq \mathbf{1}_D x_t) = 1 \quad (5.3)$$

and

$$\mathbb{P}(\mathbf{1}_D \widetilde{V}_T(x_t, p_t, \varphi^t, \mathcal{C}^t) > \mathbf{1}_D x_t) > 0 \quad (5.4)$$

then $\mathbb{P}(\mathbf{1}_D p_t > \mathbf{1}_D \widehat{p}_t(x_t, \mathcal{C}^t)) > 0$.

Similarly as in the case of $t = 0$, if condition (i) holds, then condition (ii) is equivalent to the following condition:

(iii) if p_t and there exists $(x_t, p_t, \varphi^t, \mathcal{C}^t) \in \Psi^{t, x_t}(\mathcal{C})$ such that for some event $D \in \mathcal{G}_t$

$$\mathbb{P}(\mathbf{1}_D \tilde{V}_T(x_t, p_t, \varphi^t, \mathcal{C}^t) \geq \mathbf{1}_D x_t) = 1 \quad (5.5)$$

then the following implication holds: if $\mathbf{1}_D p_t = \mathbf{1}_D \hat{p}_t(x_t, \mathcal{C}^t)$, then

$$\mathbb{P}(\mathbf{1}_D \tilde{V}_T(x_t, p_t, \varphi^t, \mathcal{C}^t) = \mathbf{1}_D x_t) = 1. \quad (5.6)$$

In the following extension of Proposition 4.8, we assume that the hedger's endowment x_t at time t is given and the contract \mathcal{C} has all cash flows on $(t, T]$ so that $\mathcal{C}^t = \mathcal{C}$. We now search for the hedger's fair price for \mathcal{C} at time t assuming that a replication strategy exists. A closely related, but not identical, pricing problem is studied in Section 5.2 where we study pricing at time t of a contract initiated at time 0.

Proposition 5.3. *Assume that a market model $\mathcal{M} = (\mathcal{S}, \mathcal{D}, \mathcal{B}, \mathcal{C}, \Psi^{t, x_t}(\mathcal{C}))$ is regular on $[t, T]$ with respect to the class \mathcal{C} . Then for every contract $\mathcal{C} \in \mathcal{C}$ such that $\mathcal{C} = \mathcal{C}^t$ which can be replicated on $[t, T]$ we have:*

- (i) *the replication cost $\hat{p}_t(x, \mathcal{C})$ is unique,*
- (ii) *$\hat{p}_t(x_t, \mathcal{C})$ is the lower bound for superhedging costs and strict superhedging costs, that is,*
 $\hat{p}_t(x_t, \mathcal{C}) = \bar{p}_t^s(x_t, \mathcal{C}) = \bar{p}_t(x_t, \mathcal{C}),$
- (iii) *$\hat{p}_t(x_t, \mathcal{C})$ is the maximal fair price and the upper bound for strict subhedging prices, that is,*
 $\hat{p}_t(x_t, \mathcal{C}) = \underline{p}_t^f(x_t, \mathcal{C}) = \underline{p}_t(x_t, \mathcal{C}).$

The proof of Proposition 5.3 is completely analogous to the proof of Proposition 4.8, so it is omitted. According to Proposition 5.3, in any market model regular on $[t, T]$, a replication cost is unique and it is a fair price at time t for the hedger with an endowment x_t .

5.2 Hedger's Ex-dividend Price at Time t

Since it was postulated that the initial endowment of the hedger at time 0 equals x , it is clear that Definition 5.1 should be complemented by a financial interpretation of the hedger's endowment x_t at time t and a relationship between the quantity x_t and the hedger's initial endowment x should be clarified.

One may consider several alternative specifications for x_t , which correspond to different financial interpretations of pricing problems under study:

1. A first natural choice is to set $x_t = x_t(x) := x\mathcal{B}_t(x)$ meaning that the hedger has not been dynamically hedging the contract between time 0 and time t (this particular convention was adopted and studied in [BR15, NR16c, NR15]). Then the quantity $\hat{p}_t(x_t, \mathcal{C}^t)$ is the future fair price at time t of the contract \mathcal{C}^t , as seen at time 0 by the hedger with the endowment x at time 0, who decided to postpone trading in \mathcal{C} to time t . This specification could be convenient if one wishes to study, for instance, the issue of pricing at time 0 of the option with the expiration date t written on the contract \mathcal{C}^t .
2. Alternatively, one may postulate that the contract was entered into by the hedger at time 0 at the price $\hat{p}_0(x, \mathcal{C})$ and he decided to keep his position unhedged. In that case, the initial price, the cash flows, and the adjustments should be appropriately accounted for when computing the actual hedger's endowment x_t at time t using a particular market model.

3. Next, one may assume that the contract was entered into by the hedger at time 0 at the price $\widehat{p}_0(x, \mathcal{C})$ and was hedged by him on $[0, t]$ through a replicating strategy $\widehat{\varphi}$, as given by Definition 4.4. Then the hedger's endowment at time $t > 0$ equals $x_t = V_t(x, \widehat{p}_0(x, A, \mathcal{X}), \widehat{\varphi}, \mathcal{C})$ and it is natural to expect that the equality $\widehat{p}_t(x_t, \mathcal{C}^t) = 0$ should hold for all $t \in (0, T]$.
4. Finally, one can simply postulate that the endowment x_t is exogenously specified. Then Definition 5.1 reduces in fact to Definition 5.1 with essentially the same financial interpretation: we define the hedger's initial price at time t for the contract A^t given his initial endowment x_t at time t . Of course, under this convention, there is no connection between the quantities x and x_t .

In the next definition, we apply Definition 5.1 to the first specification of x_t , that is, we set $x_t = x_t(x) := x\mathcal{B}_t(x)$. As in Definition 5.1, the discounted wealth is given by (3.3).

Definition 5.4. For a fixed $t \in [0, T]$, assume that p_t^e is a \mathcal{G}_t -measurable random variable. If there exists a trading strategy $(x_t(x), p_t^e, \varphi^t, \mathcal{C}^t) \in \Psi^{t, x_t(x)}(\mathcal{C})$ such that

$$\widetilde{V}_T(x_t(x), p_t^e, \varphi^t, \mathcal{C}^t) = x_t(x), \tag{5.7}$$

then $p_t^e = p_t^e(x, \mathcal{C}^t)$ is called the *hedger's ex-dividend price* at time t for the contract \mathcal{C}^t .

