MATH 380

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Mathematical Modeling

Real world behavior $\xrightarrow[\text{Simplification}]{\text{Idealization}}$ Model

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Real world behavior $\xrightarrow[\text{Simplification}]{\text{Idealization}}$ Model

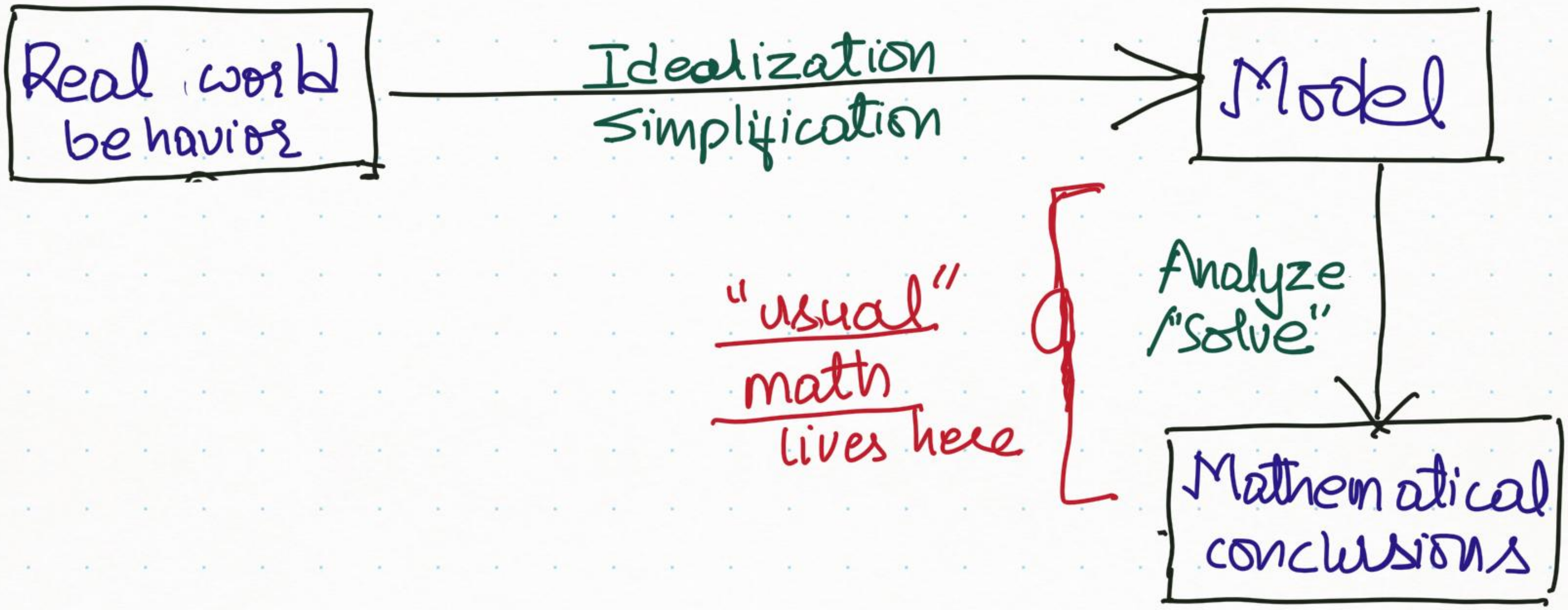
Defining the Problem

Mathematical Formulation

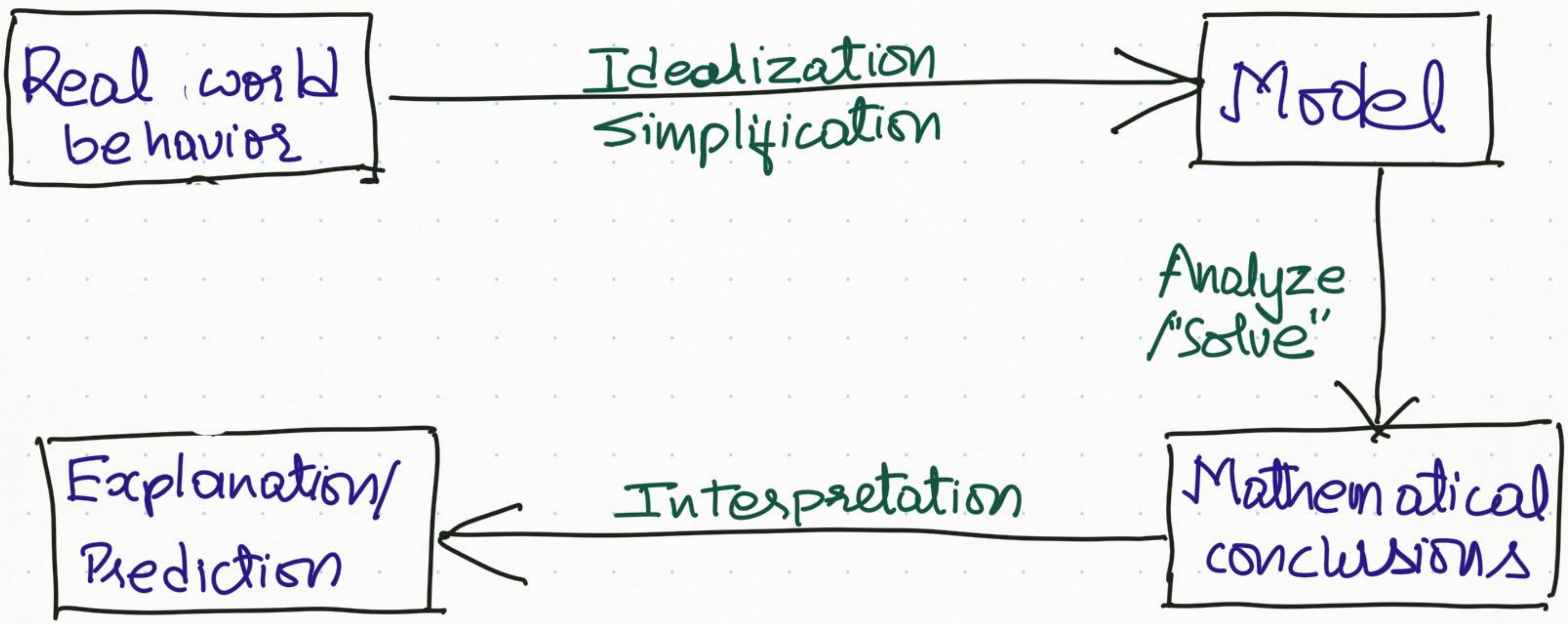
Making assumptions

Defining variables & parameters

