MATH 380

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Modeling with a Differential Equation

Recall the Discrete Difference equation: $\boxed{\Delta P_n = P_{n+1} - P_n = f(n)}$ ⊛

where $n \leftrightarrow$ "time"

$f(n) \leftrightarrow$ "change in P in time period n"

We think of P_n as "observation" made at "time $= n$"

Let rewrite ⊛ by thinking of time as $t = t_0, t_1, t_2, \cdots,$
with $t_{n+1} - t_n = \underbrace{\Delta t}_{\text{time gap between observ.}}$ $\forall n$

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We let $P \equiv P(t)$, a "continuous" process of change
with $P(t_n) = P_n$, "observations" at $t = t_n$, $n = 0, 1, \ldots,$ with
$t_{n+1} = t_n + \Delta t$

Then $\Delta P = P(t_{n+1}) - P(t_n) = P(t_n + \Delta t) - P(t_n)$

so, $$\frac{\Delta P}{\Delta t} = \frac{P(t+\Delta t) - P(t)}{\Delta t}$$

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so, $\dfrac{\Delta P}{\Delta t} = \dfrac{P(t+\Delta t) - P(t)}{\Delta t}$ if $P(t)$ is a "continuous" process
or if Δt is very small
or if <u>its easier to analyze than the discrete system</u> ← CALCULUS!

The corresponding "continuous" model becomes:

$$\lim_{\Delta t \to 0} \frac{\Delta P}{\Delta t} = \lim_{\Delta t \to 0} \frac{P(t+\Delta t) - P(t)}{\Delta t} = \frac{dP}{dt}$$ Derivative of P w.r.t. t

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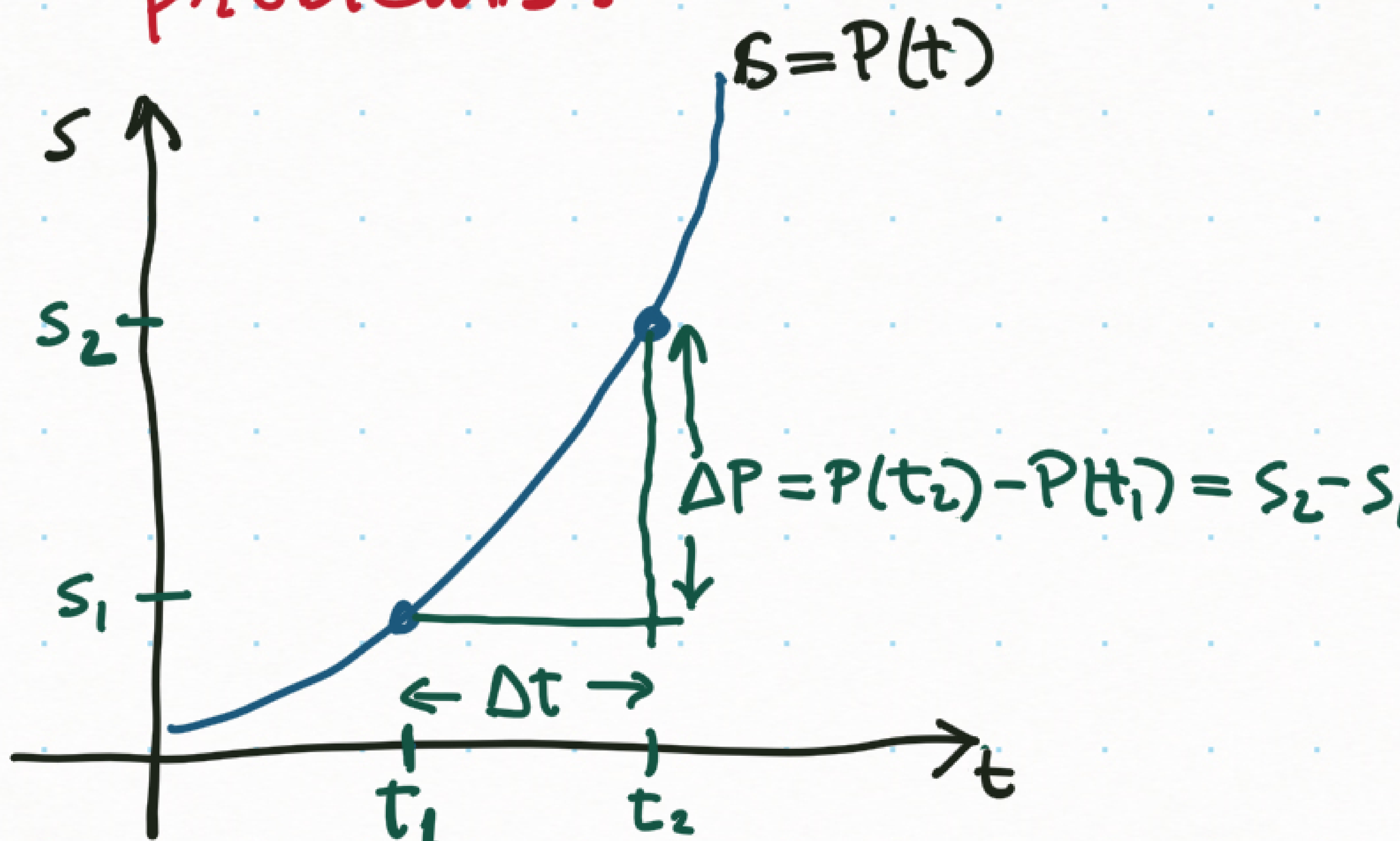
Derivative is useful to

→ represent instantaneous change in "continuous" problems

→ approximate average rate of change in "discrete" problems.



As $t_2 \to t_1$, i.e. $\Delta t \to 0$, we get $\frac{dP}{dt}$, the slope of the tangent to the curve $s = P(t)$ at $t = t_1$.

Malthusian Population Growth Model

Recall the original yeast growth model $\Delta P_n \propto P_n$

Change in population is proportional to current population

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Given $P_0 = P(t_0)$, initial population

Discrete version: $$P_{n+1} - P_n = bP_n - dP_n$$

i.e. $$\Delta P_n = (b-d) P_n$$

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Given $P_0 = P(t_0)$, initial population

Continuous version: $\frac{dP}{dt} \propto P$, i.e. $\frac{dP}{dt} = kP$ for $k = b - d$ "effective" pop. growth rate

underlying assumptions?

Let's solve this!

$$\frac{dP}{dt} = kP, \quad P(t_0) = P_0, \quad t_0 \leq t \leq t_1$$

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$\frac{dP}{dt} = kP$, seperation of variables gives

$$\int \frac{dP}{P} = \int k\,dt, \quad \text{i.e. } \ln|P| = kt + C$$

what to do?

why? How?

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what to do? why? How?

we have, $\ln P = kt + C$

i.e., $P = e^{kt+C} = (e^c)(e^k)^t$

i.e., $P(t) = A\, r^t$ for some constants $A, r > 0$ ($r = e^k$)

Using the initial value, $P(t_0) = P_0$, we get

$$P_0 = P(t_0) = A\, r^{t_0}, \quad \text{i.e., } A = P_0\, r^{-t_0}$$

$$\therefore \boxed{\begin{array}{l} P(t) = P_0\, r^{(t-t_0)} \\ \text{or } P(t) = P_0\, e^{k(t-t_0)} \quad \text{for } t \ge t_0 \end{array}}$$

Verify?

Fit to Data?

Fit to Data?

Given population data (t_i, P_i), $i = 0, 1, \ldots, m$
we seek "best" k (or r) based CAC or LSC
for the transformed model $\ln P = \ln P_0 + k(t - t_0)$

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for the Transformed model $\ln P = \ln P_0 + k(t - t_0)$

Once we have the best model(s) of this type, we can look at the "total error" based on CAC & LSC (or even use R^2) and also just a simple "sanity check" by plotting the data vs. the model

Change the model?

Different/better assumptions?

Logistic Growth Model (Pop. growth under limited resources)

$\Delta P_n \propto P_n(M - P_n)$ where $M \equiv$ max value for P

numerical solution (we saw it used for pop. growth, spread of disease, spread of rumors, etc.

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$$\boxed{\frac{dP}{dt} \propto P(M-P)}$$, continuous model

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$$\frac{dP}{dt} = r\,P(M-P), \text{ ie., } \int \frac{dP}{P(M-P)} = \int r\,dt$$

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$$\text{i.e., } \int \left(\frac{1/M}{P} + \frac{1/M}{M-P}\right) dP = \int r\,dt$$

$$\text{i.e., } \frac{1}{M}\left(\ln|P| + \ln|M-P|\right) = rt + c$$

$$\text{i.e., } \ln\left|\frac{P}{M-P}\right| = Mrt + C$$

Partial Fractions

$$\frac{1}{P(M-P)} = \frac{A}{P} + \frac{B}{M-P}$$

$$= \frac{AM - AP + BP}{P(M-P)}$$

means $1 = (B-A)P + AM$

i.e., $AM = 1$ & $B-A=0$

i.e. $B = A = 1/M$

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Given data (t_i, P_i), $i = 0, 1, \ldots, m$
we can use CAC or LSC to find
the best a and b for the model
$y = aX + b$ with transformed
data $\left(Mt_i, \ln\left|\frac{P_i}{M-P_i}\right|\right)$, $i = 1, \ldots, m$,
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We can also find C explicitly with some algebra by plugging in the initial condition $P(t_0)=P_0$ into $C = \ln\left|\frac{P}{M-P}\right| - Mrt$, i.e., $C = \ln\left|\frac{P_0}{M-P_0}\right| - Mrt_0$

& now use in the solution above

For $0 < P < M$, $\boxed{P(t) = \dfrac{MP_0}{P_0 + (M - P_0)e^{-kM(t - t_0)}}}$

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How to estimate M?

