Absence of eigenvalues for integro-differential operators with periodic coefficients

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Abstract

The absence of the point spectrum for some nonselfadjoint integro-differential operators is investigated by applying perturbation theory

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methods. The considered differential operators could be of any order and are assumed to act in $\mathbf{L}_p(\mathbb{R}_+)$ or $\mathbf{L}_p(\mathbb{R})$ $(1 \leq p < \infty)$. Finally, as an application of general results, some spectral properties of the perturbed Hill operator are derived.

Keywords: Spectral theory, perturbation theory, point spectrum, integro-differential operators, periodic coefficients, nonselfadjoint operators.

AMS Subject Classifications: Primary 47A10, 47A55, 47A75, Secondary 47G

1 Introduction

The spectral theory of integro-differential operators plays an important role in theory of neutrons scattering, plasma oscillations, quantum physics, mechanics and chaos behavior (see for instance classical works by J. Lehner and G.M. Wing [15], E.A. Catchpole [5], D. Bohm and E. Grose [3], N.G. van Kampen [20], K.M. Case [4], and recent survey with application to chaos behavior by J. McCaw and B.H.J. McKeller [16]). One of the central question in spectral theory is to describe quantitatively the spectral components of a given operator. Usually, the essential spectrum can be easily determined by applying Weyl's type theorems about stability of the essential spectrum (see T. Kato [13]). However, this is not the case for other components of the spectrum. In case of nonselfadjoint operators an important spectral component is the point spectrum (the set of all eigenvalues). Two fundamental properties related to point spectrum are: the absence and the finiteness of the point spectrum. While these problems look similar, the methods developed for their study are different. Some important results on the absence of eigenvalues of differential operators of any order have been obtained by P. Cojuhari [9]. Also, P. Cojuhari and M.M. Stanescu [10], [19] studied the same problem for integro-differential operators, with the unperturbed operator being a differential operator with constant coefficients. The absence of the point spectrum for tridiagonal opeartors have been investigated by C.G. Kokologiannaki [14].

In this paper we will investigate the problem of absence of the point spectrum for a large class of integro-differential operators. These operators are generally assumed to be non-selfadjoint, of any order, and act in one of the spaces $\mathbf{L}_p(\mathbb{R}_+)$ or $\mathbf{L}_p(\mathbb{R}),\ 1 \le p < \infty$. Applying methods from perturbation theory, we consider the original operator as a sum of a differential operator with periodic coefficients (the unperturbed operator) and an integro-differential operator (the perturbation). We establish sufficient conditions on the coefficients and kernels of the perturbation that guarantee that the point spectrum of the original operator is empty. The paper is organized as follows. In Section 2 we state the problem and derive some auxiliary results, mainly describing explicitly the spectrum and the resolvent of the unperturbed operator by applying Floquet theory. In Section 3 we prove the main result. We conclude the paper with an application of developed theory to perturbed Hill operator, hat represents an important and interesting result by itself.

The absence of the point spectrum depends on how fast the coefficients and the kernels of the perturbation vanish at infinity. The polynomial decay, with order of decay depending on the multiplicity of the corresponding Floquet multipliers, together with subdiagonal property of the kernels $(k(t,s) = 0,\ s < t)$, will guarantee the absence of the eigenvalues of the perturbed operator. The results agree with those particular cases established in [9], [10], [19], and the conditions are in some sense necessary (see for instance [7]).

2 The problem and some auxiliary results

In the space $\mathbf{L}_p(\mathbb{R}_+)$ consider the differential operator $D = i\frac{d}{dx}$ with the domain of definition determined by the set of all functions $u \in \mathbf{L}_p(\mathbb{R}_+)$ which are absolutely continuous on every bounded interval of the positive semi-axis and the generalized derivative u' belonging to $\mathbf{L}_p(\mathbb{R}_+)$.