Note that $\widehat{p}_0(x, \mathcal{C}) = p_0^e(x, \mathcal{C})$ and $\widehat{p}_T(x, \mathcal{C}) = p_T^e(x, \mathcal{C}^T) = 0$. The price given in Definition 5.4 is suitable when dealing with derivatives written on the contract \mathcal{C}^t as an underlying asset or, simply, when the hedger would like to compute the future fair price for \mathcal{C}^t without actually entering into the contract at time 0. Furthermore, it can also be used to define a proxy for the marked-to-market value of the contract \mathcal{C}^t .

It is natural to ask whether the processes $\widehat{p}_t(x, \mathcal{C})$ and $p_t^e(x, \mathcal{C}^t)$ coincide for all $t \in [0, T]$. We will argue that this holds when the pricing problem is *local*, but it is not necessarily true for a *global* pricing problem (see Proposition 6.6). The reason is that in the former case the two processes satisfy identical BSDE, whereas in the latter case one obtains a generalized BSDE for the former process and a classical BSDE for the latter. It is also intuitively clear that in the case of the global pricing problem the two processes will typically differ, since the value of $p_t^e(x, \mathcal{C}^t)$ is clearly independent of the hedger's trading strategy on $[0, t]$, as opposed to $\widehat{p}_t(x, \mathcal{C})$, which may depend on the whole history of his trading. Since most pricing problems encountered in the existing literature have a local nature, to the best of our knowledge, this particular issue was not yet examined by other authors.

5.3 Marked-to-Market Value

The issue of pricing at time t is also important when we ask the following question: in which way a contract entered into by the hedger at time 0 can be unwound by him at time t . The easiest way to unwind at time t a contract initiated at time 0 at the price $\widehat{p}_0(x, \mathcal{C})$ would be to 'assign' all obligations associated with the remaining part of the contract on $[t, T]$ to another trader. It is rather clear that the gained value $\widehat{p}_t(x, \mathcal{C})$ for $0 < t < T$ is the amount of cash, which the hedger would be willing to pay to another trader who would then take the hedger's position from time t onwards. This leads to the following definition of the marked-to-market value.

Definition 5.5. The *marked-to-market value* of a contract \mathcal{C} entered into at time 0 by the hedger with the initial endowment x is given by the equality $p_t^m(x, \mathcal{C}) := -\widehat{p}_t(x, \mathcal{C})$ for every $t \in [0, T]$.

On the one hand, Definition 5.5 reflects the market practice where the marked-to-market value is related to the concept of unwinding an existing contract at time t at its current ‘market price’. Indeed, the equality $\widehat{p}_t(x, \mathcal{C}) + p_t^m(x, \mathcal{C}) = 0$ holds for every $t \in [0, T]$ meaning that the net value of the fully hedged position is null at any moment when the contract is marked to market.

On the other hand, a practical implementation of Definition 5.5 could prove difficult when dealing with a global pricing problem, since it would require to keep track of the past cash flows and gains from the hedging strategy (of course, provided the hedging strategy has been used by the hedger). We thus contend that the proxy for the marked-to-market value $p_t^m(x, \mathcal{C}) := -p_t(x, \mathcal{C})$ could be more suitable for practical purposes when dealing with a global pricing problem. Since the equality $\widehat{p}_t(x, \mathcal{C}) = p_t(x, \mathcal{C})$ holds for a local pricing problem, the issue of a choice of the marking to market convention is immaterial in that case.

5.4 Offsetting Price

Assume that the hedger is unable to “assign” to another trader his existing position in the contract \mathcal{C} entered into at time 0. Then he may attempt to offset his future obligations associated with \mathcal{C} by taking the opposite position in an “equivalent” contract. In next definition, we postulate that the hedger attempts to unwind his long position in $\mathcal{C} = (A, \mathcal{X})$ at time t by entering into an *offsetting contract* $(-A^t, \mathcal{Y}^t)$. It is also assumed here that he liquidates at time t the replicating portfolio for \mathcal{C} so that his endowment at time t equals $\widehat{V}_t(x, \mathcal{C}) := V_t(x, \widehat{p}_0(x, \mathcal{C}), \widehat{\varphi}, \mathcal{C})$ where $\widehat{\varphi}$ is a replicating strategy for \mathcal{C} on $[0, T]$.

Definition 5.6. For a fixed $t \in [0, T]$, let p_t^o be a \mathcal{G}_t -measurable random variable. If there exists an admissible trading strategy $(\widetilde{V}_t(x, \mathcal{C}), p_t^o, \varphi^t, 0, \mathcal{X}^t + \mathcal{Y}^t)$ on $[t, T]$ such that

$$\widetilde{V}_T(\widetilde{V}_t(x, \mathcal{C}), p_t^o, \varphi^t, 0, \mathcal{X}^t + \mathcal{Y}^t) = x, \quad (5.8)$$

then $p_t^o = p_t^o(x, \mathcal{C}^t)$ is called the *offsetting price* of $\mathcal{C}^t = (A^t, \mathcal{X}^t)$ through $(-A^t, \mathcal{Y}^t)$ at time t .

Definition 5.6 takes into account the fact that the cash flows of A^t and $-A^t$ (and perhaps also some cash flows associated with the corresponding adjustments \mathcal{X}^t and \mathcal{Y}^t) offset one another and thus only the residual cash flows need to be accounted for when computing the price at which the contract \mathcal{C} can be unwound by the hedger at time t .

In the special case where the equality $\mathcal{X}^t + \mathcal{Y}^t = 0$ holds for all $t \in [0, T]$ (that is, the offsetting is perfect) we obtain the equality $p_t^o(x, \mathcal{C}^t) = -\widehat{p}_t(x, \mathcal{C})$ since then, in view of (4.9), we have that

$$\widehat{V}_t(x, \mathcal{C}) + p_t^o(x, \mathcal{C}^t) = \widehat{p}_t(x, \mathcal{C}) + x\mathcal{B}_t(x) + p_t^o(x, \mathcal{C}^t) = x\mathcal{B}_t(x)$$

where $x\mathcal{B}_t(x)$ is the cash amount that is required to replicate the null contract $(0, 0)$ on $[t, T]$. Note also that these computations are consistent with Definition 5.5 of the marked-to-market value.