We have $P' = r(M-P)P$, so $P'' = rMP' - 2rPP'$
$= rP'(M - 2P)$

that is, $P'' = 0$ when $P = M/2$

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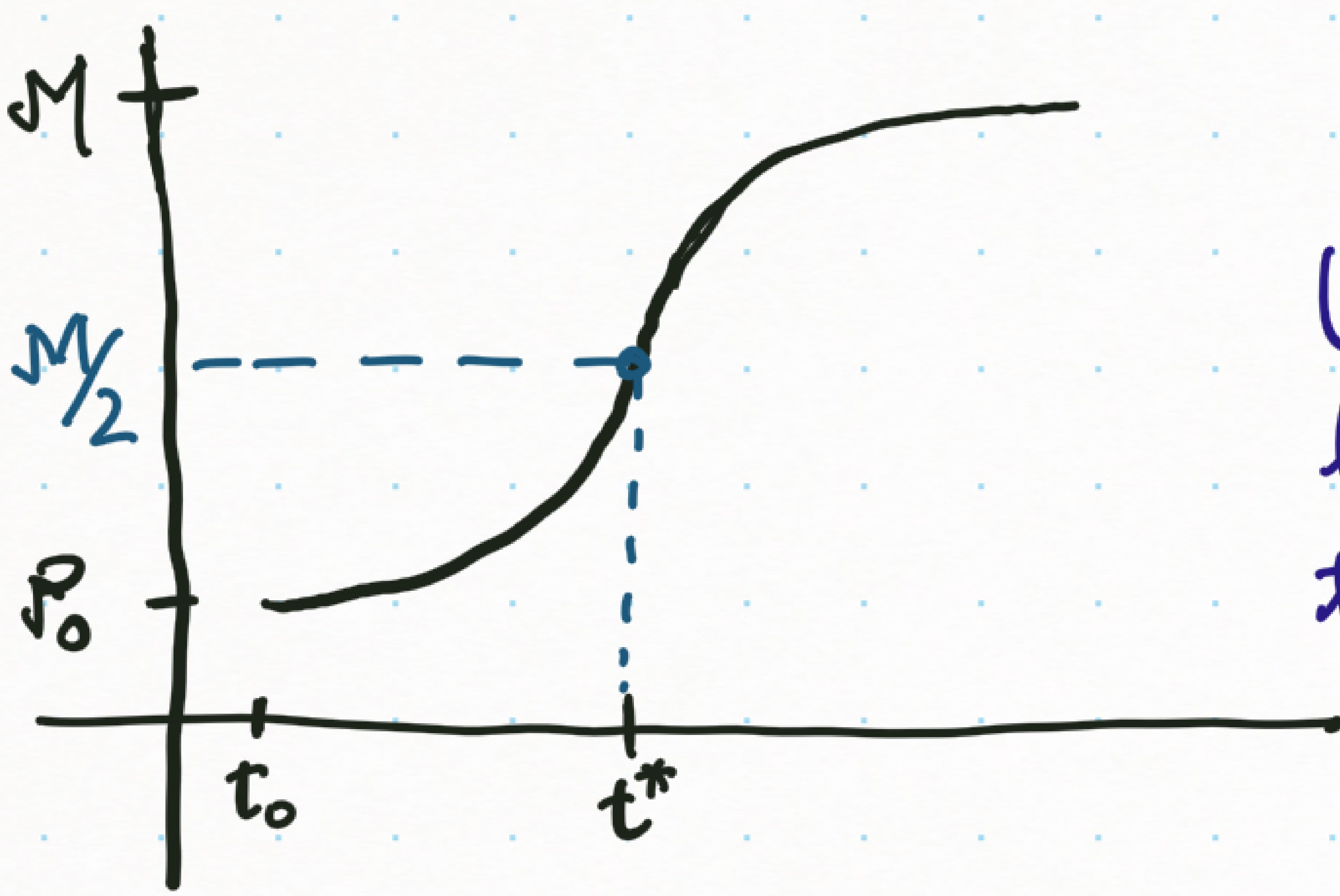

Under the assumption that growth is logistic, we can estimate $M/2$ as the population when rate of growth is maximum (point at which P'' changes from >0 to <0).

Look at Table 11.1 for the "Growth of yeast" example and how well the data fits the logistic growth model.

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How about Human Population growth?

→ HW problem to use logistic growth model to estimate US population growth.

What are the shortcomings of this model for this task?

Missing factors?

Understanding behavior of a Differential Eqn. Model

Quantitatively, we can find numerical approximate solutions —

ODE $\frac{dy}{dx} = g(x, y)$ ⟵ Think of y as function of x ("x ⟵ time")

starting at point (x_0, y_0), can be approximated as

$$\frac{y(x_0+\Delta x) - y(x_0)}{\Delta x} \approx g(x_0, y_0)$$

i.e., $y(x_0+\Delta x) - y(x_0) \approx (\Delta x)(g(x_0, y_0))$

i.e., $y(x_0+\Delta x) \approx y(x_0) + (\Delta x)\, g(x_0, y_0)$

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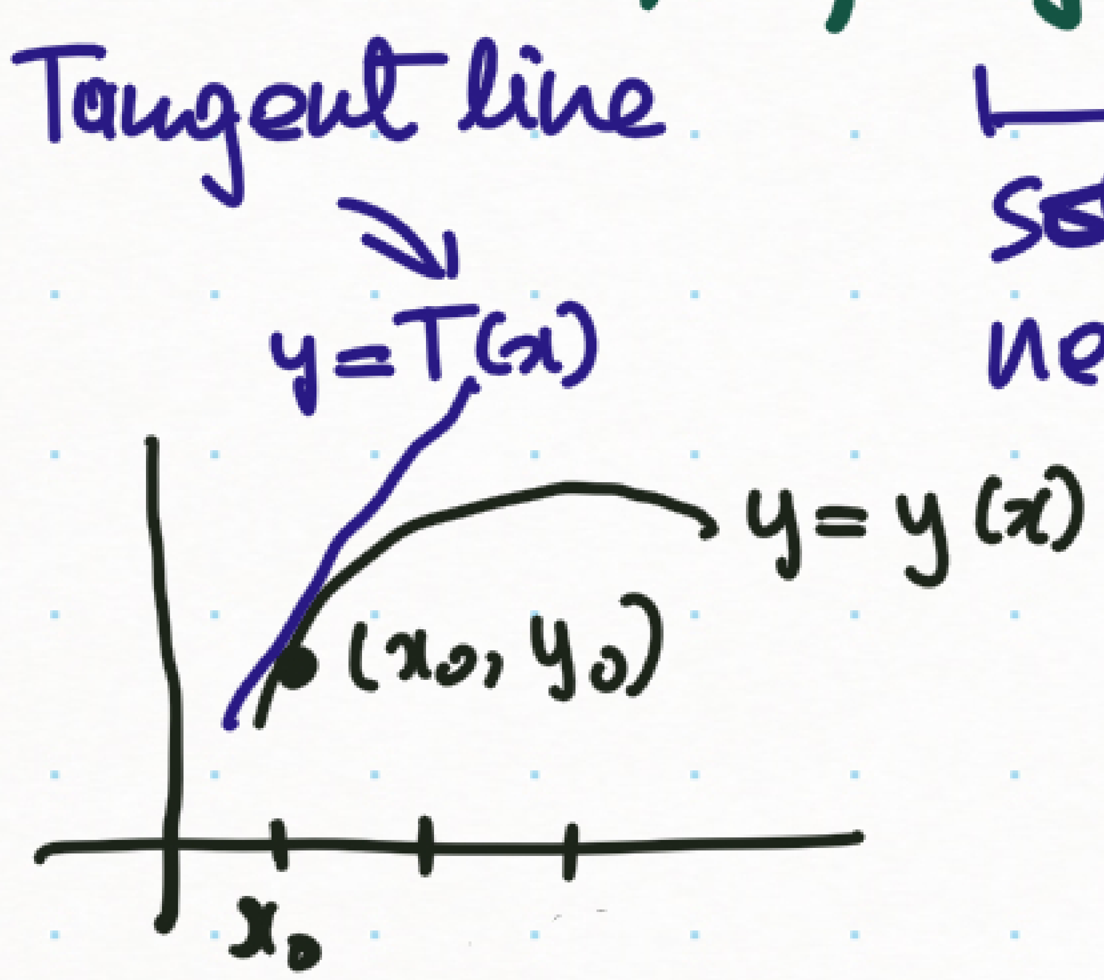

$y = y_0 + (x - x_0)\, g(x_0, y_0)$

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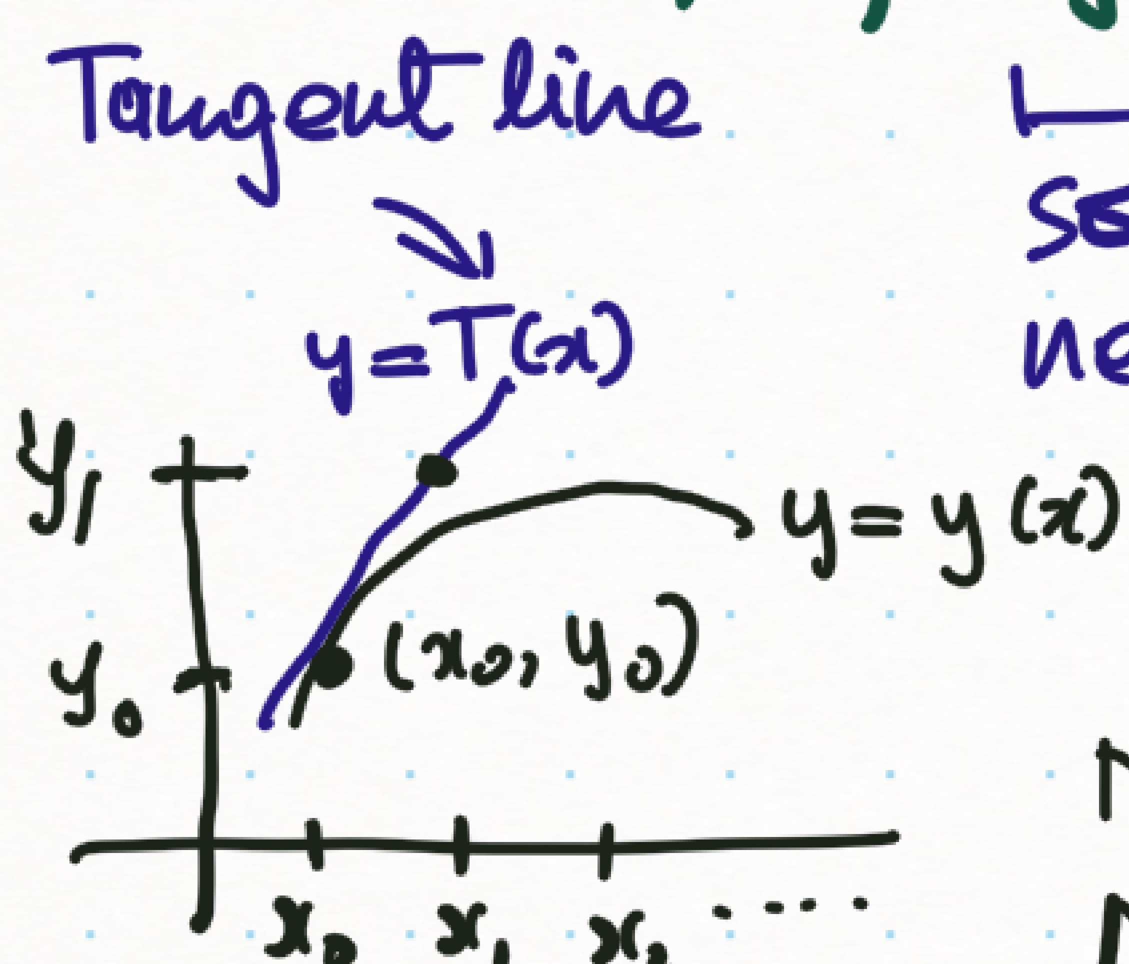

$y = y_0 + (x - x_0)\, g(x_0, y_0)$

Next step $x_1 = x_0 + \Delta x$: $y_1 = y_0 + \Delta x\, g(x_0, y_0)$

Next step $x_2 = x_1 + \Delta x$: $y_2 = y_1 + \Delta x\, g(x_1, y_1)$ & so on.

Understanding behavior of a Differential Eqn. Model

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$$\frac{y(x_0+\Delta x) - y(x_0)}{\Delta x} \approx g(x_0, y_0)$$

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Approximate the solution at each step by the Tangent line at that pt.

$$\boxed{\begin{array}{l} y(x_n) \approx y(x_{n-1}) + h\, g(x_{n-1}, y(x_{n-1})) \\ n \equiv \text{step } \#n,\ n = 0, 1, 2, \dots, \quad \text{and } h \equiv \text{step-size} \end{array}}$$

"Euler's Method"

e.g. $\frac{dy}{dx} = 1 + y$, $y(0) = 1$

To Find approximate solutions :

$x_0 = 0$, $y_0 = 1$

Choose step size $h = \Delta x = 0.1$

Then $x_1 = 0.1$, $x_2 = 0.2$, $x_3 = 0.3$, ...