Let H be an integro-differential operator of the form

$$H = \sum_{j=0}^{n} H_j D^j, \tag{2.1}$$

where

$$(H_j u)(t) = h_j(t)u(t) + \int_{\mathbb{R}_+} k_j(t,s)u(s)ds\,, \quad j = 0,\dots,n\,,$$

the functions $h_j(t)$ and the kernels $k_j(t,s),\ j = 0,\dots,n;\ t,s \in \mathbb{R}_+$, are complex-valued functions and smooth as it will be necessary. We consider the operator H on its maximal domain, i.e. on the set of all functions $u \in \mathbf{W}_p^n(\mathbb{R}_+)$, $1 \leq p < +\infty$, such that $(H_j D^j)(u) \in \mathbf{W}_p^n(\mathbb{R}_+)$, $j = 0, \ldots, n$, where $\mathbf{W}_p^n(\mathbb{R}_+)$ denotes the Sobolev space of order n over $\mathbb{R}_+$.

Assume that the functions h_j have the representation $h_j(t) = a_j(t) + b_j(t)$ for $t \in \mathbb{R}_+$, $j = 0, \ldots, n$, such that a_j are periodic functions of period T, $a_j(t+T) = a_j(t)$, and suppose that $a_n(t) \equiv 1$. The operator H will be considered as a perturbation of the differential operator $A = \sum_{j=0}^{n} A_j D^j$ by the operator $B = \sum_{j=0}^{n} B_j D^j$, where A_j and B_j, $j = 0, \ldots, n$, are operators acting in $\mathbf{L}_p(\mathbb{R}_+)$ and defined by

$$(A_j u)(t) = a_j(t) u(t) \ , \quad (B_j u)(t) = b_j(t) u(t) + \int_{\mathbb{R}_+} k_j(t,s) u(s) ds.$$

Under above notations, $H = A + B$, where A is a differential operator with periodic coefficients and B is an integro-differential operator.

The problem is to find sufficient conditions on the coefficients b_j and kernels k_j, $j = 1, \ldots, n$, that guarantee that *the point spectrum* (the set of all eigenvalues, including those on the continuous spectrum) of the perturbed operator H *is absent*. To apply perturbation methods from operator theory, we need to have at hand a manageable representation of the resolvent function $(A - \lambda I)^{-1}$ of the unperturbed operator A.

The spectral properties of the operator A have been investigated by many authors (see for instance [17, 18] and the references therein). In [18] the operator A is considered in the space $L_2(\mathbb{R})$, while in [17] in $L_p(\mathbb{R})$, $1 \leq p \leq \infty$. In these papers it is shown that the operator A has a purely continuous spectrum which coincides with the set of those values λ (the zone of relative stability) for which the equation $Au = \lambda u$ has a non trivial solution, bounded on the whole real line. Although the spectrum of the operator H_0 is well-known (see for instance [17, 18]), we will present here a different method for describing explicitly the resolvent of A, which relies on Floquet-Liapunov theory for linear differential equations with periodic coefficients (see for instance [12, 21]).

Without loss of generality we can assume that $T = 1$.

Let us consider the equation

$$A\varphi = \lambda \varphi \,, \tag{2.2}$$

where λ is a complex number, or in vector form

$$\frac{dx}{dt} = A(t, \lambda)\, x\,, \tag{2.3}$$

where

$$A(t, \lambda) = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 & 1 \\ \lambda - A_0 & -A_1 & -A_2 & \dots & -A_{n-2} & -A_{n-1} \end{pmatrix},$$

and $x = (u, Du, \dots, D^{n-1}u)^t$.

Denote by $U(t)$ $(= U(t, \lambda))$ the matriciant of the equation (2.3), i.e. the matrix which satisfies the following system of differential equations

$$\frac{dU(t)}{dt} = A(t, \lambda)\, U(t), \quad U(0) = E_n\,,$$

where E_n is $n \times n$ identity matrix. The matrix $U(1)$ is called the monodromy matrix of the equation (2.3) and the eigenvalues $\rho_1(\lambda), \dots, \rho_m(\lambda)$ of the matrix $U(1)$ are called the multipliers (Floquet multipliers). Also, we will say that $U(1)$ is the monodromy matrix and $\rho_1(\lambda), \dots, \rho_m(\lambda)$ are multipliers of the operator $A - \lambda I$.