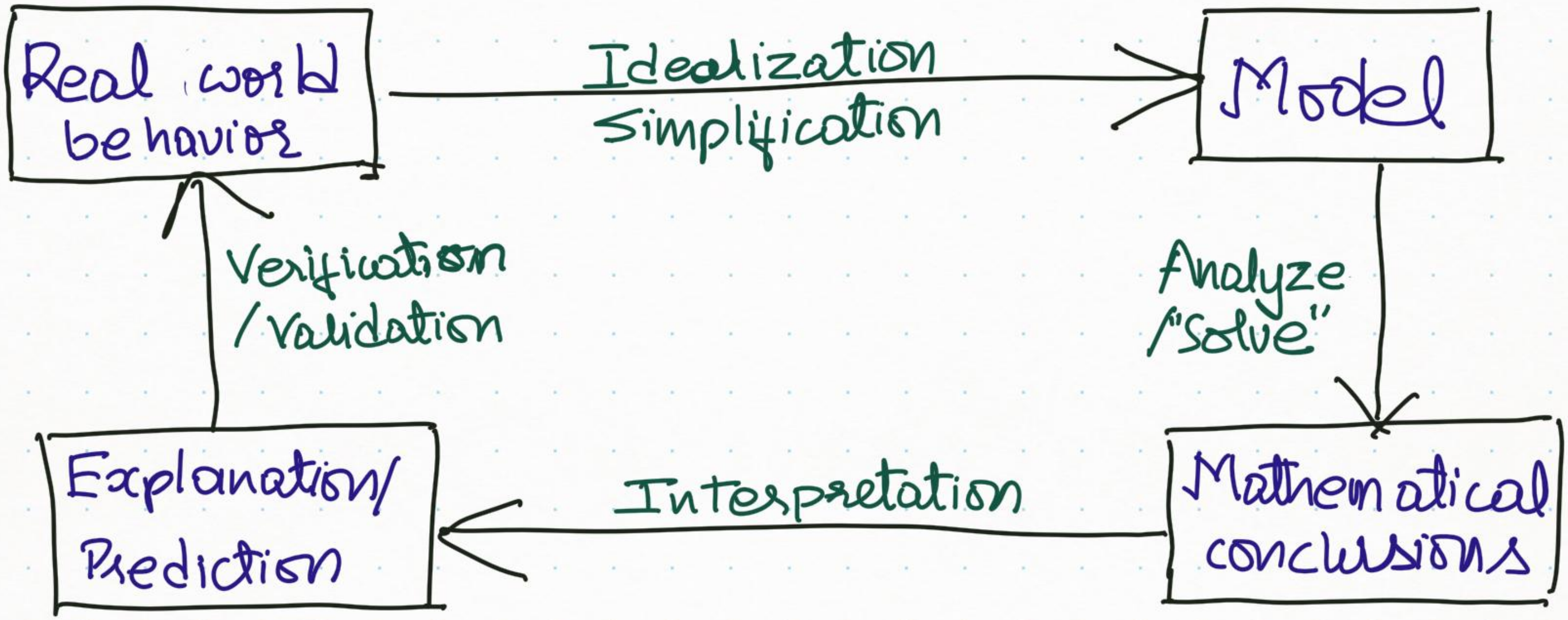
Mathematical Modeling



Mathematical Modeling



Mathematical Modeling



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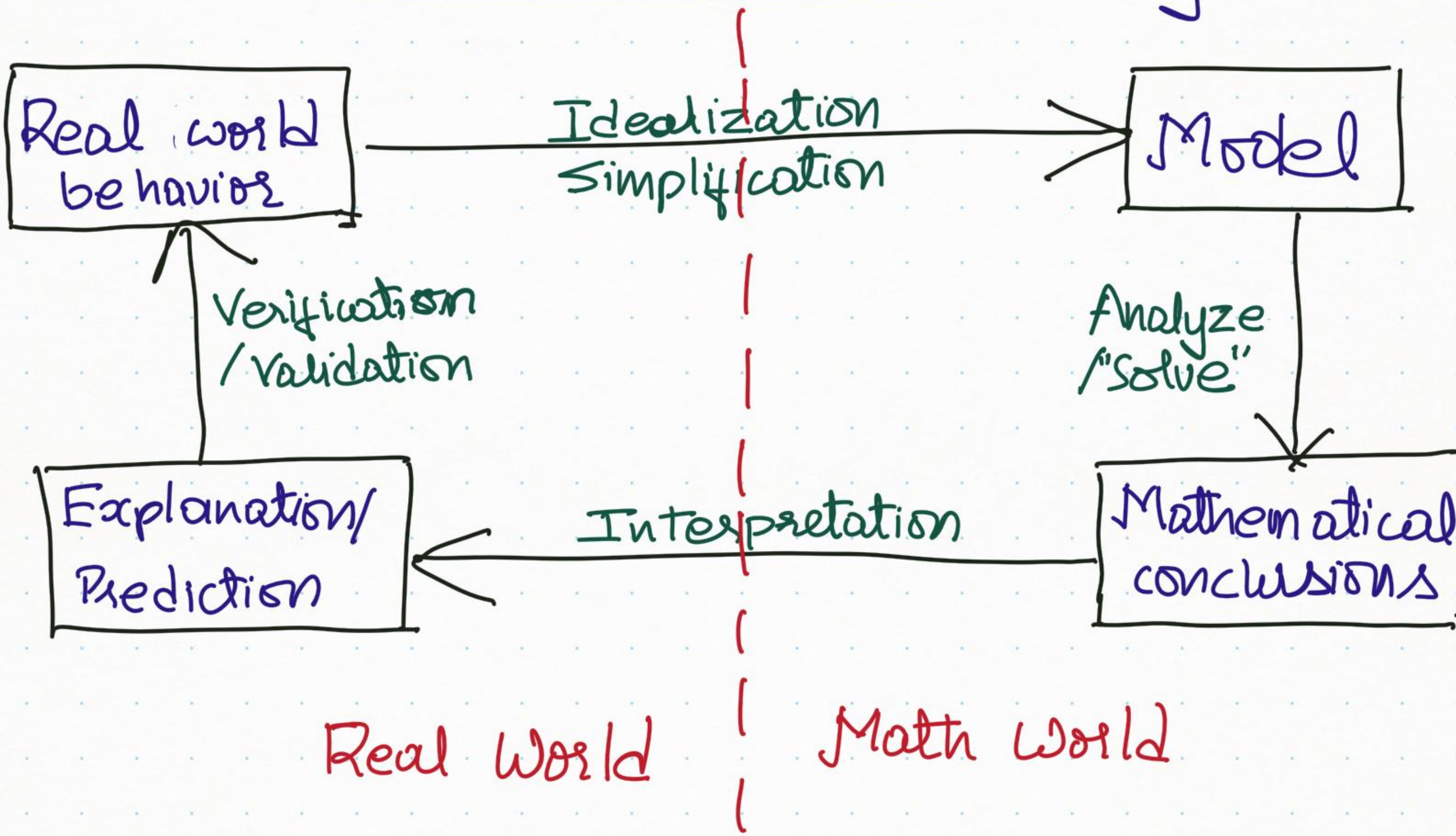

e.g. Throw an apple

"Every model is wrong, but some are useful"