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Choose step size $h = \Delta x = 0.1$

Then $x_1 = 0.1$, $x_2 = 0.2$, $x_3 = 0.3$, ...

approx. solution at $x = 0.1$

$$y_1 = y_0 + g(x_0, y_0)\,\Delta x$$
$$= y_0 + (1 + y_0)\,\Delta x = 1 + (1+1)(0.1) = 1.2$$

at $x = 0.2$

$$y_2 = y_1 + g(x_1, y_1)\,\Delta x$$
$$= y_1 + (1 + y_1)\,\Delta x = 1.2 + (1+1.2)(0.1) = 1.42$$

at $x = 0.3$

$$y_3 = y_2 + g(x_2, y_2)\,\Delta x$$
$$= y_2 + (1 + y_2)\,\Delta x = 1.42 + (1+1.42)(0.1) = 1.662$$

$\vdots$

Understanding behavior of a Differential Eqn. model

$$\frac{dy}{dx} = g(x,y)$$

→ Euler's method for approximate solution

→ Graphical visualization of the solution curve

Given initial condition $y(x_0)=y_0$ of $y'=g(x,y)$, the solution curve has to pass through (x_0, y_0) and have slope $g(x_0,y_0)$ there.

This can be visualized as short line segments of slope $g(x,y)$ at point (x,y)

Understanding behavior of a Differential Eqn. model

$$\frac{dy}{dx} = g(x,y)$$

→ Analze solutions around "equilibrium values" using a "phase line".

Defn Autonomous differential equation is a DE of form $\frac{dy}{dx} = g(y)$
(e.g. $\frac{dP}{dt} = kP$ or $kP(M-P)$)

For such a DE $\frac{dy}{dx} = g(y)$,
the values of y for which $\underbrace{\frac{dy}{dx} = 0}_{\text{points where } y \text{ is at "rest"/not changing}}$ are called **equilibrium values**

Phase line shows the DE's equillibrium values and the behavior of $\frac{dy}{dx}$ & $\frac{d^2y}{dx^2}$ around them on the y-axis.

e.g. $\frac{dy}{dx} = (y+1)(y-2)$

1. $\frac{dy}{dx} = 0 \iff (y+1)(y-2) = 0 \iff y = -1$ or 2

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2. Establish where $y' > 0$ & $y' < 0$

$y' = (y+1)(y-2)$ means

y' ? when $y < -1$

y' ? when $-1 < y < 2$

y' ? when $y > 2$

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$y' = (y+1)(y-2)$ means $(-ve)(-ve) = +ve$ when $y < -1$

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$(+ve)(+ve) = +ve$ when $y > 2$

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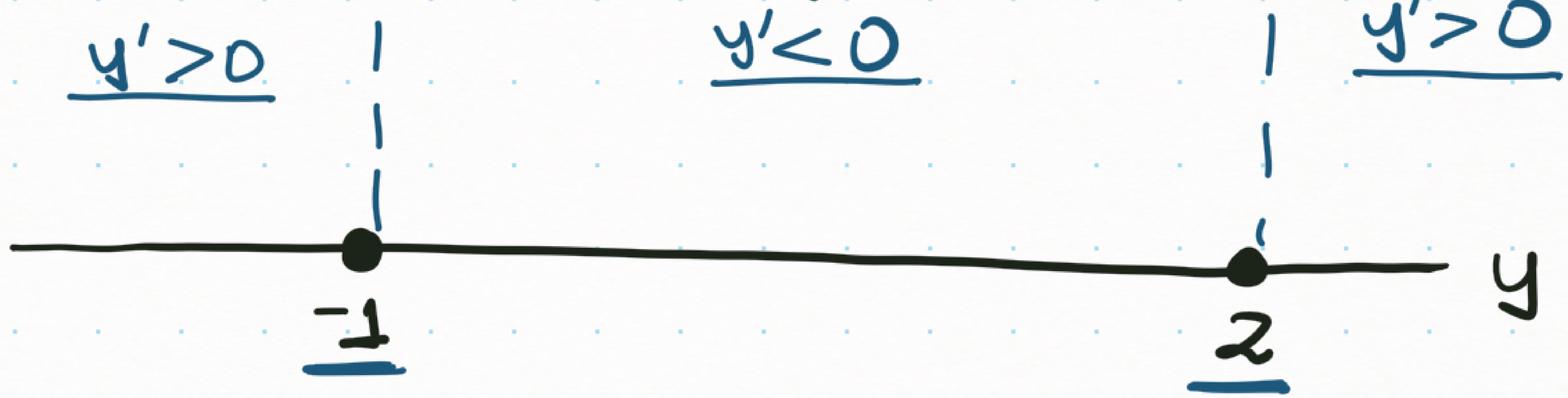

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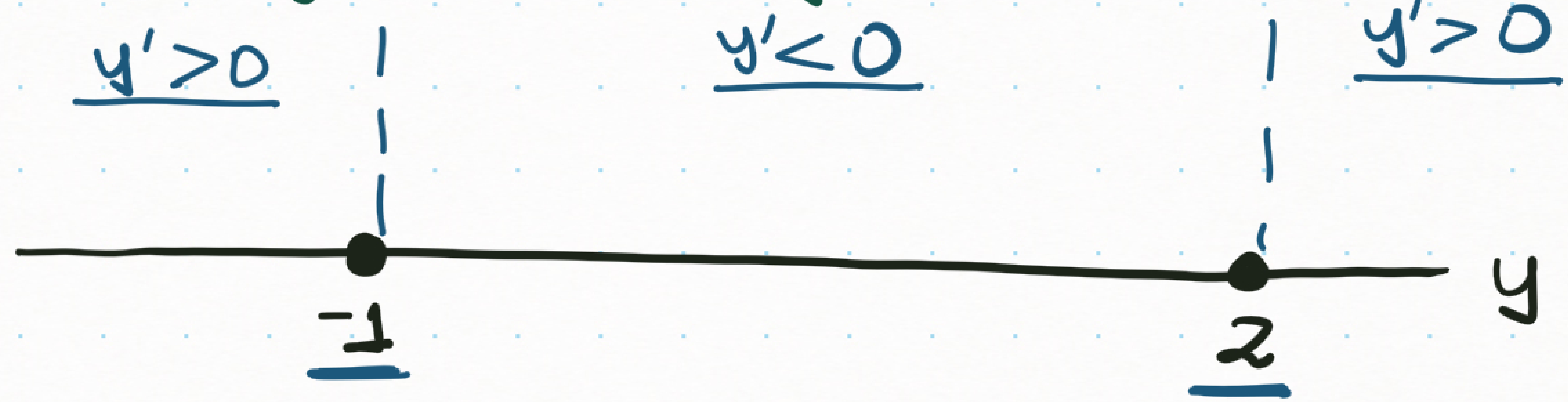

2. Establish where $y'>0$ & $y'<0$

$y' = (y+1)(y-2)$ means $(-ve)(-ve) = +ve$ when $y<-1$

$(+ve)(-ve) = -ve$ when $-1<y<2$

$(+ve)(+ve) = +ve$ when $y>2$

3. Establish where $y''>0$ & $y''<0$

$y'' = \frac{d}{dx}(y') = \frac{d}{dx}(y^2-y-2) = 2yy' - y' = (2y-1)y' = (2y-1)(y+1)(y-2)$

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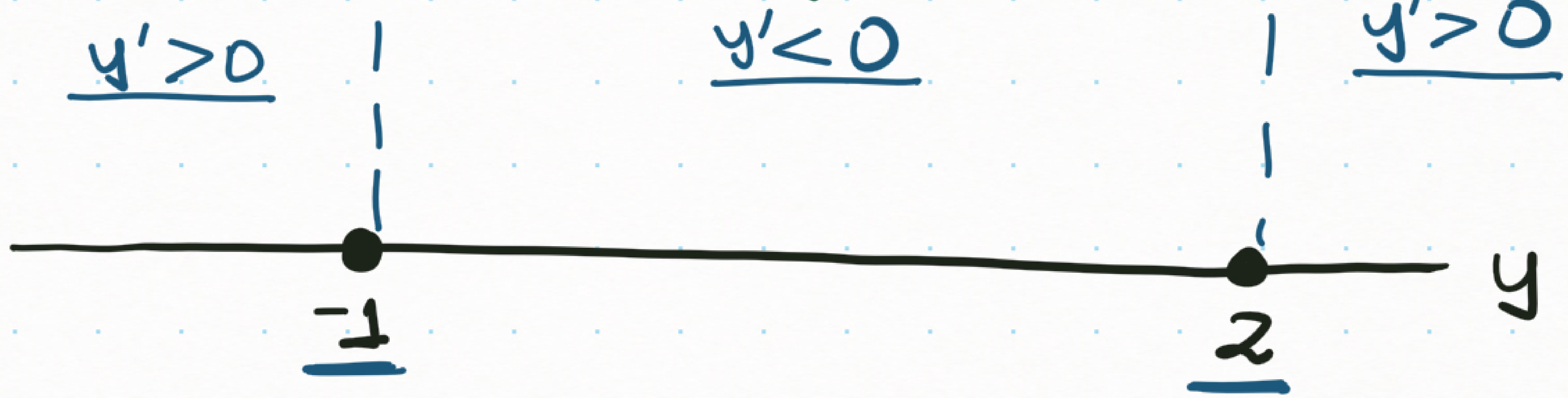

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3. Establish where $y'' > 0$ & $y'' < 0$

$y'' = (2y-1)(y+1)(y-2)$

zero at $-1, \frac{1}{2}, 2$; so sign changes there.

$(-ve)(-ve)(-ve) = (-ve)$ when $y < -1$
$(+ve)(-ve)(-ve) = (+ve)$ when $-1 < y < \frac{1}{2}$
$(+ve)(+ve)(-ve) = (-ve)$ when $\frac{1}{2} < y < 2$
$(+ve)(+ve)(+ve) = (+ve)$ when $y > 2$

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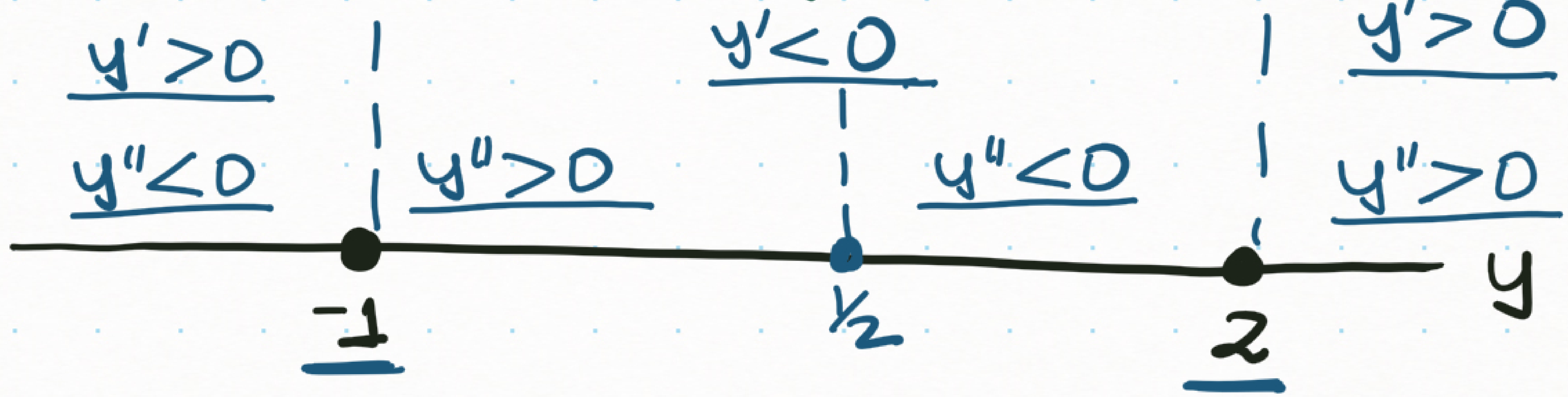

2. Establish where $y' > 0$ & $y' < 0$

3. Establish where $y'' > 0$ & $y'' < 0$

$y'' = (2y-1)(y+1)(y-2)$

zero at $-1, \frac{1}{2}, 2$; so sign changes there.