Consider the matrix $\Gamma = \ln U(1)$, where Γ is one of the solutions of equation $e^Y = U(1)$. Note that Γ exists since the monodromy matrix is nonsingular. Hence, the matrix $U(t)$ admits the Floquet representation

$$U(t) = F(t)e^{t\Gamma}, \tag{2.4}$$

where $F(t)$ is a nonsingular, differentiable matrix of period $T = 1$. The change of variables $x = F(t)y$ in (2.3) gives

$$\frac{dy}{dt} = \Gamma y, \tag{2.5}$$

where Γ depends on λ only. The solution of the Cauchy equation (2.3) with initial condition $y(0) = y_0$ has the form

$$y(t) = e^{t\Gamma} y_0\ . \tag{2.6}$$

Let us describe explicitly the structure of matrix $\exp(\Gamma t)$. For this, we write the matrix Γ in its Jordan canonical form, $\Gamma = SJS^{-1}$, where $J = \mathrm{diag}[J(1), \ldots, J(m)]$, and $J(\alpha),\ \alpha = 1, \ldots, m$, are the Jordan canonical blocks

$$J(\alpha) = \begin{pmatrix} \lambda_\alpha & 1 & 0 & \ldots & 0 \\ 0 & \lambda_\alpha & 1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & 1 \\ 0 & 0 & 0 & \ldots & \lambda_\alpha \end{pmatrix}.$$

Thus,

$$\exp(Jt) = \mathrm{diag}[t \exp J(1), \ldots, t \exp J(m)]\ , \tag{2.7}$$

with

$$\exp\left(tJ(\alpha)\right) = \exp(t\lambda_\alpha) \begin{pmatrix} 1 & t & \ldots & \frac{t^{p_\alpha - 1}}{(p_\alpha - 1)!} \\ 0 & 1 & \ldots & \frac{t^{p_\alpha - 2}}{(p_\alpha - 2)!} \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & 1 \end{pmatrix},$$

where p_α is the dimension of the Jordan block $J(\alpha),\ \alpha = 1, \ldots, m$.

From (2.5)-(2.7), we conclude that the components of the general solution $y(t)$ of (2.5) are linear combinations of $\exp(\lambda_1 t), \ldots, \exp(\lambda_m t)$ with polynomial coefficients in t.

Note that if $\mathrm{Re}(\lambda) > 0$, then $|t^k \exp(t\lambda)| \to \infty$, for $k = 1, 2, \ldots$, and if $\mathrm{Re}(\lambda) = 0$, then $|t^k e^{t\lambda}| \to \infty$ for $k = 1, 2, \ldots$ and $|t^k e^{t\lambda}| \to 1$ for $k = 0$. By spectral mapping theorem, for each eigenvalue $\lambda_\alpha,\ \alpha = 1, \ldots, m$, of the matrix Γ the corresponding multiplier $\rho_\alpha = \exp(\lambda_\alpha),\ \alpha = 1, \ldots, m$, is in interior, exterior or on the unit circle if $\mathrm{Re}(\lambda_\alpha) < 0$, $\mathrm{Re}(\lambda_\alpha) > 0$, or $\mathrm{Re}(\lambda_\alpha) = 0$.

Remark 2.1. The solution $y(t)$ of equation (2.5) belongs to $\mathbf{L}_p^n(\mathbb{R}_+)$ if the coefficients of the terms $\exp(t\lambda_\alpha)$ with $\mathrm{Re}\lambda_\alpha \geq 0$ are zero. Thus, if we have multipliers inside the unit circle (and only in this case), then the equations (2.3) has a nontrivial solutions in the space $\mathbf{L}_p(\mathbb{R}_+)$, and the inverse operator $(A - \lambda I)^{-1}$ does not exist.

Suppose that λ is such that all corresponding multipliers satisfy the condition $|\rho| \geq 1$. Then the inverse operator (possible unbounded) of $A - \lambda I$ exists, and to describe its structure, we consider the equation $Au - \lambda u = \nu$, where ν is an arbitrary element from $\mathrm{Ran}(A - \lambda I)$. Similarly to (2.3), we write the last equation in its vector-form

$$\frac{dx}{dt} = A(t,\lambda)x + f, \tag{2.8}$$

where $f = (0, \dots, \nu)^t$. The change of variable $x = F(t)y$ in equation (2.8) implies

$$\frac{dy}{dt} = \Gamma y + F^{-1}(t)f. \tag{2.9}$$

The vector-valued function

$$y(t) = -\int_t^\infty \exp(\Gamma(t-s))F^{-1}(s)f(s)ds$$

is the solution of nonhomogeneous equation (2.9), and hence the solution of equation (2.8) has the form