- A good model reveals relationships that may not be apparent superficially
- Mathematical analysis builds strategies / courses of action that are more sophisticated / powerful than a naive approach
- Allows for experimentation (simulation) when its impossible or too expensive in the real world.

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Skills → Mathematical knowledge & expertize
→ Computational experience
→ Effective communication to/in a group

A simple (yet powerful) tool for simplification:

<u>Proportionality</u> Two variables are proportional (to each other) if one is a multiple of the other, i.e.

$\exists\, k \neq 0$ s.t. $y = kx$. We write $y \propto x$.

- What is the (function) graph of y vs. x?

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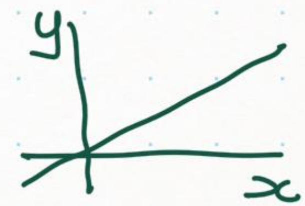
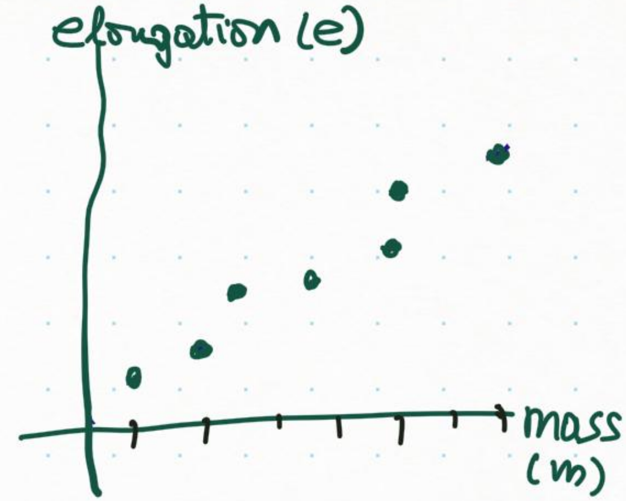


- What is the (function) graph of y vs. x?
- Linear?

e.g.

mass	elongation
50	1.000
100	1.875
150	2.750
200	3.250
250	4.375
⋮	⋮



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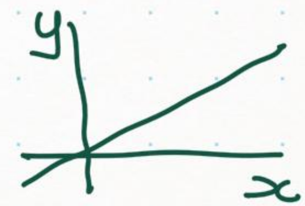

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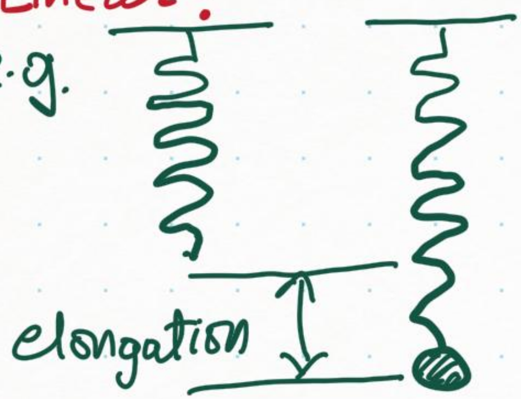

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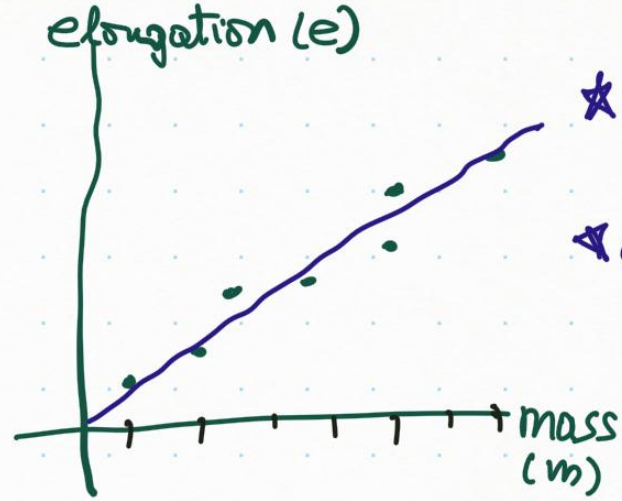