6 A BSDE Approach to Nonlinear Pricing

For each definition of the price, one may attempt to derive the corresponding backward stochastic differential equation (BSDE) by combining their definitions with dynamics (2.11) of the hedger’s wealth or, even more conveniently, with dynamics (3.9) of his discounted wealth. Subsequently, each particular pricing problem can be addressed by solving a suitable BSDE. In addition, one may use a BSDE approach to establish the regularity property of a market model at hand. To this end, one may either use the existing comparison theorems for solutions to BSDEs or establish original comparison results.

6.1 BSDE for the Gained Value

We will first derive a generic BSDE associated with the hedger's gained value $\widehat{p}(x, A, \mathcal{X})$ introduced in Definition 4.4. For concreteness, we focus on the case of $x \geq 0$ so that the equality $\mathcal{B}(x) = B^l$ is valid. To simplify the presentation, we also postulate that $B^{i,l} = B^{i,b} = B^i$ for $i = 1, 2, \dots, d$ and we consider trading strategies satisfying the funding constraint $\widehat{\xi}_t^i S_t^i + \widehat{\psi}_t^i B_t^i = 0$ for all $i = 1, 2, \dots, d$ and every $t \in [0, T]$. Recall that (see (3.8))

$$\widetilde{S}_t^{i,cld}(x) := (\mathcal{B}_t(x))^{-1} S_t^i + \int_0^t (\mathcal{B}_u(x))^{-1} dD_u^i.$$

Lemma 6.1. *Assume that a strategy $(x, \widehat{p}_0, \widehat{\varphi}, \mathcal{C}) \in \Psi^{0,x}(\mathcal{C})$ replicates a contract \mathcal{C} . Then the processes $\widehat{Y} := \widetilde{V}^l(x, \widehat{p}_0, \widehat{\varphi}, \mathcal{C}) = (B^{0,l})^{-1} V^l(x, \widehat{p}_0, \widehat{\varphi}, \mathcal{C})$ and $\widehat{Z}^i := \widetilde{B}^{i,l} \widehat{\xi}^i$ satisfy the BSDE*

$$\begin{aligned} d\widehat{Y}_t = & \sum_{i=1}^d \widehat{Z}_t^i d\widehat{S}_t^{i,cld} - (B_t^{0,b})^{-1} \left(\widehat{Y}_t B_t^{0,l} + \sum_{k=1}^n \alpha_t^k X_t^k \right)^- dB_t^{0,b,l} \\ & + (B_t^{0,l})^{-1} dA_t - \sum_{k=1}^n \widehat{X}_t^k d\widetilde{\beta}_t^{k,l} + \sum_{k=1}^n (1 - \alpha_t^k) X_t^k d(B_t^{0,l})^{-1} \end{aligned} \quad (6.1)$$

with the terminal condition $\widehat{Y}_T = x$.

Proof. Under the present assumptions, (3.9) and (3.10) imply

$$\begin{aligned} d\widetilde{V}_t^l(x, \widehat{p}_0, \widehat{\varphi}, \mathcal{C}) = & \sum_{i=1}^d \widehat{\xi}_t^i \widetilde{B}_t^{i,l} d\widehat{S}_t^{i,cld} + \widehat{\psi}_t^{0,b} dB_t^{0,b,l} + (B_t^{0,l})^{-1} dA_t - \sum_{k=1}^n \widehat{X}_t^k d\widetilde{\beta}_t^{k,l} \\ & + \sum_{k=1}^n (1 - \alpha_t^k) X_t^k d(B_t^{0,l})^{-1}, \end{aligned} \quad (6.2)$$

where $\widetilde{B}^{i,l} := (B^{0,l})^{-1} B^i$, $B^{0,b,l} := (B^{0,l})^{-1} B^{0,b}$, $\widetilde{\beta}^{k,l} := (B^{0,l})^{-1} \beta^k$. Equation (2.10) and conditions $\widehat{\psi}^{0,l} \geq 0$, $\widehat{\psi}^{0,b} \leq 0$ and $\widehat{\psi}^{0,l} \widehat{\psi}^{0,b} = 0$ yield

$$\widehat{\psi}_t^{0,l} = (B_t^{0,l})^{-1} \left(V_t(x, \widehat{p}_0, \widehat{\varphi}, \mathcal{C}) + \sum_{k=1}^n \alpha_t^k X_t^k \right)^+ \quad (6.3)$$

and

$$\widehat{\psi}_t^{0,b} = -(B_t^{0,b})^{-1} \left(V_t(x, \widehat{p}_0, \widehat{\varphi}, \mathcal{C}) + \sum_{k=1}^n \alpha_t^k X_t^k \right)^-. \quad (6.4)$$

Note that the process $\widehat{\psi}^{0,l}$ does not appear in (6.2) and the process $\widehat{\psi}^{0,b}$ can also be eliminated from (6.2) by using (6.4). If we set $\widehat{Y} := \widetilde{V}^l(x, \widehat{p}_0, \widehat{\varphi}, \mathcal{C})$, then (6.2) can be represented as the BSDE

$$\begin{aligned} d\widehat{Y}_t = & \sum_{i=1}^d \widehat{\xi}_t^i \widetilde{B}_t^{i,l} d\widehat{S}_t^{i,cld} - (B_t^{0,b})^{-1} \left(\widehat{Y}_t B_t^{0,l} - \sum_{i=1}^d \widehat{\xi}_t^i S_t^i - \sum_{i=1}^d \widehat{\psi}_t^i B_t^i + \sum_{k=1}^n \alpha_t^k X_t^k \right)^- dB_t^{0,b,l} \\ & + (B_t^{0,l})^{-1} dA_t - \sum_{k=1}^n \widehat{X}_t^k d\widetilde{\beta}_t^{k,l} + \sum_{k=1}^n (1 - \alpha_t^k) X_t^k d(B_t^{0,l})^{-1}, \end{aligned} \quad (6.5)$$

with the terminal condition $\widehat{Y}_T = \widetilde{V}_T^l(x, \widehat{p}_0, \widehat{\varphi}, \mathcal{C}) = x$. In view of equality (3.10), BSDE (6.5) further simplifies to (6.1). \square

In the next result, we focus on a market model satisfying regularity conditions introduced in Definition 5.2. From the regularity of a model, it follows that the hedger's gained value $\widehat{p}_t(x, \mathcal{C})$ is unique for each fixed $t \in [0, T]$. However, this does not suffice to define the process $\widehat{p}(x, \mathcal{C})$ and thus in the next result we will make assumptions regarding BSDE (6.1). First, we postulate that for a given $x \geq 0$ and any contract $\mathcal{C} \in \mathcal{C}$ there exists a unique solution $(\widehat{Y}, \widehat{Z})$ to (6.1) in a suitable space of stochastic processes. Second, we assume that BSDE (6.1) enjoys the following variant of the strict comparison property.