$$x(t) = -F(t)\int_t^\infty \exp(\Gamma(t-s))F^{-1}(s)f(s)ds. \tag{2.10}$$

Taking into account relations (2.4)-(2.7) and representation (2.10) we get

$$\left((A-\lambda I)^{-1}\nu\right)(t) = \sum_{\alpha=1}^{m}\sum_{k=0}^{p_\alpha} g_{\alpha k}(t)\int_t^\infty (t-s)^k \exp(\lambda_\alpha(t-s))h_{\alpha k}(s)\nu(s)ds, \tag{2.11}$$

where $g_{\alpha k}$ and $h_{\alpha k}$ are some continuous and periodic functions, with period $T = 1$.

Remark 2.2. If the unperturbed operator A acts in $\mathbf{L}_p(\mathbb{R})$, then $\lambda \in \sigma(A)$ if and only if there exists at least one multiplier which belongs on the unit circle $\mathbb{T} = \{z \in \mathbb{C} \ : \ |z| = 1\}$. Moreover, the point spectrum of A is absent (for details, see for instance [6]).

3 The main result

In this section will present some general results about the absence of the point spectrum of the perturbed operator $H = A + B$. A natural condition, typical for perturbation methods, is to assume that the perturbation B is subordinated, in some sense, to the unperturbed operator A. In what follows, we assume that $b_n(t) = 0$ and $k_n(t,s) = 0$, for every $t, s \in \mathbb{R}_+$.

By Weyl's type theorem, if the perturbation B is a compact operator, then the essential spectrum of operators H and A coincide. This is true, for example, if the coefficients b_j's decay fast enough to zero, as $t \to \infty$, and the kernels k_j's are completely continuous. However, even if the unperturbed operator A has no eigenvalues, the operator H can have infinitely many eigenvalues, including on continuous spectrum. Some more restrictive conditions on the coefficients and kernels will imply the absence of point spectrum of H.

The following result holds true.

Theorem 3.1. *Let $\rho_\alpha = \rho_\alpha(\lambda)$, $\alpha = 1, \ldots, m$, be the Floquet multipliers corresponding to the operator $A - \lambda I$ such that $|\rho_\alpha| \geq 1$, $\alpha = 1, \ldots, m$. Assume that l is the maximum order of canonical Jordan blocks corresponding to unit multipliers $|\rho_\alpha| = 1$. If there exists $\delta > l$ such that*

$$(1+t)^\delta b_j(t) \in \mathbf{L}_\infty(\mathbb{R}_+), \quad j = 0, \ldots, n\ ,$$

the integral operators with kernels

$$(1+t)^\delta k_j(t,s)\,, \quad \delta > l, j = 0, \ldots, n\,,$$

are bounded in $\mathbf{L}_p(\mathbb{R}_+)$, and

$$k_j(t,s) = 0, \quad t > s,\ j = 0, \ldots, n\,,$$

then λ is not an eigenvalue of the perturbed operator H.

Proof. To simplify the presentation of the proof, we will introduce several auxiliary notations. Denote by $\mathcal{C}$ the Banach space obtained as the direct sum of n copies of $\mathbf{L}_p(\mathbb{R}_+)$, i.e. $\mathcal{C} = \oplus_{j=0}^{n-1} \mathbf{L}_p(\mathbb{R}_+)$. We define the norm in $\mathcal{C}$ as follows $\|\psi\|_{\mathcal{C}} = \sum\limits_{j=0}^{n-1} \|\psi_j\|_{L_p(\mathbb{R}_+)}$, with $\psi := (\psi_j)_{j=0}^{n-1} \in \mathcal{C}$.

Let S de the operator acting on $W_p^n(\mathbb{R}_+)$ with values in $\mathcal{C}$, and defined by

$$Su = (u, Du, \ldots, D^{n-1}u), \quad u \in W_p^n(\mathbb{R}_+)\,.$$

We also consider the following family of operators

$$(T_j u)(t) = b_j(t)u(t) + \int\limits_{\mathbb{R}_+} k_j(t,s)u(s)ds, \quad t \in \mathbb{R}_+, j = 0, \dots, n-1\,,$$

which, obviously, are bounded in $\mathbf{L}_p(\mathbb{R}_+)$. We associate to this family the operator T acting in the space $\mathcal{C}$ and defined by

$$T\psi = \sum_{j=0}^{n-1} T_j \psi_j, \quad \psi = (\psi_j)_{j=0}^{n-1} \in \mathcal{C}\,.$$

Note that $B = TS$ and $H = A + TS$.