★ avg ratio of e/m

★ avg slope of graph

gives $\frac{e}{m} = 0.017$

Modeling change with Difference Equations

Future value = present value + change

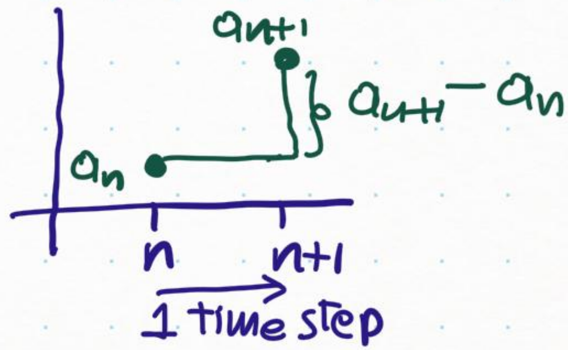
i.e., change = future value − present value

If time is measured in discrete steps: Difference Equation

If time is measured continuously: Differential Equation.

Defn For a sequence of numbers (our data) $A = \{a_0, a_1, \cdots\cdots\}$, the **first order differences** are

$$\Delta a_0 = a_1 - a_0$$
$$\Delta a_1 = a_2 - a_1$$
$$\vdots$$
$$\Delta a_n = a_{n+1} - a_n$$



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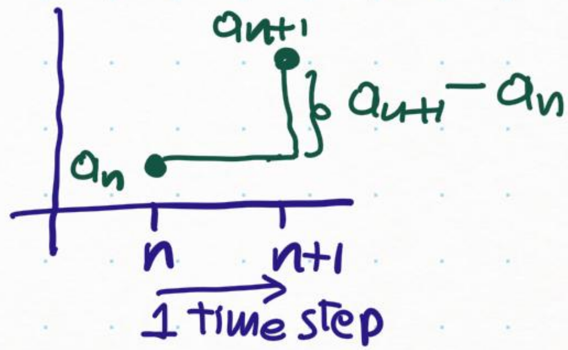

__Example__ a savings certificate with \$1000 initially accumulates interest at rate of 1% per month

$$A = \{1000, ?, ?, ?, \cdots\}$$

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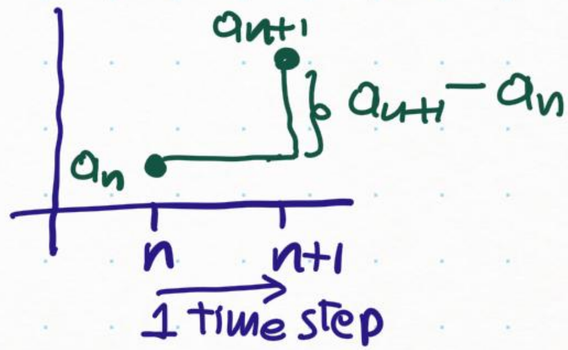

Example a savings certificate with \$1000 initially accumulates interest at rate of 1% per month

$$A = \{1000, 1010, 1020.10, 1030.30, \cdots\cdots\}$$

$$\left.\begin{aligned} \Delta a_0 &= 10 \\ \Delta a_1 &= 10.10 \\ \Delta a_2 &= 10.20 \\ &\vdots \end{aligned}\right\}$$ interest earned in that time period

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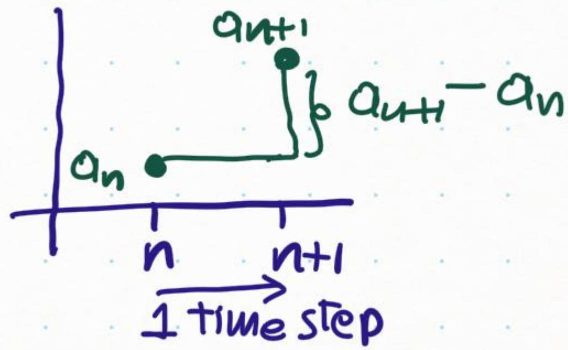

Example a savings certificate with \$1000 initially accumulates interest at rate of 1% per month

$A = \{1000, 1010, 1020.10, 1030.30, \cdots\cdots\}$

$\Delta a_0 = 10$
$\Delta a_1 = 10.10$
$\Delta a_2 = 10.20$
$\vdots$

interest earned in that time period

$\Delta a_n = a_{n+1} - a_n = (0.01)a_n$

i.e., $a_{n+1} = a_n + (0.01)a_n$

i.e., $\boxed{\begin{array}{l} a_{n+1} = (1.01)a_n, \; n \geq 0 \\ a_0 = 1000 \end{array}}$

$$a_{n+1} = (1.01)\, a_n\,, \quad n = 0, 1, 2, \ldots$$
$$a_0 = 1000$$

Dynamical system model

Often Change $= \Delta a_n =$ some function f

$= f$ (terms in sequence, external factors, etc.)