(−ve)(−ve)(−ve) = (−ve) when $y < -1$
(+ve)(−ve)(−ve) = (+ve) when $-1 < y < \frac{1}{2}$
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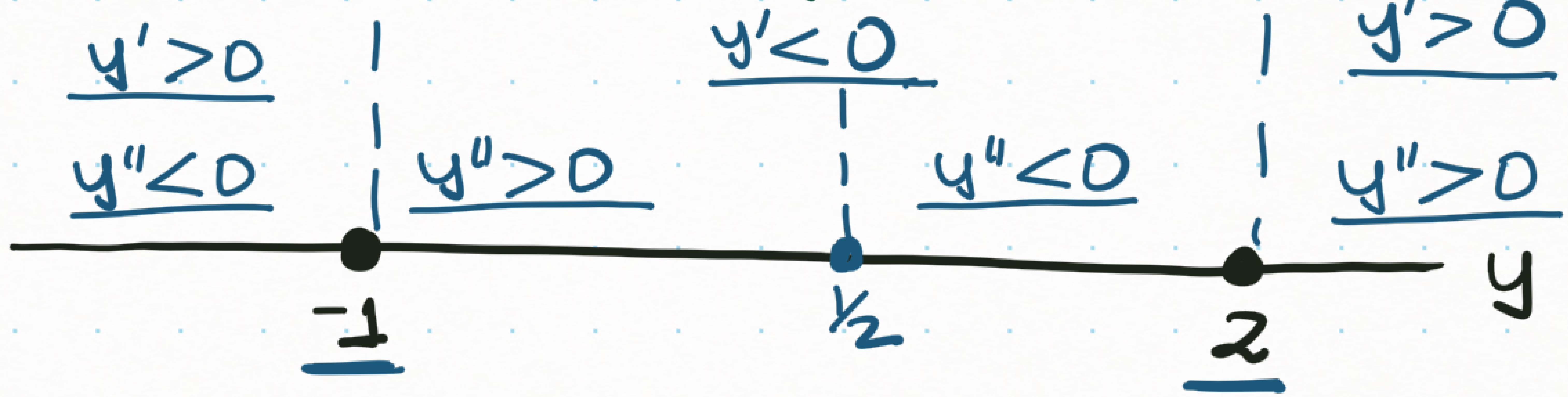

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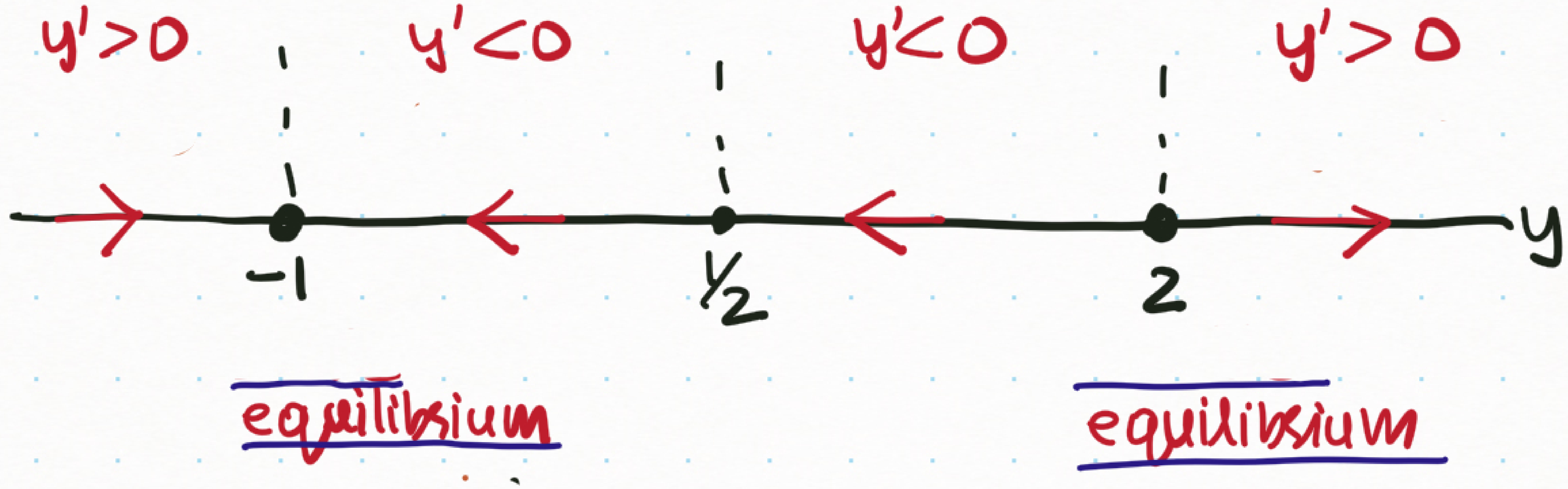
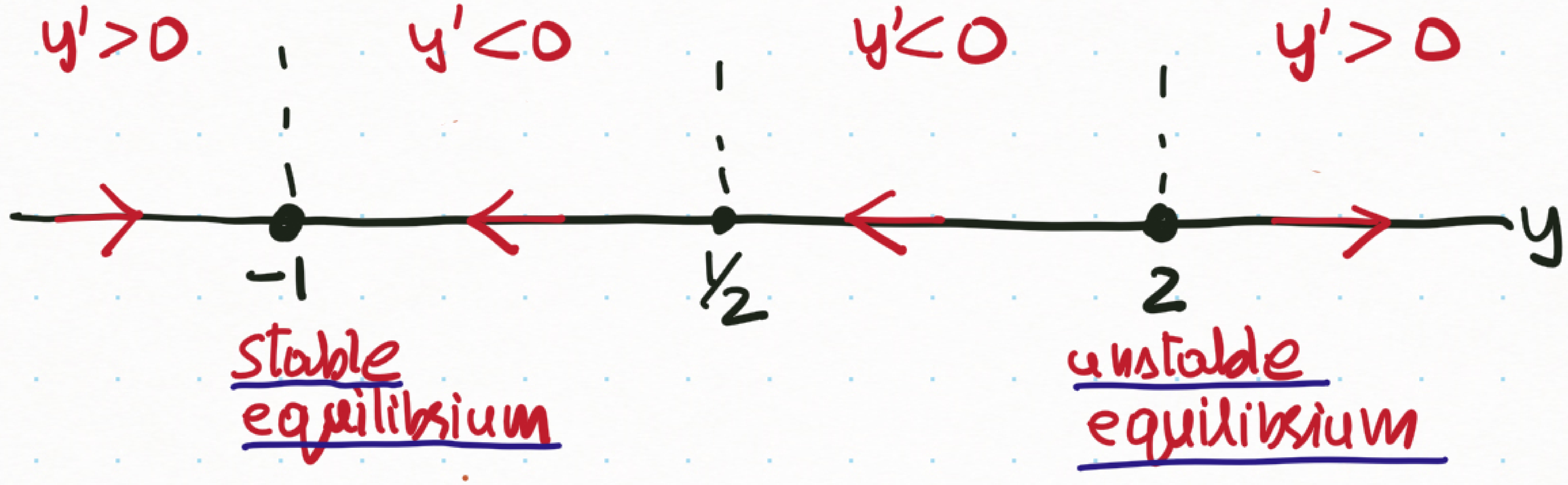
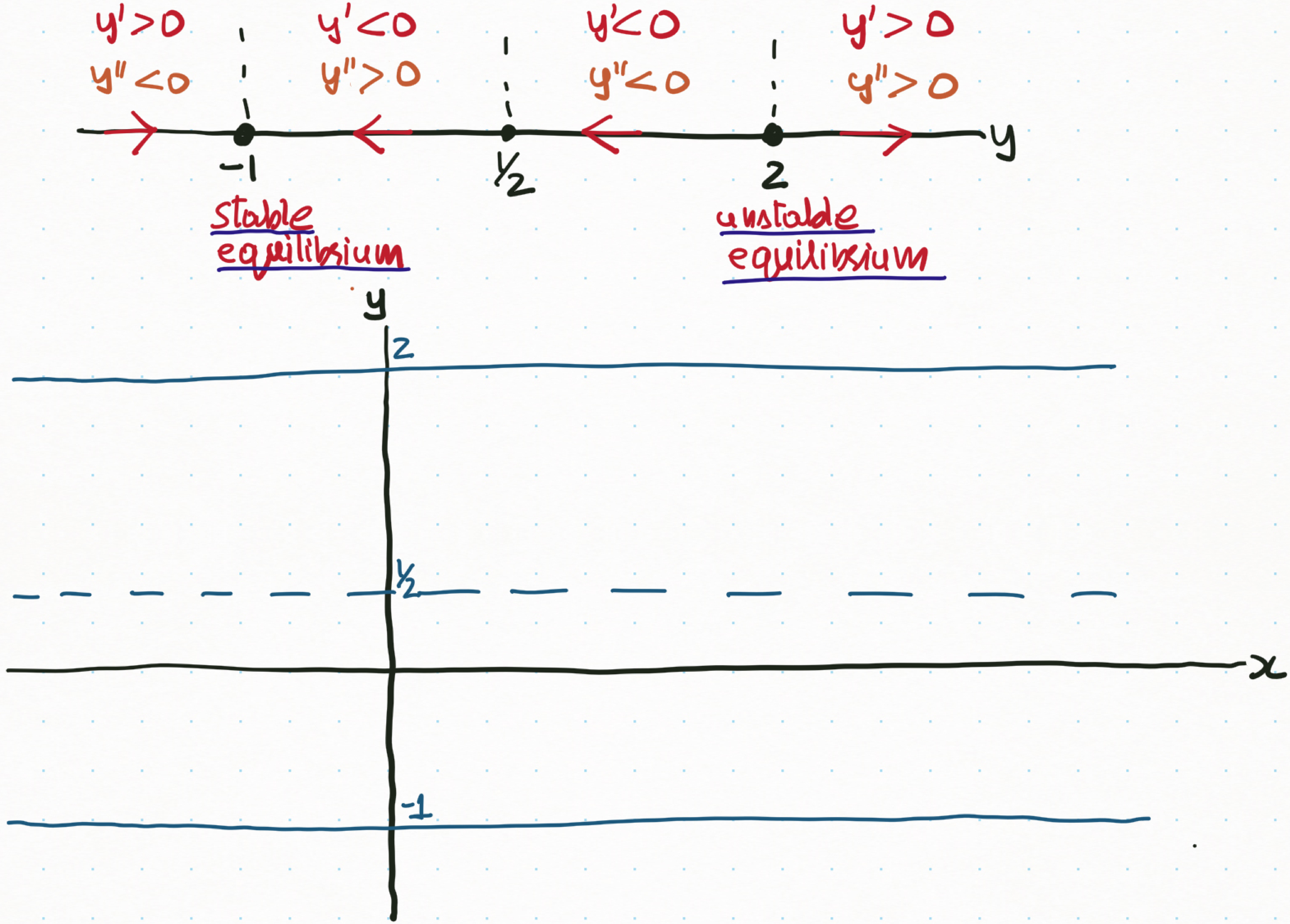
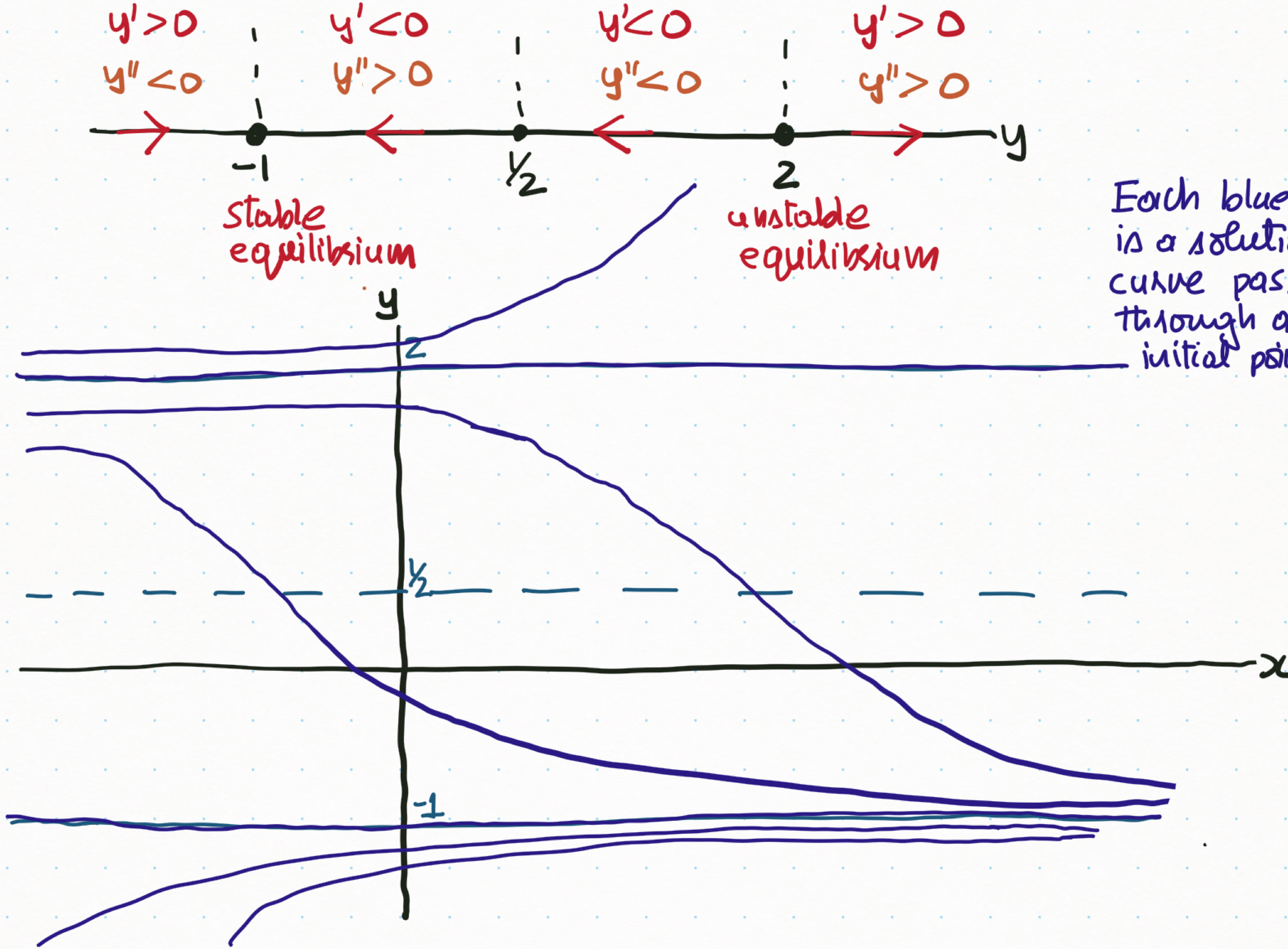
4. Sketch some solution curves using the phase diagram on the xy-plane