For every $\tau \geq 0$, we define

$$(L_\tau x)(t) = (1+t)^\tau x(t), \quad t \in \mathbb{R}_+\,,$$

and for every $p \in [1, \infty)$, we consider the following family of spaces

$$\mathbf{L}_{p,\tau}(\mathbb{R}_+) := \{u \in \mathbf{L}_p(\mathbb{R}_+) \mid L_\tau u \in \mathbf{L}_p(\mathbb{R}_+)\},$$

with corresponding norm $\|u\|_{p,\tau} := \|L_\tau^{-1} u\|$.

Suppose by the contrary, that λ is an eigenvalue of H, i.e. there exists an element $u \in \mathbf{L}_p(\mathbb{R}_+)$, $u \neq 0$, such that

$$Hu = \lambda u\,. \tag{3.1}$$

Taking into account that $H = A + TS$, and since λ cannot be an eigenvalue of A, the equation (3.1) implies

$$Su + S(A - \lambda I)^{-1} TSu = 0\,.$$

We note that $Su \neq 0$, since otherwise the equation (3.1) would imply that $Au = \lambda u$ with $u \neq 0$, that is a contradiction. In what follows we denote $x = Su$. The equation (3.1), written in vector form, implies

$$\frac{dx}{dt} = A(t,\lambda)x + B(t)x, \tag{3.2}$$

where

$$B(t) = \begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & . \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & . \\ -B_0 & -B_1 & \dots & -B_{n-1} \end{pmatrix},\ x = \begin{pmatrix} u \\ Du \\ \dots \\ D^{n-1}u \end{pmatrix}.$$

The change of variables $x = F(t)y$ in (3.2) implies

$$\frac{dy}{dt} = \Gamma y + F^{-1}(t)B(t)y$$

and consequently, we deduce

$$x(t) = -F(t)\int_t^\infty \exp(\Gamma(t-s))F^{-1}(s)B(s)x(s)ds. \tag{3.3}$$

Note that the vector's components from the right hand side of (3.3) are the sums of the following quantities

$$(K_\alpha x_j)(t) = q(t)\int_t^\infty (t-s)^{l_\alpha - 1}\exp(\lambda_\alpha(t-s))p(s)B_{n-j+1}(s)x_j(s)ds\,,$$

where $p(t)$ and $q(t)$ are continuous periodic functions of period 1, l_α takes one of the values $1, \dots, p_\alpha$, and $\alpha = 1, \dots, m,\ j = 0, \dots, n-1$.

To complete the proof will use the following result.

Suppose that operators A and B act in a Banach space $\mathcal{D}$, and assume that:

(i) $\sigma_p(A) = \emptyset$;

(ii) $B = TS$, with S acting from $\mathcal{D}$ into $\mathcal{C}$, and T acting from $\mathcal{C}$ into $\mathcal{D}$, provided that $\mathrm{Dom}(S) \supset \mathrm{Dom}(T)$;

(iii) There exists a family of operators $L_\tau,\ \tau \geq 0$, on $\mathcal{C}$, such that for every $\tau \geq 0$ the operator L_τ is one-to-one, i.e. $\mathrm{Ker}(L_\tau) = 0$. In addition, $L_0 = I_{\mathcal{C}}$ ($I_{\mathcal{C}}$ is the identity operator on the space $\mathcal{C}$).