→ Plot change

→ Observe a pattern

→ Describe it mathematically

Discrete time: changes that happen at fixed times

vs.

Continuous time: changes that happen instantaneously

Observ-ations?

Growth of yeast

n	P_n
0	9.6
1	18.3
2	29.0
3	47.2
4	71.1
5	119.1
6	174.6
7	257.3

Observations from a lab

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not linear

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4	71.1	48.0
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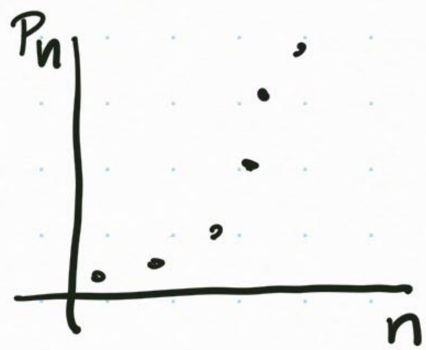

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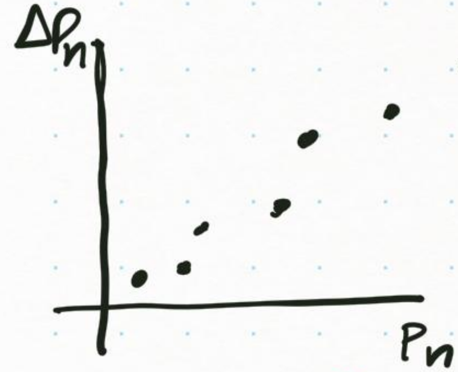

roughly linear

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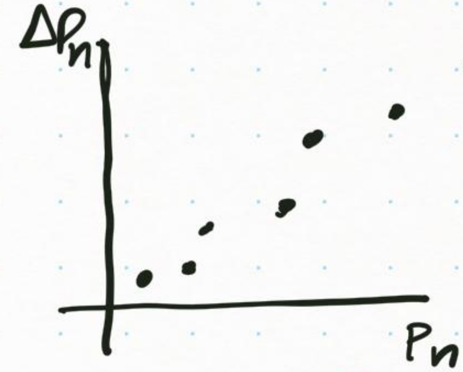

roughly linear

$$\frac{\Delta P_n}{P_n} \cong 0.6057 \quad \left(\text{avg. of } \frac{\Delta P_i}{P_i}\right)$$

i.e., $\Delta P_n = (0.6057) P_n$

i.e., $P_{n+1} - P_n = (0.6057) P_n$

i.e., $\boxed{P_{n+1} = (1.6057) P_n}$

Does the model match reality?

Growth of yeast

n	P_n	$\Delta P_n = P_{n+1} - P_n$
0	9.6	8.7
1	18.3	10.7
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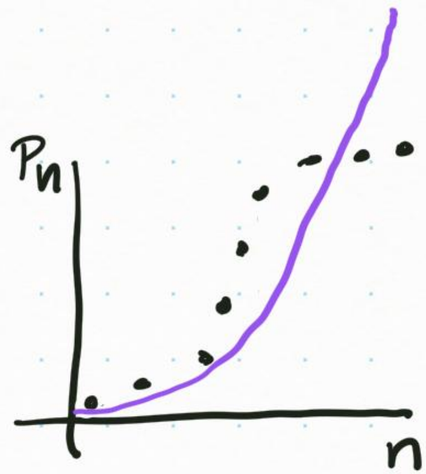
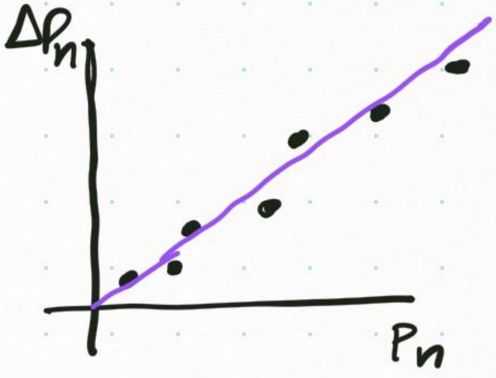