y'>0
y'<0
y'<0
y'>0
-1
1/2
2
y
equilibrium
equilibrium


y'>0
y'<0
y'<0
y'>0
y
-1
1/2
2
stable equilibrium
unstable equilibrium



y'>0
y''<0
y'<0
y''>0
y'<0
y''<0
y'>0
y''>0
y
-1
1/2
2
stable equilibrium
unstable equilibrium
y
2
1/2
x
-1

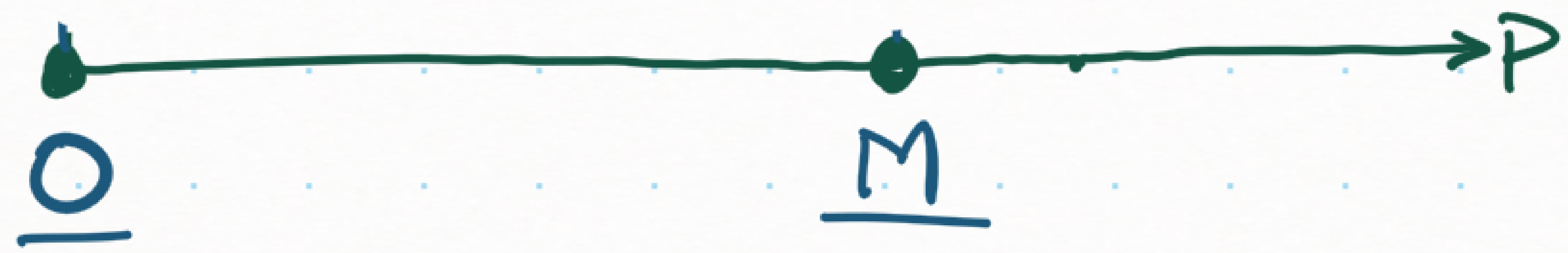


Each blue curve is a solution curve passing through a particular initial point (x_0, y_0)

e.g. Logistic Growth Model

$$\frac{dP}{dt} = r(M-P)P$$

Equillibrium values are $r(M-P)P = 0 \Leftrightarrow P = 0$ or M



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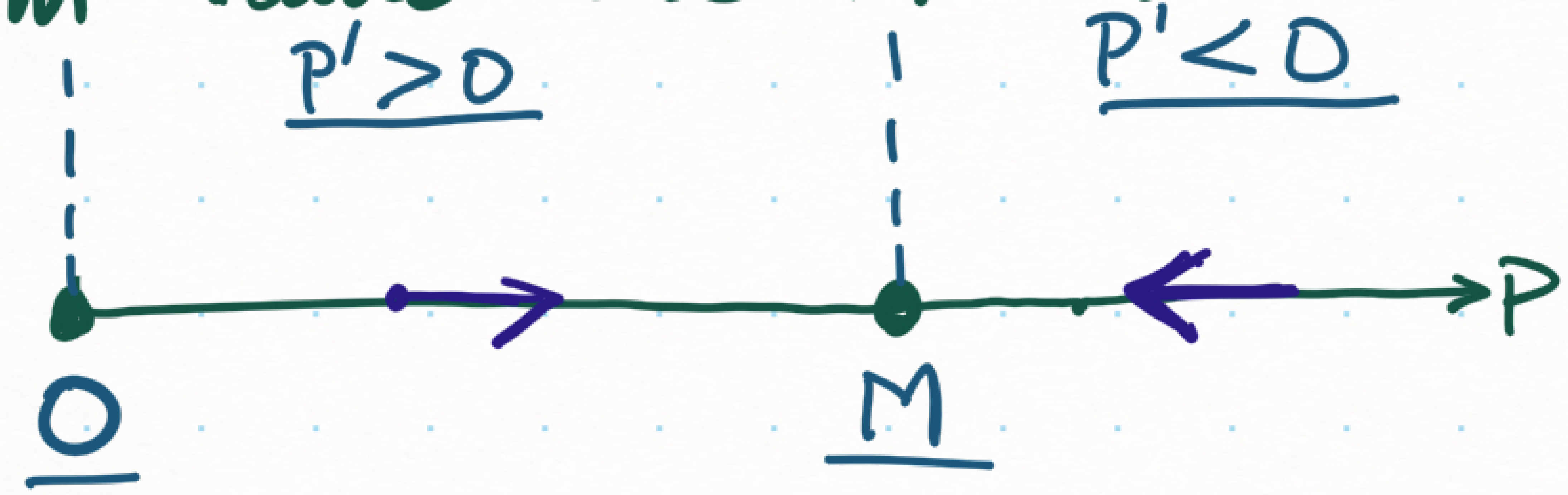

$P' = r(M-P)P$
(+ve) (+ve) if $0 < P < M$
(−ve) (+ve) if $M < P$

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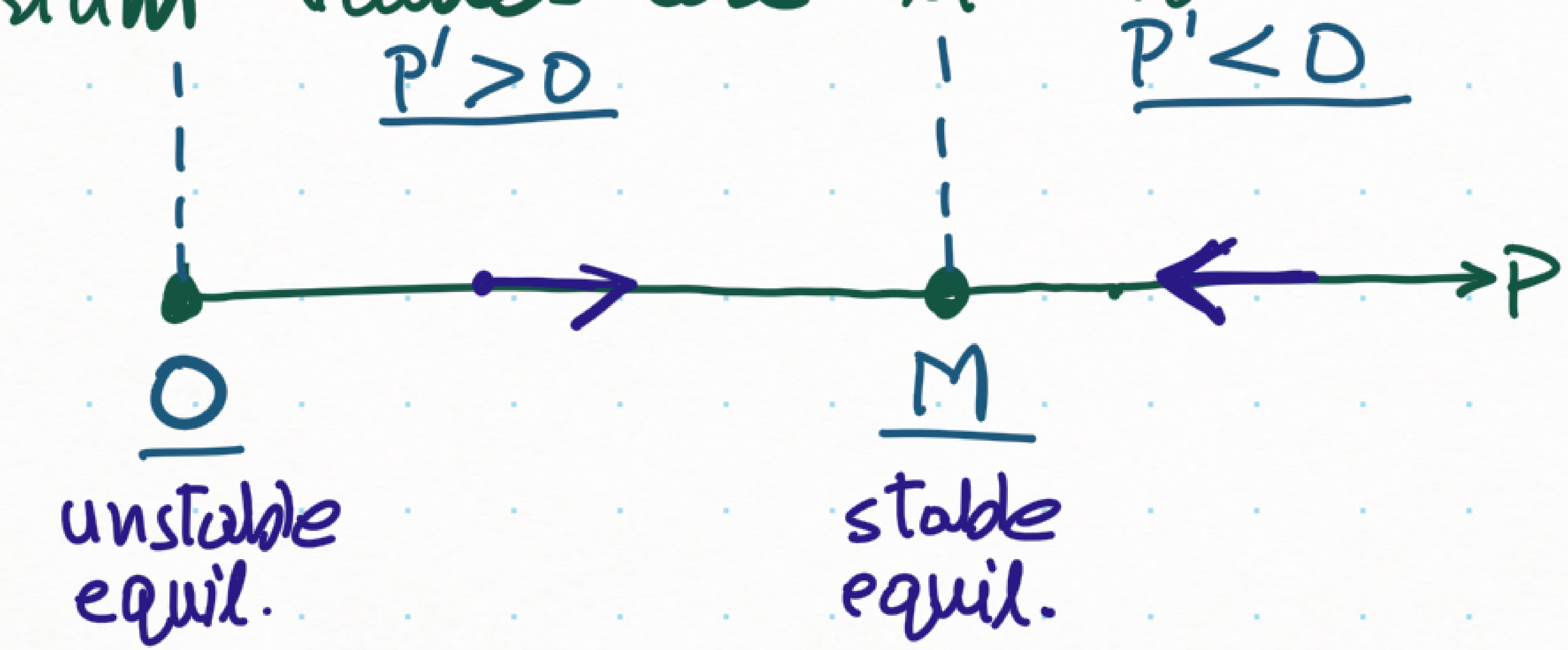