(iv) There exists $\tau \geq 0$ such that, if $\psi \in \mathcal{C}$ and $T\psi \in \mathrm{Ran}(A - \lambda I)$, then $\psi \in \mathcal{C}_\tau$, $S(A-\lambda I)^{-1}T\psi \in \mathcal{C}_\tau$, and

$$\|S(A-\lambda I)^{-1}T\psi\|_{\mathcal{C},\tau} \leq a\|\psi\|_{\mathcal{C},\tau}, \quad 0 < a < 1\ ,$$

where $|u|_\tau := \|L_\tau u\|_{\mathcal{C}}$ for $u \in D_\tau := \mathrm{Dom}(L_\tau)$;

(v) For every $\psi \in \mathcal{C}_\tau$ such that $T\psi \in \mathrm{Ran}(A - \lambda I)$, the following inequality holds true

$$\|S(A-\lambda I)^{-1}T\psi\|_{\mathcal{C},\tau} \leq c\|\psi\|_{\mathcal{C},\tau'}\ ,$$

where $\tau > \tau' \geq 0$ and c is a positive constant independent of ψ.

Conditions (i)-(v) imply that λ is not an eigenvalue of the perturbed operator $A+B$. For detailed proof see for instance [9].

Following the same notations, we observe that our operators satisfy conditions (i)-(iii). To check the conditions (iv) and (v) we define the operator

$$(R(\lambda)u)(t) = \int_t^\infty \exp(\lambda(t-s))u(s)ds, \quad 0 < t < \infty\,.$$

For all $\tau \geq 0$ and $\mathrm{Re}(\lambda) > 0$, the operator $L_\tau^{-1}R(\lambda)L_\tau$ is bounded in $\mathbf{L}_p(\mathbb{R}_+)$, since $\|L_\tau^{-1}R(\lambda)L_\tau\| \leq (\mathrm{Re}(\lambda))^{-1}$ (see for instance Lemma 1 and 2 from [8]). Moreover, for all $\varepsilon > 0$ we have $a(\tau) = \|L_\tau^{-1}R(\lambda)L_{\tau+\varepsilon}\| \to 0$, when $\tau \to \infty$. If $\mathrm{Re}\lambda = 0$, the operator $L_\tau^{-1}R(\lambda)L_{\tau+1}$ is bounded in $\mathbf{L}_p(\mathbb{R}_+)$, given that $\|L_\tau^{-1}R(\lambda)L_{\tau+1}\| \leq 2$. Note that

$$(R^m(\lambda)u)(t) = (-1)^{m-1}\int_t^\infty (t-s)^{m-1}\exp(\lambda(t-s))u(s)ds\,.$$

Let us estimate the norm $\|(K_\alpha x_k)(t)\|_{p,\tau} = \|L_\tau^{-1}(K_\alpha x_k)(t)\|$. For $\mathrm{Re}(\lambda_\alpha) > 0$, using the assumptions on the functions b_j and kernels $k_j,\ j = 0,\ldots,n$, we obtain the following estimate

$$\|(K_\alpha x_k)(t)\|_{p,\tau} = ca(\tau)\|x_k\|_{p,\tau}. \tag{3.4}$$

For $\mathrm{Re}(\lambda_\alpha) = 0$ the following equalities hold true

$$\begin{aligned}(I-iD)R(1)x &= x,\\ (I-iD)R(\lambda)x &= x + (1-\lambda)R(\lambda)x\,,\end{aligned}$$

where $x \in \mathrm{Dom}(R(\lambda))$. The above, together with initial assumptions, implies (3.4). Hence, (3.4) is satisfied for every λ_α. Consequently, we get

$$\|x\|_{\mathcal{C},\tau} \leq ca(\tau)\|x\|_{\mathcal{C},\tau}. \tag{3.5}$$

Similarly to (3.5), for $\mathrm{Re}(\lambda_\alpha) \geq 0$ we obtain

$$\|(K_\alpha x_k)(t)\|_{p,\tau} = c(\tau)\|x_k\|_{p,\tau'}\ , \tag{3.6}$$

and thus

$$\|x\|_{\mathcal{C},\tau} \leq c(\tau)\|x\|_{\mathcal{C},\tau'} \tag{3.7}$$

where c is a constant, and $\tau > \tau' \geq 0$.