Definition 6.2. We say that the *strict comparison* property holds for the BSDE (6.1) if for any contract $\mathcal{C} \in \mathcal{C}$ if $(\widehat{Y}^1, \widehat{Z}^1)$ and $(\widehat{Y}^2, \widehat{Z}^2)$ are solutions with \mathcal{G}_T -measurable terminal conditions $\xi_T^1 \geq \xi_T^2$, respectively, then the equality $\mathbb{1}_D Y_t^1 = \mathbb{1}_D Y_t^2$ for some $t \in [0, T)$ and some $D \in \mathcal{G}_t$ implies that $\mathbb{1}_D \xi_T^1 = \mathbb{1}_D \xi_T^2$.

It is also important to note that one needs to examine the manner in which the inputs in BSDE (6.1) (that is, the stochastic processes introduced in Assumption 2.1) may possibly depend on the unknown processes \widehat{Y} and \widehat{Z} .

Definition 6.3. The pricing problem is *local* if $X_t^k = h^k(t, \widehat{Y}_t, \widehat{Z}_t)$ and $d\widetilde{\beta}_t^{k,l} = h^{k,l}(t, \widehat{Y}_t, \widehat{Z}_t) dt$ for some functions h^k and $h^{k,l}$ and $k = 1, 2, \dots, n$. The pricing problem is called *global* if $X_t^k = h^k(t, \widehat{Y}, \widehat{Z})$ and $d\widetilde{\beta}_t^{k,l} = h^{k,l}(t, \widehat{Y}, \widehat{Z}) dt$ for some non-anticipating functionals h^k and $h^{k,l}$.

We deduce from Lemma 6.1 that local problem can be solved using classical BSDEs. In contrast, the situation where the inputs depend on the past history of processes is harder to address, since it leads to a global pricing problem, which requires to study generalized BSDEs with progressively measurable functionals. Since both situations are covered by Proposition 6.4, we refer the reader to Cheridito and Nam [CN15] for existence and uniqueness results for generalized BSDEs. It is worth noting that, to the best of our knowledge, strict comparison theorems for generalized BSDE are not yet available, as opposed to the case of classical BSDEs.

Proposition 6.4. *Assume that the BSDE (6.1) has a unique solution $(\widehat{Y}, \widehat{Z})$ for any contract $\mathcal{C} \in \mathcal{C}$ and the strict comparison property for solutions to (6.1) holds. Then the following assertions are valid:*

- (i) *the market model is regular on $[t, T]$ for every $t \in [0, T]$;*
- (ii) *the hedger's gained value satisfies $\widehat{p}(x, \mathcal{C}) = B^{0,l}(\widehat{Y} - x)$ where $(\widehat{Y}, \widehat{Z})$ is a solution to BSDE (6.1) with the terminal condition $\widehat{Y}_T = x$;*
- (iii) *a unique replicating strategy $\widehat{\varphi}$ for \mathcal{C} satisfies $\widehat{\xi}^i = (\widetilde{B}^{i,l})^{-1} \widehat{Z}^i$ and the cash components $\widehat{\psi}^{0,l}$ and $\psi^{0,b}$ are given by (6.3) and (6.4), respectively, with $V(x, \widehat{p}_0, \widehat{\varphi}, \mathcal{C})$ replaced by $B^{0,l} \widehat{Y}$.*

Proof. In view of Definition 5.2, it is clear that the existence, uniqueness and the strict comparison property for the BSDE (6.1) imply that the market model is regular on $[t, T]$ for every $t \in [0, T]$. To establish (ii), we recall that the regularity of a model implies that the hedger's gained value $\widehat{p}_t(x, \mathcal{C})$ is unique. Moreover, we also know that $\widehat{p}(x, \mathcal{C})$ satisfies for every $t \in [0, T]$ (see (4.9))

$$\widehat{p}_t(x, \mathcal{C}) = V_t(x, \widehat{p}_0(x, \mathcal{C}), \widehat{\varphi}, \mathcal{C}) - x \mathcal{B}_t(x) = B_t^{0,l} \widehat{Y}_t - x \mathcal{B}_t(x) = B_t^{0,l} (\widehat{Y}_t - x),$$

which establishes the asserted equality $\widehat{p}(x, \mathcal{C}) = B^{0,l}(\widehat{Y} - x)$. In particular, the hedger's replication cost $\widehat{p}_0(x)$ satisfies $\widehat{p}_0(x) = \widehat{Y}_0 - x$ for any fixed initial endowment $x \geq 0$. Finally, part (iii) is an immediate consequence of Lemma 6.1. \square

Of course, one needs to check for which models the assumptions of Proposition 6.4 are satisfied. For general results regarding BSDEs driven by one- or multi-dimensional continuous martingales, the reader is referred to Carbone et al. [CFS08], El Karoui and Huang [EKH97] and Nie and Rutkowski [NR16b] and the references therein. Typically, a suitable variant of the Lipschitz continuity of a generator to a BSDE is sufficient to guarantee the desired properties of its solutions. Several instances of nonlinear market models with BSDEs satisfying the comparison property were studied in [NR15, NR16a, NR16c], although the concept of a regular model was not formally stated therein. In particular, they analyzed contracts with an endogenous collateral, meaning that an adjustment process X^k explicitly depends on a solution \widehat{Y} (or even on solutions to the pricing problems for the hedger and the counterparty).