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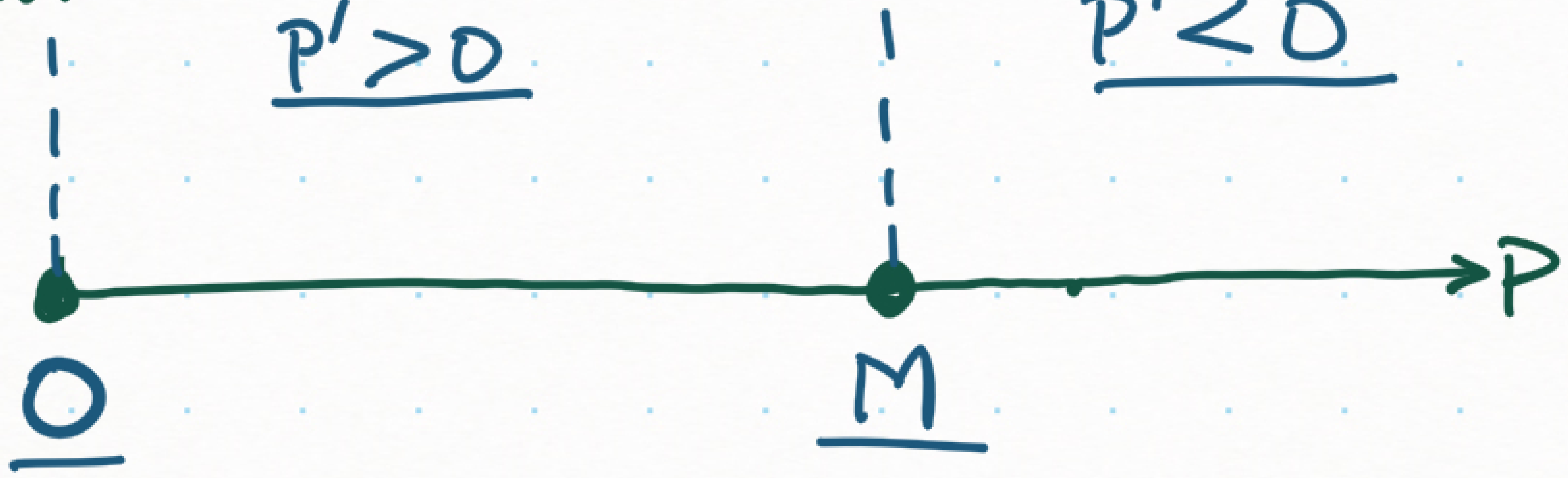

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$$= r^2(M-2P)(M-P)P$$

(+ve) (+ve) (+ve) if $0 < P < \frac{M}{2}$

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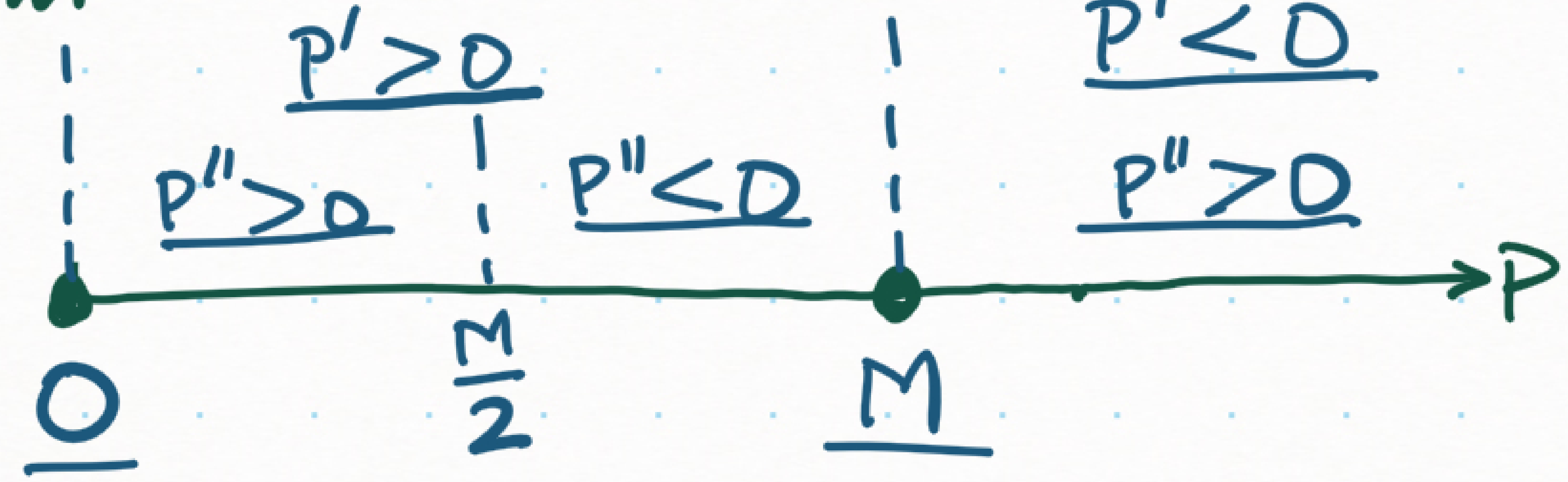

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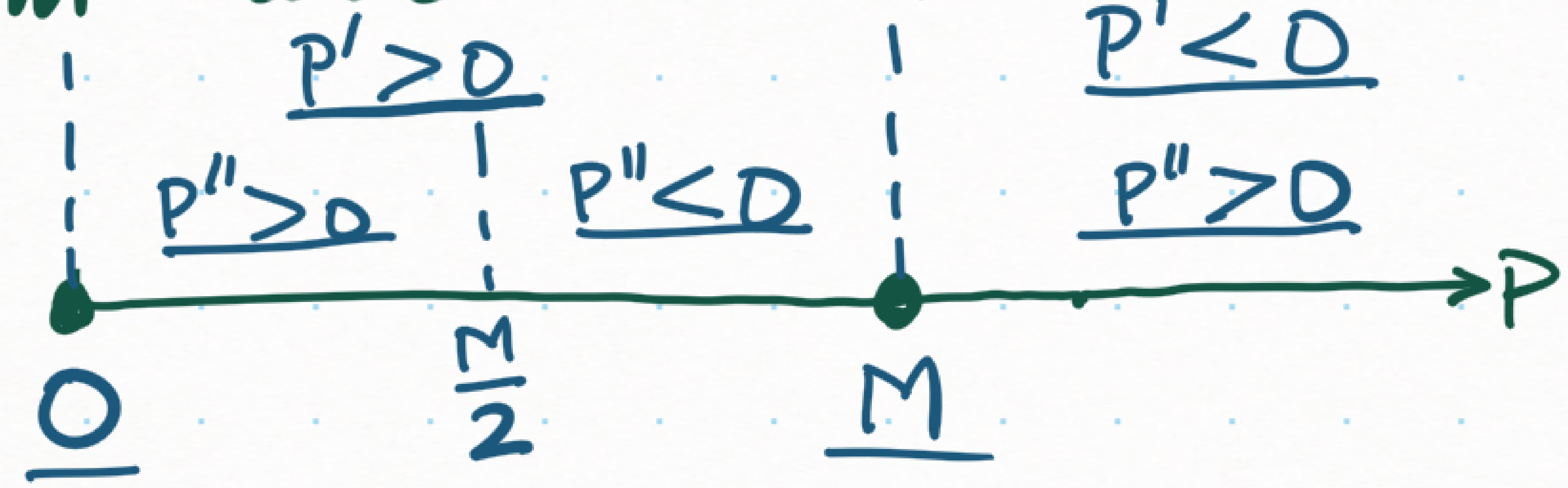

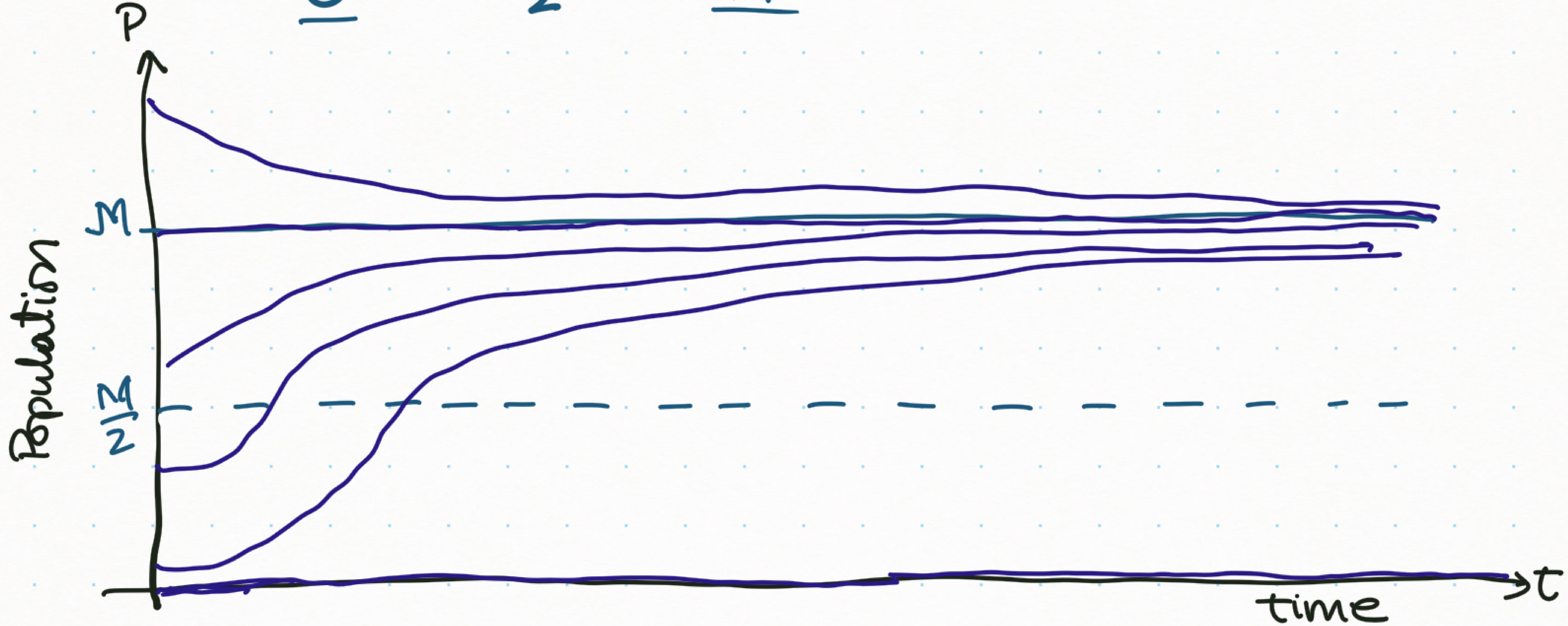

System of Differential Equations

We will focus (for now) on qualitative understanding of behavior of a system of D Eqns of the form:

Think as time

Autonomous 1st-order Diff. eqns

$$\begin{cases} \dfrac{dx}{dt} = f(x,y) \\ \dfrac{dy}{dt} = g(x,y) \end{cases}$$

→ x, y depend on the independent var. t

→ rates of change depend only on current states but not on time t.

e.g. Predator–prey (Owls & mice)

competing populations (Owls & Hawks)

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<u>Solution curve</u> is $x = x(t)$, $y = y(t)$., parametric curve whose derivatives satisfy $x' = f(x,y)$, $y' = g(x,y)$.