Let $\tau' = \tau - \varepsilon,\ \varepsilon > 0$. From estimate (3.7), it follows that $\|x\|_{\mathcal{C},\varepsilon} < \infty$. Hence, again from (3.7), we get $\|x\|_{\mathcal{C},2\varepsilon} < \infty$ and, in general $\|x\|_{\mathcal{C},n\varepsilon} < \infty$. Since ε can be chosen arbitrarily, we have that $\|x\|_{\mathcal{C},\varepsilon} < \infty$, for every $\tau \geq 0$. However, as we mentioned above, $x \neq 0$, and by (3.6), we get $1 \leq c(\tau)$. This is a contradiction, since $c(\tau) \to 0$, as $\tau \to \infty$. The proof is complete. □

We conclude this section with the case of the whole real line. Using Remark 2.2, by similar arguments as in Theorem 3.1, one can prove the following

Theorem 3.2. *Assume that the operator H acts in the space $L_p(\mathbb{R})$, and $\rho_\alpha = \rho_\alpha(\lambda),\ \alpha = 1, \dots, m$, are all unimodular multipliers. Suppose that l is the maximum value for the orders of canonical Jordan blocks corresponding to the multipliers $\rho_\alpha,\ \alpha = 1, \dots, m$. If $(1+|t|)^\delta b_j(t) \in \mathbf{L}_{\infty,\delta}(\mathbb{R}_+),\ \delta > l, j = 0, \dots, n$, and the kernels $k_j(t,s),\ j = 0, \dots, n$, are such that $k_j(t,s) = 0$ for $|t| > |s|$, and the integral operators with kernels $(1+|t|)^\delta k_j(t,s),\ \delta > l, j = 0, \dots, n$, are bounded in the space $\mathbf{L}_p(\mathbb{R})$, then λ is not an eigenvalue of the operator H.*

4 Application

In this section we will apply the general results from Section 3 to perturbed Hill operator.

In the space $\mathbf{L}_p(\mathbb{R}_+),\ 1 \leq p < \infty$, we consider the following integro-differential operator

$$\begin{aligned}(Hu)(t) &= (D^2u)(t) + p(t)u(t) + b_1(t)(Du)(t) + b_2(t)u(t) + \\ &+ \int_0^\infty k_1(t,s)(Du)(s)ds + \int_0^\infty k_2(t,s)u(s)ds\,, \\ & \quad 0 < t < \infty, u \in W_p^2(\mathbb{R}_+),\end{aligned}$$

where $p(t+1) = p(t),\ b_j(t) \in \mathbf{L}_\infty(\mathbb{R}_+),\ j = 1,2$, and kernels $k_j(t,s) \in \mathbf{L}_\infty(\mathbb{R}_+ \times \mathbb{R}_+),\ j = 1,2$.

The unperturbed operator

$$(Au)(t) = (D^2u)(t) + p(t)u(t)$$

is Hill operator (see for example [11]). It is known (see for instance [18] or [11]) that the multipliers corresponding to $\lambda \in \sigma(A)$ are simple and of modulus 1. Hence, by Theorem 3.1, we have the following result.

Proposition 4.1. *If*

$$(1+t)^{\delta} b_j(t) \in \mathbf{L}_{\infty,\delta}(\mathbb{R}_+), \quad \delta > 1, j = 1, 2;$$

the kernels $k_j(t,s)$, $j = 1, 2$, are such that $k_j(t,s) = 0$ for $t > s$; the integral operators with kernels

$$(1+t)^{\delta} k_j(t,s), \quad \delta > 1; j = 1, 2; \ t, s \in \mathbb{R}_+,$$

are bounded on the space $L_p(\mathbb{R}_+)$, then the inner point of the continuous spectrum of the operator H is not an eigenvalue.

If λ is an end point of the continuous spectrum of H, then there exists only one multiplier equal to 1 or -1 and it has multiplicity two (see [18] or [11]). Thus, based on Theorem 1 we obtain the following statement.

Proposition 4.2. *If $(1+t)^{\delta} b_j(t) \in \mathbf{L}_{\infty}(\mathbb{R}_+)$, $\delta > 2, j = 1, 2$; the kernels $k_j(t,s)$, $j = 1, 2$, are such that $k_j(t,s) = 0$, $t > s$; the integral operators with kernels $(1+t)^{\delta} k_j(t,s)$, $\delta > 2$; $j = 1, 2$; $t, s \in \mathbb{R}_+$, are bounded on the space $L_p(\mathbb{R}_+)$, then the end points of the continuous spectrum of H cannot be eigenvalues.*

Similar results can be proved for the case when operator H is considered on the whole real line $\mathbb{R}$. Some particular cases of integro-differential operators (second and forth order) are discussed in [1] and [2].

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