Let us finally mention that since the model examined in this section is a special case of the model studied in Sections 3.5 and 3.6, it follows from Proposition 3.9 that to ensure that the model is arbitrage-free for the trading desk, it suffices to assume that there exists a probability measure \mathbb{Q} , which is equivalent to \mathbb{P} on (Ω, \mathcal{G}_T) and such that the processes $\widehat{S}^{i,\text{cld}}$, $i = 1, 2, \dots, d$ given by (5.5) are \mathbb{Q} -local martingales. This assumption is also convenient if one wishes to prove the existence and uniqueness result for BSDE (6.1).

6.2 BSDE for the Ex-dividend Price

Our next goal is to derive the BSDE for the ex-dividend price $p^e(x, \mathcal{C})$ introduced in Definition 5.4. We maintain the assumption that $x \geq 0$. Recall that, for a fixed t , the hedger's ex-dividend price is implicitly given by the equality $\widetilde{V}_T^l(x_t(x), p_t^e, \varphi^t, \mathcal{C}^t) = x_t(x)$ where $x_t(x) = xB_t^l$ and the discounting is done using the process $\mathcal{B}^t(x_t(x))$ given by (3.2). We henceforth assume that the pricing problem is local. This assumption is essential for validity of Lemma 6.5 and Proposition 6.6, so it cannot be relaxed.

Lemma 6.5. *Assume that a strategy $(x_t(x), p_t^e, \varphi^t, \mathcal{C}^t) \in \Psi^{t, x_t(x)}(\mathcal{C})$ replicates a contract \mathcal{C}^t on $[t, T]$. Then the processes $\bar{Y}_u := \widetilde{V}_u(x_t(x), p_t^e, \varphi^t, A^t, \mathcal{X}^t)$ and $\bar{Z}_u^i := \widetilde{B}_u^{i,l}(\xi_u^t)^i$, $u \in [t, T]$, satisfy the following BSDE, for all $u \in [t, T]$*

$$\begin{aligned} d\bar{Y}_u = & \sum_{i=1}^d \bar{Z}_u^i d\widehat{S}_u^{i,\text{cld}} - (B_u^{0,b})^{-1} \left(\bar{Y}_u B_u^{0,l} + \sum_{k=1}^n \alpha_u^k X_u^k \right)^- dB_u^{0,b,l} \\ & + (B_u^{0,l})^{-1} dA_u - \sum_{k=1}^n \widehat{X}_u^k d\widetilde{\beta}_u^{k,l} + \sum_{k=1}^n (1 - \alpha_u^k) X_u^k d(B_u^{0,l})^{-1}, \end{aligned} \quad (6.6)$$

with the terminal condition $\bar{Y}_T = x$.

Proof. Arguing as in the proof of Lemma 6.1, we conclude that the dynamics of the discounted wealth $\widetilde{V}_u(x_t(x), p_t^e, \varphi^t, A^t, \mathcal{X}^t)$ for $u \in [t, T]$ are given by (6.2) the thus (6.6) is satisfied by \widehat{Y} and \widehat{Z} with the terminal condition $\widehat{Y}_T = x$. \square

Although BSDEs (6.1) and (6.6) have the same shape, the features of their solutions heavily depend on a specification of the processes X^k and $\beta^{k,l}$. The next result shows that the gained value and the ex-dividend price coincide when the pricing problem is local, so that the pricing BSDEs are classical. In contrast, this property will typically fail to hold when a pricing problem is global, so that (6.1) becomes a generalized BSDE. In that case, equation (6.6) needs to be complemented by additional conditions regarding the processes X^k and $\beta^{k,l}$.

Proposition 6.6. *Under the assumptions of Proposition 6.4, if a pricing problem is local, then for any contract $\mathcal{C} \in \mathcal{C}$ the hedger's gained value and the hedger's ex-dividend price satisfy $p_t^e(x, \mathcal{C}) = \widehat{p}_t(x, \mathcal{C}^t)$ for all $t \in [0, T]$.*

Proof. On the one hand, under the postulate of uniqueness of solutions to BSDE (6.1) (and thus also to BSDE (6.6)), the equality $\widehat{Y}_t = \bar{Y}_t$ is manifestly satisfied for all $t \in [0, T]$. On the other hand, from Definition 5.4, we obtain the equality $x_t(x) + p_t^e(x, \mathcal{C}^t) = B_t^l \bar{Y}_t$, which in turn yields $p_t^e(x, \mathcal{C}^t) = B_t^l(\bar{Y}_t - x)$. Since $\widehat{Y}_t = \bar{Y}_t$, we conclude that the gained value $\widehat{p}_t(x, \mathcal{C}) = B_t^l(\widehat{Y}_t - x)$ and the ex-dividend price $p_t^e(x, \mathcal{C}^t)$ coincide for all $t \in [0, T]$. \square

The property of a local pricing problem established in Proposition 6.6 is fairly general: its validity hinges on the existence and uniqueness of a common BSDE for the gained value and the ex-dividend price. This should be contrasted with the case of the global pricing problem where the equality $\widehat{p}_t(x, \mathcal{C}) = p_t^e(x, \mathcal{C}^t)$ is always satisfied for $t = 0$, but it is unlikely to hold for $t > 0$.

6.3 BSDE for the CCR Price

We now address the question raised in Section 2.8.2: can we disentangle the clean pricing of a counterparty credit risky contract from the CRR pricing? Although this is true in the linear setup, where the price additivity is known to hold, the answer to this question is unlikely to be positive within a nonlinear framework. On the one hand, according to Proposition 2.10, the counterparty risky contract (A^\sharp, \mathcal{X}) admits the following decomposition

$$(A^\sharp, \mathcal{X}) = (A, \mathcal{X}) + (A^{\text{CCR}}, 0), \quad (6.7)$$

where the first component is not subject to the counterparty credit risk (although it may include the margin account) and thus it is referred to as the *clean* contract, whereas the second component is concerned exclusively with the CCR (see Definition 2.9 for the specification of the CCR cash flow A^{CCR}). On the other hand, however, in a nonlinear framework, the price of the full contract (A^\sharp, \mathcal{X}) is unlikely to be equal to the sum of prices of its components appearing in the additive decomposition of the full contract.