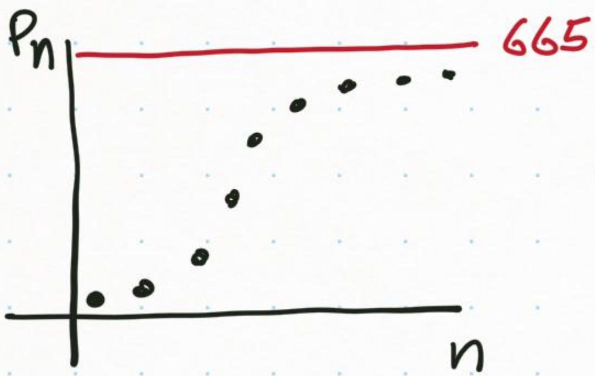
$$\boxed{\text{Model: } P_{n+1} = (1.6057)P_n}$$

Predicted value

How can we improve this model?

A fundamental implicit assumption

Limitation of resources (e.g. food) leads to limits on maximum population that can be supported.



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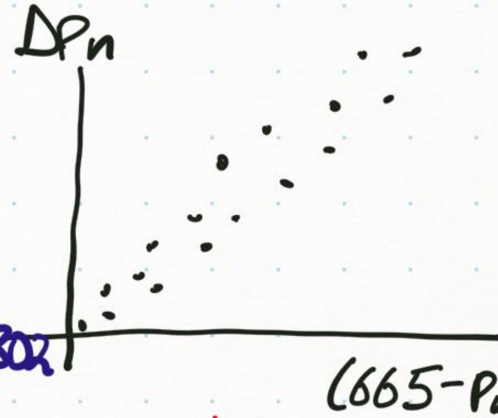


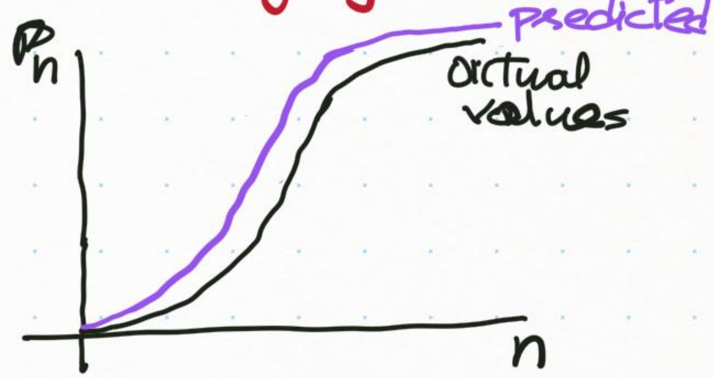
roughly linear

New model $\Delta P_n \propto (665 - P_n)\, P_n$

avg. of $\dfrac{\Delta P_i}{(665-P_i)P_i}$ gives $\dfrac{\Delta P_n}{(665-P_n)P_n} \cong 0.000802$

i.e. $P_{n+1} - P_n = (0.000802)(665 - P_n)P_n$

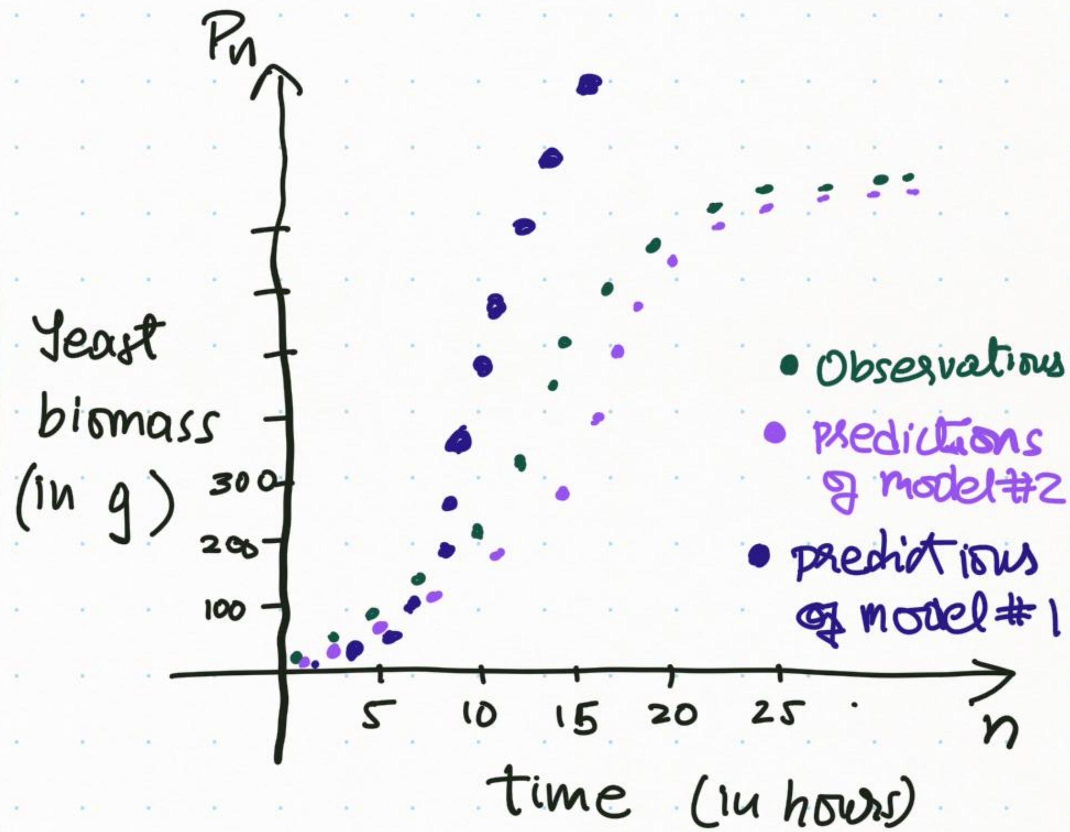
i.e. $P_{n+1} = P_n + (0.000802)(665 - P_n)P_n$
with $P_0 = 9.6$