We visualize it as <u>$(x(t), y(t))$ as t varies over time.</u>

trajectory or orbit ... of the system

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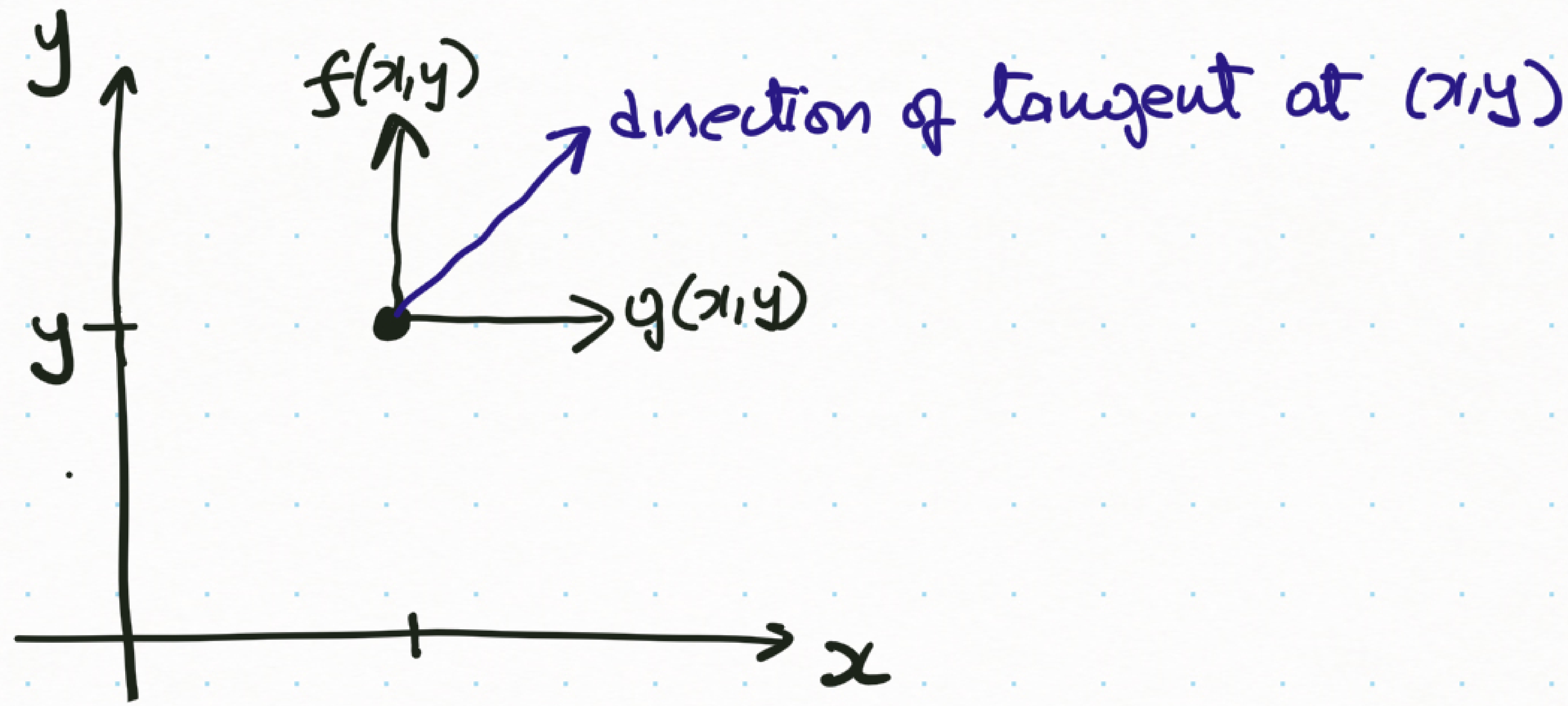
trajectory or orbit ... of the system

<u>Equilibrium point</u> (x,y) where $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} = 0$

We can sketch the $(x(t), y(t))$ trajectories of the solutions in the phase plane (xy-plane) by plotting the tangent line segments (using $g(x,y)$ & $f(x,y)$) with direction given by increasing time t.



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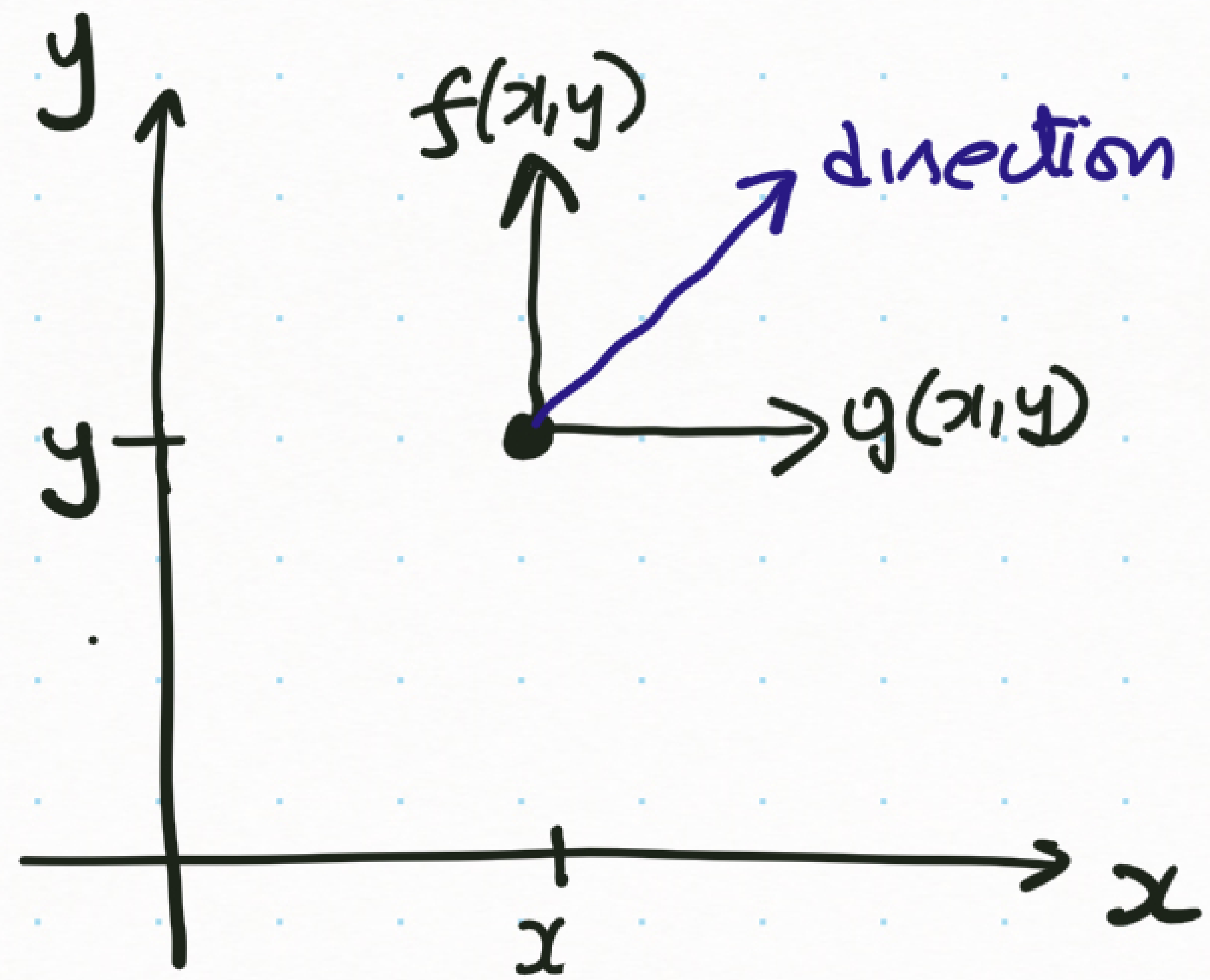


And we classify equilibrium points as stable / asymptotically stable / unstable based on the behavior of trajectories close to them.

x'= y-0.5*x

y'= sin(x)

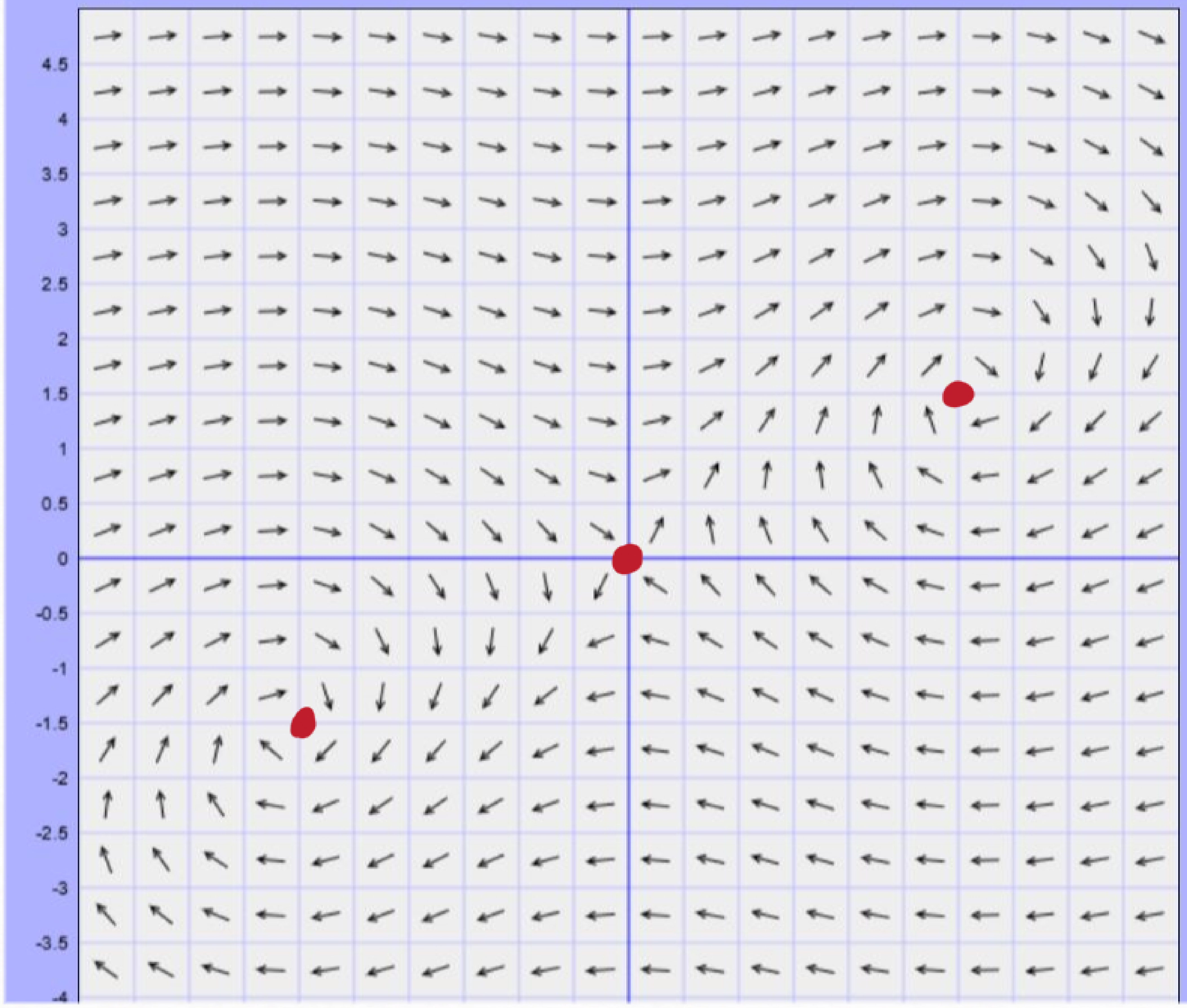
The direction field solver knows about trigonometric, logarithmic and exponential functions, but multiplication and evaluation must be entered explicitly (2*x and sin(x), not 2x and sin x).