To analyze this issue a bit further, let us assume that the underlying market model is sufficiently rich to allow for the replication of the full contract (A^\sharp, \mathcal{X}) , as well as for the replication of its components (A, \mathcal{X}) and $(A^{\text{CCR}}, 0)$. Of course, one could alternatively focus on the decomposition $(A^\sharp, \mathcal{X}) = (A, 0) + (A^{\text{CCR}}, \mathcal{X})$ in which the trading adjustments (in particular, the margin account) are assumed to affect the CCR part, rather than the clean contract $(A, 0)$. The choice of a decomposition should be motivated by practical considerations; one may argue that collateralization is a standard covenant in most contracts, not necessarily closely related to the actual level of the counterparty credit risk in a given contract.

If we denote by τ^h and τ^c the default times of the hedger and the counterparty, respectively, then $\tau = \tau^h \wedge \tau^c$ is the moment of the first default and thus the effective maturity of (A^\sharp, \mathcal{X}) and $(A^{\text{CCR}}, 0)$ is the random time $\widehat{T} = \tau \wedge T$. For the clean contract (A, \mathcal{X}) , it is convenient to formally assume that T is its maturity date, since this component of the full contract is not affected by the counterparty credit risk.

By a minor extension of Lemma 6.1, we obtain the following BSDE for the full contract (A^\sharp, \mathcal{X})

$$\begin{aligned} d\widehat{Y}_t &= \sum_{i=1}^d \widehat{Z}_t^i d\widehat{S}_t^{i,cld} - (B_t^{0,b})^{-1} \left(\widehat{Y}_t B_t^{0,l} + \sum_{k=1}^n \alpha_t^k X_t^k \right)^- dB_t^{0,b,l} \\ &\quad + (B_t^{0,l})^{-1} dA_t^\sharp - \sum_{k=1}^n \widehat{X}_t^k d\widetilde{\beta}_t^{k,l} + \sum_{k=1}^n (1 - \alpha_t^k) X_t^k d(B_t^{0,l})^{-1}, \end{aligned} \quad (6.8)$$

with the terminal condition $\widehat{Y}_{\widehat{T}} = x$. Let $x = x_1 + x_2$ be an arbitrary split of the hedger's endowment. Then we obtain the following BSDE corresponding to the clean contract (A, \mathcal{X})

$$\begin{aligned} d\widehat{Y}_t^1 &= \sum_{i=1}^d \widehat{Z}_t^{1,i} d\widehat{S}_t^{i,cld} - (B_t^{0,b})^{-1} \left(\widehat{Y}_t^1 B_t^{0,l} + \sum_{k=1}^n \alpha_t^k X_t^k \right)^- dB_t^{0,b,l} \\ &\quad + (B_t^{0,l})^{-1} dA_t - \sum_{k=1}^n \widehat{X}_t^k d\widetilde{\beta}_t^{k,l} + \sum_{k=1}^n (1 - \alpha_t^k) X_t^k d(B_t^{0,l})^{-1}, \end{aligned} \quad (6.9)$$

with $\widehat{Y}_{\widehat{T}}^1 = x_1$. The BSDE associated with the CRR component $(A^{\text{CCR}}, 0)$ reads

$$d\widehat{Y}_t^2 = \sum_{i=1}^d \widehat{Z}_t^{2,i} d\widehat{S}_t^{i,cld} - (B_t^{0,b})^{-1} (\widehat{Y}_t^2 B_t^{0,l})^- dB_t^{0,b,l} + (B_t^{0,l})^{-1} dA_t^{\text{CCR}}, \quad (6.10)$$

with $\widehat{Y}_{\widehat{T}}^2 = x_2$. If the initial endowment $x = 0$, then we may take x_1 and x_2 to be null as well.

The question formulated at the beginning of this section may be rephrased as follows: under which conditions the equality $\widehat{Y}_0 = \widehat{Y}_0^1 + \widehat{Y}_0^2$ holds for solutions to BSDEs (6.8), (6.9) and (6.10), so that the replication costs satisfy the following equality

$$\widehat{p}_0(x, A^\sharp, \mathcal{X}) = \widehat{p}_0(x_1, A, \mathcal{X}) + \widehat{p}_0(x_2, A^{\text{CCR}}, 0),$$

which formally corresponds to decomposition (6.7) of the full contract and the split $x = x_1 + x_2$ of the hedger's initial endowment. Since this equality is unlikely to be satisfied (even when $x = x_1 = x_2 = 0$, as was implicitly assumed in most existing papers on nonlinear approach to credit risk), one could ask, more generally, whether the quantities \widehat{Y}_0 and $\widehat{Y}_0^1 + \widehat{Y}_0^2$ are close to each other, so that an approximate equality is satisfied by the replication costs. Of course, an analogous question can also be formulated for the corresponding replicating strategies.

One needs first to address the issues of regularity and completeness of market models with default times. To this end, one may employ the existence and uniqueness results, as well as the strict comparison theorems, obtained for BSDEs with jumps generated by the occurrence of random times. BSDEs of this form are relatively uncommon in the existing literature on the theory of BSDEs, but they were studied in papers by Peng and Xu [PX09] and Quenez and Sulem [QS13]. Of course, to be in a position to use any result from [PX09] or [QS13], one needs to be more specific about the price dynamics for non-defaultable risky assets S^1, \dots, S^{d-2} (usually driven by a multidimensional Brownian motion) and the manner in which default times (hence also the prices of defaultable assets S^{d-1} and S^d) are defined. Note the latter problem is tackled in [PX09] or [QS13] using the so-called intensity-based approach, which was extensively studied in the credit risk literature. Moreover, it is convenient to assume that the cash and funding accounts as well as remuneration processes have absolutely continuous sample paths, so that BSDEs can be represented in the following generic form

$$dY_t = -g(t, Z_t, Y_t) dt + \sum_{i=1}^{d-2} Z_t^i dW_t^i + \sum_{i=d-1}^d Z_t^i dM_t^i + d\bar{A}_t,$$

where M^1 and M^2 are purely discontinuous \mathbb{G} -martingales associated with S^{d-1} and S^d , respectively, and the generator g is obtained from (6.8), (6.9) and (6.10) by straightforward computations.

