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FINITENESS OF EIGENVALUES OF SOME INTEGRO-DIFFERENTIAL OPERATORS

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Abstract. In this note, results on finiteness of point spectrum of operators generated by integro-differential expression of arbitrary order are announced.

The problem of finiteness of the point spectrum of a certain class of nonselfadjoint operators has been studied in many works [1–6]. Usually, in such cases, it is used the method of holomorphic extension of resolvent of unperturbed operator through continuous spectrum [2,6,8]. In many works the problem has been reduced to the investigation of analyticity of function (see, for example [1,4]), as well as to solving some integral equations, using Fredholm-type determinants [8].

In the present note the problem is solved by straightforward studying of analyticity of resolvent function. The self-adjoint case has been investigated in [7].

It should be mentioned that the case of the space $L_2(\mathbb{R}_+)$ is considered, but all results obtained here are also true in the case of the whole real axis.

I. Let H be the integro-differential operator defined by

$$H = \sum_{\alpha,\beta=0}^{n} D^{*\alpha} M_{\alpha\beta} D^{\beta}, \tag{1}$$

where

$$(M_{\alpha\beta}u)(x) = a_{\alpha\beta}u(x) + q_{\alpha\beta}(x)u(x) + \int\limits_{\mathbb{R}_+} k_{\alpha\beta}(x,y)u(y)dy \quad (x \in \mathbb{R}_+; \alpha, \beta = 0, \dots, n).$$

Here $a_{\alpha\beta}$ are complex numbers, $a_{\alpha\beta} = \overline{a}_{\beta\alpha}$ $(\alpha, \beta = 0, 1, \dots, n; a_{nn} = 1)$; $q_{\alpha\beta}$ and $k_{\alpha\beta}$ are measurable (generally speaking, complex-valued) functions on $\mathbb{R}_+$ and $\mathbb{R}_+ \times \mathbb{R}_+$ $(\alpha, \beta = 0, 1, \dots, n; q_{nn}(x) = 0, k_{nn}(x,y) = 0, x, y \in \mathbb{R}_+)$ respectively.

It is assumed that the operator H acts in the space $L_2(\mathbb{R}_+)$ and differential operator $D = i\frac{d}{dx}$ is defined on the set of all functions $u \in L_2(\mathbb{R}_+)$ which are absolutely continuous on every bounded interval of the positive semi-axis and whose derivative u' (in the sense of distributions) belongs to $L_2(\mathbb{R}_+)$ and $u(0) = 0$.

Let A be the following differential operator with constant coefficients

$$A = \sum_{\alpha,\beta=0}^{n} a_{\alpha\beta} D^{*\alpha} D^{\beta}.$$

The operator A is self-adjoint and its spectrum $\sigma(A)$ is the set of values of the polinomial

$$A(\xi) = \sum_{\alpha,\beta=0}^{n} a_{\alpha\beta}\xi^{\alpha+\beta}, \quad -\infty < \xi < +\infty.$$

Hence $\sigma(A) = [a, +\infty)$, where $a = min\{A(\xi)|\xi \in \mathbb{R}\}$ and, moreover, the operator A has not eigenvalues.

Operator H may be considered as a perturbation of the operator A by the integro-differential operator B, i.e. $H = A + B$, where

$$B = \sum_{\alpha,\beta}^{n} D^{*\alpha}(q_{\alpha\beta}(x) + K_{\alpha\beta})D^{\beta},$$

$K_{\alpha\beta}$ are the integral operators defined by

$$(K_{\alpha\beta}u)(x) = \int\limits_{\mathbb{R}_+} k_{\alpha\beta}(x,y)u(y)dy \quad (x \in \mathbb{R}_+; \alpha, \beta = 0, 1, \ldots, n).$$

Further, we will provide conditions on the functions $q_{\alpha\beta}$ and the kernels $k_{\alpha\beta}$ $(\alpha, \beta = = 0, 1, \ldots, n)$ for finiteness of the point spectrum $\sigma_p(H)$ of the operator H. Moreover, under these conditions every possible eigenvalue has a finite multiplicity.

Due to Weyl-type theorems, essential spectrum of the operator H coincides with essential spectrum of unperturbed operator A, if the functions $q_{\alpha\beta}$ $(\alpha, \beta = = 0, 1, \ldots, n)$ are approaching zero as $x \to +\infty$ and the integral operators $K_{\alpha\beta}$ $(\alpha, \beta = 0, 1, \ldots, n)$ are compact in the space $L_2(\mathbb{R}_+)$.

Theorem. *Let H be an integro-differential operator as in (1). If the functions $q_{\alpha\beta}$ $(\alpha, \beta = 0, 1, \ldots, n)$ are such that*

$$q_{\alpha\beta}(x)e^{\tau x} \in L_\infty(\mathbb{R}_+), \quad \tau > 0$$

and the integral operators with kernels

$$e^{\tau(x+y)}k_{\alpha\beta}(x,y) \quad (\alpha, \beta = 0, 1, \ldots, n)$$

are bounded on the space $L_2(\mathbb{R}_+)$ for $\tau > 0$, then the point spectrum of the operator H is at most a finite set. Moreover, the possible eigenvalues have finite multiplicity.

The proof of the theorem is based on both the studying of analyticity of an operator valued function and the application of the theorem of operator valued analytic function (see, for instance [9, theorem XII.13] and [10, theorem 5.1]).

II. Examples

1. Let H be an operator of the form

$$(Hu)(x) = -\frac{d^2u}{dx^2} + q_1(x)\frac{du}{dx} + q_0(x)u(x) + \int\limits_{\mathbb{R}_+} k(x,y)u(y)dy, \quad (2)$$

where q_α $(\alpha = 0, 1)$ and k are measurable complex-valued functions. The domain of the operator H is considered maximal in the space $L_2(\mathbb{R}_+)$.

Corollary. *If the functions* $q_\alpha e^{\delta x} \in L_\infty(\mathbb{R}_+)$ $(\alpha = 0, 1; \delta > 0)$ *and if the integral operator with kernel* $e^{\delta(x+y)}k(x, y)$ $(\delta > 0)$ *is bounded on* $L_2(\mathbb{R}_+)$*, then the operator (2) has a finite set of eigenvalues, each of them being of finite multiplicity.*

2. Let L be an operator defined in the space $L_2(\mathbb{R}_+)$ as follows:

$$(Lu)(x) = \frac{d^4u}{dx^4} + p(x)u(x) + \int\limits_{\mathbb{R}_+} k(x, y)\frac{d^4u(y)}{dy^4}dy, \tag{3}$$

where p and k are measurable (generally speaking, complex-valued) functions on $\mathbb{R}_+$ and $\mathbb{R}_+ \times \mathbb{R}_+$ respectively.

Corollary. *If* $p(x)e^{\delta x} \in L_\infty(\mathbb{R}_+)$ $(\delta > 0)$ *and if the integral operator with kernel* $e^{\delta(x+y)}k(x, y)$ $(\delta > 0)$ *is bounded on* $L_2(\mathbb{R}_+)$*, then the operator (3) has at most a finite set of eigenvalues. The possible eigenvalues have finite multiplicity.*

This example is similar to that from [4].

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