The Display:

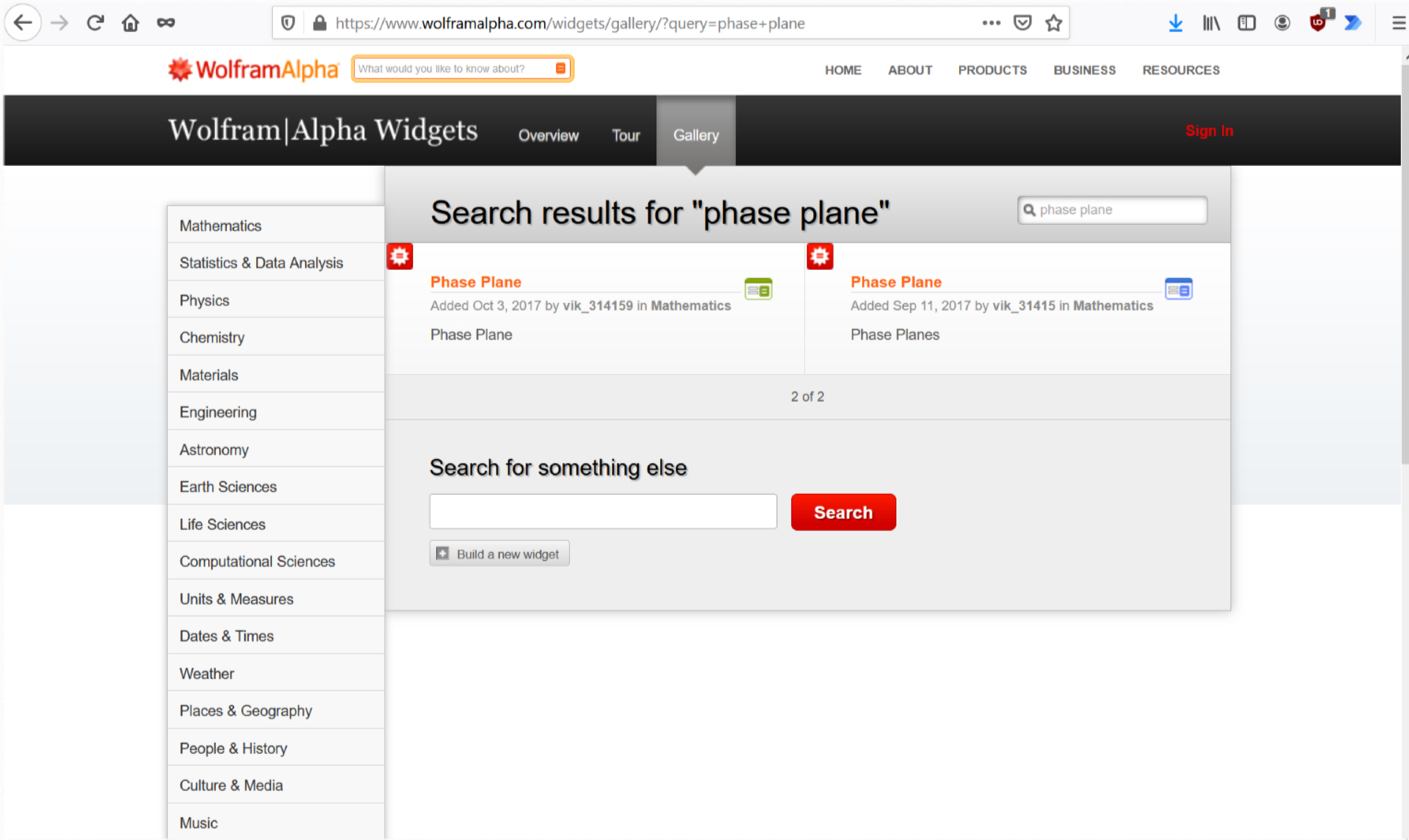
Minimum x: -5 Minimum y: -5 Arrow length: 15 ☐ Variable length arrows

Maximum x: 5 Maximum y: 5 Number of arrows: 20

Graph Phase Plane



see also Figure 12.2 in textbook for another asymptotically unstable equillibrium value at origin.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Phase Planes
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e.g. $\left.\begin{aligned} \frac{dx}{dt} &= -x+y \\ \frac{dy}{dt} &= -x-y \end{aligned}\right\}$ has a solution $\begin{aligned} x &= e^{-t}\sin t \\ y &= e^{-t}\cos t \end{aligned}$

easy to verify: $\frac{dx}{dt} = \frac{d}{dt}(e^{-t}\sin t) =$

$= -e^{-t}\sin t + e^{t}\cos t = -x+y$

similarly, $\frac{dy}{dt} = \cdots = -x-y$

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Equilibrium points:

$\left.\begin{array}{l} -x+y=0 \\ -x-y=0 \end{array}\right\} \Rightarrow \begin{array}{l} x=0 \\ y=0 \end{array}$

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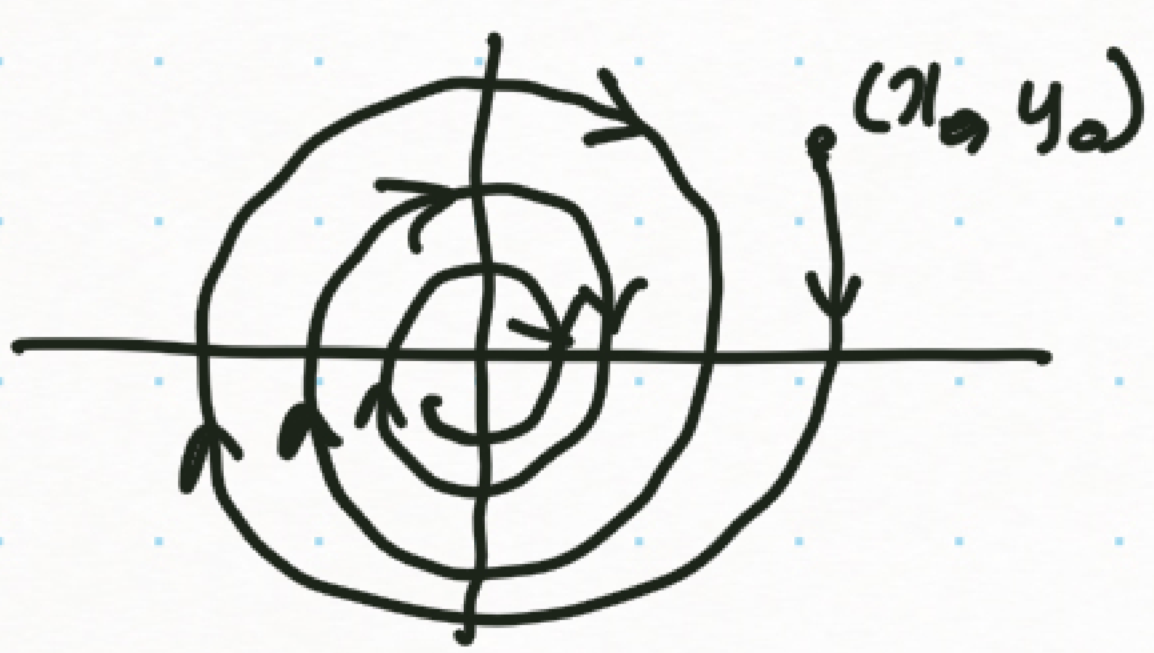
Consider

$$x^2+y^2 = e^{-2t}\sin^2 t + e^{-2t}\cos^2 t = e^{-2t}(\sin^2 t + \cos^2 t) = e^{-2t}$$

$x^2+y^2 = e^{-2t}$, spiral whose radius is continuously decreasing as time increases.



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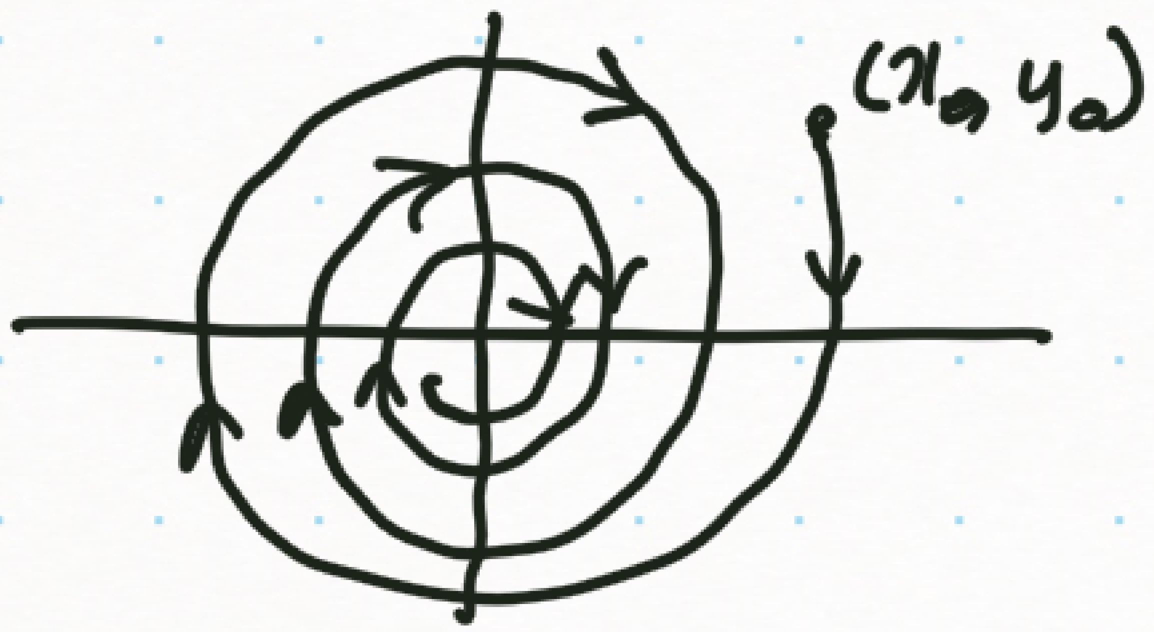

$(0,0)$ is an asymptotically stable rest point.