From the financial perspective, to ensure the completeness of a market, one needs to postulate that some defaultable securities (typically, defaultable bonds issued by the two parties) are among primary traded assets. Finally, it is necessary to explicitly specify the closeout valuation process Q (see Remark 2.8) and the collateral process C . When dealing with a local pricing problem, the most natural postulate would be to set $Q_t := p_t^e(x_1, \mathcal{C}) = \widehat{p}_t(x_1, \mathcal{C}^t)$ and $C_t := p_t^e(x, A^\#, \mathcal{X}) = \widehat{p}_t(x, (A^\#)^t, \mathcal{X}^t)$ (see Proposition 6.6), although the latter convention of the *endogenous collateral* is a bit cumbersome to handle, even when dealing with BSDEs driven by continuous martingales (see [NR16a]). Note also that it would require to replace C_τ by $C_{\tau-}$ when specifying the closeout payoff \mathfrak{K} (hence also the process $A^\#$) in Section 2.8.1. For technical problems for BSDEs with jumps arising in this context and related to the so-called *immersion hypothesis*, as well as the way in which they can be overcome, the interested reader is referred to papers by Crépey and Song [CS16, CS15].

For the case of linear market models, the issue of completeness and various methods for replication in such models were studied in several works (see, in particular, Bielecki et al. [BJR04, BJR06a, BJR06b, BJR06c, BJR08]). In contrast, only a few papers devoted to nonlinear models of credit risk are available. More recently, Crépey [Cré15a, Cré15b], Dumitrescu et al. [DQS15] and Bichuch et al. [BCS15] used BSDEs with jumps to solve the pricing and hedging problems for contracts with the counterparty credit risk. In [BCS15] and [DQS15], the authors focus on pricing and hedging of the full contract, whereas in [Cré15a, Cré15b], the problem of the approximate additivity credit valuation adjustment is addressed.

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7 Appendix: Examples of Non-Regular Models

We give here two examples of models that fail to be regular, but they do satisfy either the no-arbitrage property with respect to the null contract (see Section 7.1) or even the stronger no-arbitrage property for the trading desk (see Section 7.2). Recall that, in essence, the *regularity* of a market model means that if a model is arbitrage-free in a properly specified sense, then an extended model obtained by adding a derivative asset with the price process defined by the cost of replication remains arbitrage-free.

7.1 Model with Trading Constraints

To illustrate the issue of the model regularity, we start by considering the Black-Scholes model (B, S^1) with the null interest rate in which, however, the borrowing of cash is precluded. We assume that the hedger’s initial endowment is null, that is, $x = 0$. Obviously, the model is arbitrage-free in the classical sense and the hedger is able to replicate without borrowing of cash the short position in the put option maturing at T written on the stock S^1 . Similar arguments show that he may fairly price the contract $A = (A, 0)$ in which $a_1 = P_U(K)$ for some $0 < U < T$ and $a_2 = -P_T(K) = -(K - S_T^1)^+$, where $P_t(K)$ denotes the Black-Scholes price at time $t \leq T$ of the put option. Observe that the initial price for A is null not only in the classical Black-Scholes model, but also in the present model with no borrowing of cash.

Let us now extend the model by introducing the second risky asset with the following price process

$$S_t^2 = \mathbf{1}_{[0, T]}(t) + 2K(t - U)(P_U(K))^{-1} \mathbf{1}_{[U, T]}(t).$$

One may check that the model (B, S^1, S^2) is still arbitrage-free in the sense of Definition 3.3 if the borrowing of cash is not allowed, since the only way of investing in the second asset is to sell short the first one. Furthermore, the price of the contract A based on the concept of replication is still equal to 0. However, the hedger who enters into the contract A at time 0 at null price has now an obvious arbitrage opportunity, since he may now use the cash amount $P_U(K)$ received at time U to buy the asset S^2 . This simple strategy is obviously self-financing and admissible since its wealth process is non-negative. Furthermore, it yields

the amount $2K(T - U)$ at time T , which strictly dominates the hedger's liability $P_T(K)$ provided that $T - U \geq 0.5$.

The present example makes it clear that Definition 3.3 of no-arbitrage, which is based on trading in primary assets only, is too weak to eliminate non-regular nonlinear models. In contrast, one may check that the model (B, S^1, S^2) would be rejected if we apply Definition 3.7 of an arbitrage opportunity for the trading desk.

From the mathematical point of view, the issue with the extended model $\widetilde{\mathcal{M}} = (B, S^1, S^2)$ where S^3 is that the strict comparison property for BSDE governing wealth process fails to hold. Hence a superhedging strategy for the claim $P_T(K)$ can be obtained using the same initial wealth as is needed for its replication. Indeed, it is clear that starting from the null initial endowment, one may either replicate $\xi_1 := P_T(K)$ or strictly super-replicate this claim by producing the terminal payoff $\xi_2 = 2K(T - U) > \xi_1 = P_T(K)$. This is inconsistent with the strict comparison property for solutions of BSDEs, since it states that if $\xi_2 \geq \xi_1$ and the two solutions coincide at time 0, $V_0^1 = V_0^2$, then necessarily $\xi_1 = \xi_2$. In our case, we have that $V_0^1 = V_0^2 = 0$ but $V_T^2 > V_T^1$.

7.2 Model Without Trading Constraints

We start by placing ourselves within the framework of Bergman's [Ber95] model (B^l, B^b, S^1) with differential borrowing and lending interest rates and we assume that the hedger's endowment $x = 0$. This means, in particular, that the stock price S^1 is driven by the Black-Scholes dynamics and the interest rates satisfy $r^b \geq r^l$. It is straightforward to verify that the model (B^l, B^b, S^1) is arbitrage-free, in the sense of Definition 3.3. Moreover, the hedger is able to replicate, with no borrowing of cash, the short position in the put option on the stock S^1 with maturity T and any strike price $K > 0$. The price of the put is thus given by the Black-Scholes formula and it is denoted as $P_t(K)$ for $t \in [0, T]$. We now fix $K > 0$ and $0 < U < T$, and we introduce an additional risky asset S^2 with the price process

$$S_t^2 = \mathbf{1}_{[0, T]}(t) + (K(t - U))(P_U(K))^{-1} \mathbf{1}_{\{2P_U(K) > K\}} \mathbf{1}_{[U, T]}(t).$$

To ensure that $\mathbb{P}(2P_U(K) > K) > 0$, we may set $r^l = 0$ (of course, it is enough to assume that r^l is low enough). For concreteness, we henceforth take $T - U = 1$. It is obvious that $S_T^2 = 1$ on the event $\{2P_U(K) < K\}$. On the event $\{2P_U(K) > K\}$ we have

$$1 < S_T^2 = 1 + K(P_U(K))^{-1} < 3 = e^{\ln 3}.$$

One may check that the model $\mathcal{M} = (B^l, B^b, S^1, S^2)$ is still arbitrage-free, in the sense of Definition 3.3, provided that the borrowing rate r^b is set to be high enough. Specifically, it suffices to take $r^b > \ln 3$ in order to ensure that the interest rate r^b is higher than the rate of return on the asset S^2 . Of course, the price of the put option with any strike K in the model (B^l, B^b, S^1, S^2) can still be defined by replication (using in fact B^l and S^1 only), and thus it is still given by the Black-Scholes formula $P_t(K)$.