n	Actual Data	Predicted values	Error
	P_n	$\tilde{P}_n$	$\lvert P_n - \tilde{P}_n \rvert$
0	9.6	9.6	0
1	18.3	14.8	3.5
2	29.0	22.6	6.4
3	47.2	34.5	12.7
4	71.1	52.4	18.7
5	119.1	78.7	40.4
⋮	⋮	⋮	⋮
15	651.1	652.3	1.2
16	655.9	659.1	3.2
17	659.8	662.3	1.5

model#2

$$\boxed{\tilde{P}_{n+1} = \tilde{P}_n + (0.00802)(665 - \tilde{P}_n)\tilde{P}_n \quad \text{with } \tilde{P}_0 = 9.6}$$



State conclusion in words.

Drug Dosage A patient is prescribed 250 mg of a drug every 4 hours. 30% of the drug in the bloodstream is eliminated by the patients body every 4 hours.
How much drug will be in the patient's bloodstream after 72 hours? Long term?

Step 1

Step 2

Step 3

Step 4

Step 5

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Step 1 Identify the problem

Step 2 Assumptions/ simplifications, & variables, etc.

Step 3 Construct the model

Step 4 Solve & interpret the model

Step 5 Validate the prediction vs. real data/observations

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Determine the relationship between the amount of drug in the bloodstream and time.

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Step 2 Variables, assumptions / simplifications

time?

drug?

assumptions?

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Determine the relationship between the amount of drug in the bloodstream and time.

Step 2 Variables, assumptions/simplifications

time? n = number of 4-hour time periods. 0, 1, 2, ...

drug? $a(n)$ = amount of drug in the bloodstream after period n, $n = 0, 1, 2, \ldots$

assumptions?

Step 1 Identify the problem

Determine the relationship between the amount of drug in the bloodstream and time.

Step 2 Variables, assumptions/simplifications

time? n = number of 4-hour time periods. 0, 1, 2, ...

drug? $a(n)$ = amount of drug in the bloodstream after period n, $n = 0, 1, 2, \ldots$

assumptions?
- patient does not have any abnormalities
- no other drugs/interactions in the bloodstream
- no internal/external factors that affect drug absorption
- patient takes the drug at the correct time with correct dosage
  - drug is immediately ingested into the bloodstream

Step3 The Model

Change = dose − loss from the system

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$$\Delta a(n) = 250 - (0.3)\,a(n)$$

i.e., $a(n+1) - a(n) = 250 - (0.3)\,a(n)$

i.e., $\boxed{a(n+1) = (0.7)\,a(n) + 250}$

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Step4 Solve the model & interpret

It can be solved exactly as: $\boxed{a(n) = \frac{2500}{3} - \frac{2500}{3}(0.7)^n}$

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After 72 hours:

Long term:

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It can be solved exactly as: $\boxed{a(n) = \frac{2500}{3} - \frac{2500}{3}(0.7)^n}$

After 72 hours: $a(18) = 831.98\,mg$

Long term: $\lim_{n \to \infty} a(n) = \frac{2500}{3} = 833.33\,mg$

Is this acceptable?