We claim that the nonlinear model $\mathcal{M} = (B^l, B^b, S^1, S^2)$ is not regular with respect to the put option with strike K . Specifically, the hedger who sells the put option at time 0 at the price $P_0(K)$ can construct an arbitrage opportunity. To see this, assume that the hedger uses the replicating strategy for the put on the interval $[0, U]$. On the event $\{2P_U(K) \leq K\}$, he continues to replicate the put till its maturity date T . On the event $\{2P_U(K) > K\}$, he buys $P_U(K)/S_U^2 = P_U(K)$ of shares of the asset S^2 and holds it till T . Then the hedger's wealth at T , after he delivers the cash amount $(K - S_T^1)^+$ to the buyer of the put, equals

$$\begin{aligned} V_T(0, P_0(K), \varphi, -P(K)) &= \left(\frac{P_U(K)}{S_U^2} S_T^2 - (K - S_T^1)^+ \right) \mathbf{1}_{\{2P_U(K) > K\}} + 0 \cdot \mathbf{1}_{\{2P_U(K) \leq K\}} \\ &= (P_U(K)(1 + K(P_U(K))^{-1}) - (K - S_T^1)^+) \mathbf{1}_{\{2P_U(K) > K\}} > \\ &= (1.5K - (K - S_T^1)^+) \mathbf{1}_{\{2P_U(K) > K\}} > 0.5K \mathbf{1}_{\{2P_U(K) > K\}}. \end{aligned}$$

Since $\mathbb{P}(2P_U(K) > K) > 0$, and it is clear that the wealth is always non-negative (hence the strategy is admissible), we obtain an arbitrage opportunity for the hedger.

From the mathematical point of view, the issue with the extended model $\widetilde{\mathcal{M}} = (B^l, B^b, S^1, S^2, S^3)$, where $S^3 = P(K)$ is the replication price for the put, is due to the fact that the strict comparison property for the

wealth process fails to hold in the model $\mathcal{M} = (B^l, B^b, S^1, S^2)$. In particular, a strict superhedging strategy for the claim $P_T(K)$ can be obtained using the initial wealth equal to the replication cost. Indeed, it is clear that starting from the null initial endowment, the hedger may either replicate $\xi_1 := P_T(K)$ or strictly super-replicate this claim by producing the terminal payoff

$$\xi_2 = P_T(K)\mathbb{1}_{\{2P_U(K) \leq K\}} + P_U(K)(1 + K(P_U(K))^{-1})\mathbb{1}_{\{2P_U(K) > K\}}.$$

It is easy to check that $\xi_2 \geq \xi_1$ and $\mathbb{P}(\xi_2 > \xi_1) > 0$. Once again, this is manifestly inconsistent with the strict comparison property for solutions of BSDEs. This model is arbitrage-free with respect to the null contract and with respect to the trading desk.

8 Appendix: Local and Global Pricing Problems

Market adjustments, represented by \mathcal{X} , may depend both on the cash flow process A and the trading strategy φ . By the same token, the trading strategy φ typically depends on the trading adjustments. So, there could be a feedback effect between φ and \mathcal{X} potentially present in the trading universe, that needs to be accounted for in valuation and hedging. Furthermore, it is important to distinguish between the case where the dependence is only on the current composition of the hedging strategy and the current level of the wealth process and where the dependence is on the history of these processes. If the contract (A, \mathcal{X}) , the cash and funding accounts, and the prices of risky assets do not depend on the strict history (i.e., the history not including the current values of processes of interest) of a trading strategy φ and its value process $V^p(\varphi)$, then we say that the pricing problem is *local*. Otherwise, it is referred to as a *global* pricing problem. As one might guess, the two pricing problem will typically have very different properties for any date $t \in (0, T)$. In particular, they will typically correspond to different classes of BSDEs. It is important to stress that the distinction between the local and global problems is not related to the concept of path-independent contingent claims or to a Markovian property of the underlying model for risky assets. It is only related to the above-mentioned (either local or global) feedback effect between the hedger's trading decisions and the market conditions inclusive of particular adjustments for the contract at hand.

Example 8.1. As a stylized example of a global pricing problem, let us consider a contract, which lasts for two months (for the sake of concreteness, let us say that it is a combination of the put and the call on the stock S^1 with maturities one month and two months, respectively). The borrowing rate for the hedger is set to be 5% per annum, rising to 6% after one month if the hedger borrows any cash during the first month and it will stay at 5% if he doesn't. Similarly, the lending rate initially equals 3% per annum and drops to 2% if the hedger borrows any cash during the first month. It is intuitively clear that the pricing/hedging problem is here global, since its solution on $[t, T]$ will depend on the strict history of trading. In contrast, if a model has possibly different, but fixed, borrowing and lending rates, then the pricing problem for any contract will be local in the sense introduced above, unless the adjustments depend on the strict history of trading. For instance, if the only adjustment is the collateral with the current amount specified by the hedger's valuation and with a constant remuneration rate, then the problem is local.

Precise statements and formal definitions of local and global pricing problems are given in Section 6 where we examine the BSDEs approach to the nonlinear pricing. Note that most pricing problems examined in the existing financial literature are local and thus they can be solved using existing results for classical BSDEs. By contrast, the global pricing problems are much harder to analyze since they require to use novel classes of BSDEs (see Cheridito and Nam [CN15] and the references therein).