MATH 380

Hemanshu Kaul

kaul@iit.edu

Recall from previous example

- The previous weight gain of 5lbs/day may not be accurate

Different weight gain assumptions?

Growth rate is not a constant, but is proportional to its current weight.

$$\frac{dw}{dt} = cw \quad \text{for a constant } c > 0.$$

with $w(0) = 200$

Then $\frac{dw}{w} = c\,dt \Rightarrow w(t) = 200\,e^{ct}$

What is c?

we can estimate c as $\frac{dw}{dt} \approx 5$ lbs/day when $w = 200$ lbs

so, $c \approx \frac{5}{200} = 0.025$

Profit $P(t) = (65-t)(200e^{ct}) - 45t$

For any fixed c, the optimum occurs at t for $P'(t) = 0$

$P'(t) = 200c\,e^{ct}(65-t) - 200e^{ct} - 45$

How is $P'(t) = 0$ solved?

Another Calculus idea

Given a differentiable function $F(x)$,
and an approximation x_0 to a root of $F(x)$ (i.e., x_0 is close to x s.t. $F(x)=0$)

Found using a "Global" method.

Now we want to improve x_0 & bring it closer to a root.

We know $F'(x_0) = \lim_{x \to x_0} \dfrac{F(x)-F(x_0)}{x-x_0}$,

For x close to x_0, $F'(x_0)(x-x_0) \approx F(x)-F(x_0)$

i.e., $F(x) \approx F(x_0) + F'(x_0)(x-x_0)$ [Linear approximation of $F(x)$]

To find x s.t. $F(x)=0$, set $F(x_0)+F'(x_0)(x-x_0)=0$

i.e., $x = x_0 - F(x_0)/F'(x_0)$

new approximation

Another Calculus idea

Given a differentiable function $F(x)$, and an $\underbrace{\text{approximation } x_0 \text{ to a root of } F(x) \text{ (i.e., } x_0 \text{ is close to } x \text{ s.t. } F(x)=0)}_{\text{Found using a "Global" method.}}$

Numerical Approximation using Newton's method

$x(n)$ = approx. location of root of $F(x)$ after n iterations

N = # iterations

Input $x(0)$, N

Process For $n=1$ to N DO $\quad x(n) = x(n-1) - \dfrac{F(x(n-1))}{F'(x(n-1))}$

Output $x(N)$

Under what conditions?

Another Calculus idea

Given a differentiable function $F(x)$,
and an $\underbrace{\text{approximation } x_0 \text{ to a root of } F(x) \text{ (i.e., } x_0 \text{ is close to } x \text{ s.t. } F(x)=0)}_{\text{Found using a "Global" method.}}$

Numerical Approximation using Newton's method

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Output $x(N)$

Under what conditions? $F'(x)$ exists & is continuous locally
$F'(x) \neq 0$
$\vdots$

Multivariable non-linear Optimization models

We make 40" & 50" TVs.
Manufacturing costs: \$195 per 40", \$225 per 50", plus \$400000 fixed costs.
Suggested retail price: \$339 for 40" & \$399 for 50".

To sell all produced TVs in a competitive market:
we drop prices by 1¢ per unit of that type sold, and additionally
price of 40" drops 0.3¢ per unit of 50" sold,
price of 50" drops 0.4¢ per unit of 40" sold.

How many units of each type of TV should we manufacture?

Multivariable non-linear Optimization models

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How many units of each type of TV should we manufacture?

Let x_1 = # 40" TVs , x_2 = # 50" TVs

Maximize Profit [Profit = Revenue − Costs]

Costs, $C(x_1, x_2) = 195x_1 + 225x_2 + 400000$

Revenue, $R(x_1, x_2) = (339 - 0.01x_1 - 0.003x_2)x_1 + (399 - 0.004x_1 - 0.01x_2)x_2$

Maximize $P(x_1, x_2) = R(x_1, x_2) - C(x_1, x_2)$ over $x_1 \geq 0, x_2 \geq 0$

Costs, $C(x_1,x_2) = 195x_1 + 225x_2 + 400000$

Revenue, $R(x_1,x_2) = (339 - 0.01x_1 - 0.003x_2)x_1 + (399 - 0.004x_1 - 0.01x_2)x_2$

Maximize $P(x_1,x_2) = R(x_1,x_2) - C(x_1,x_2)$ over $x_1 \geqslant 0, x_2 \geqslant 0$

<u>Method 1</u> Set $\nabla P = \left(\frac{\partial P}{\partial x_1}, \frac{\partial P}{\partial x_2}\right) = \vec{0}$

$\boxed{\text{Gradient}}$

$$\frac{\partial P}{\partial x_1} = 339 - 0.02x_1 - 0.003x_2 - 0.004x_2 - 195 = 0$$

$$\frac{\partial P}{\partial x_2} = -0.003x_1 + 399 - 0.004x_1 - 0.02x_2 - 225 = 0$$

Solve : $x_1 \approx 4735$, $x_2 \approx 7043$

Verify it's really the maximum – 2nd deriv. test or sketch graph.

Profit $\approx \$553641$

Costs, $C(x_1,x_2) = 195x_1 + 225x_2 + 400000$

Revenue, $R(x_1,x_2) = (339 - 0.01x_1 - 0.003x_2)x_1 + (399 - 0.004x_1 - 0.01x_2)x_2$

Maximize $P(x_1,x_2) = R(x_1,x_2) - C(x_1,x_2)$ over $x_1 \geq 0, x_2 \geq 0$

<u>Method 2</u> <u>Gradient Method of Steepest ascent/descent</u>

Costs, $C(x_1,x_2) = 195x_1 + 225x_2 + 400000$

Revenue, $R(x_1,x_2) = (339 - 0.01x_1 - 0.003x_2)x_1 + (399 - 0.004x_1 - 0.01x_2)x_2$

Maximize $P(x_1,x_2) = R(x_1,x_2) - C(x_1,x_2)$ over $x_1 \geq 0, x_2 \geq 0$

Method 2 Gradient Method of Steepest ascent/descent

Idea ∇P at a point in domain of a differentiable function $P(x,y)$ always points in the direction of max rate of increase of P

See Table 13.1 in textbook

Iterative process Need (x_0, y_0) initial approximation and a sequence of step-sizes λ_k.

For $k = 0, 1, \ldots, N$

$$x_{k+1} = x_k + \lambda_k \frac{\partial P}{\partial x}(x_k, y_k)$$

$$y_{k+1} = y_k + \lambda_k \frac{\partial P}{\partial y}(x_k, y_k)$$

How to choose λ_k?
As $(x_k, y_k) \to$ extreme pt.,
$\nabla P \to (0,0)$, so we will
need λ_k to be larger...
e.g. $\lambda_k = \lambda_0 \delta^k$ for fixed $\delta > 1$

Costs, $C(x_1, x_2) = 195x_1 + 225x_2 + 400000$

Revenue, $R(x_1, x_2) = (339 - 0.01x_1 - 0.003x_2)x_1 + (399 - 0.004x_1 - 0.01x_2)x_2$

Maximize $P(x_1, x_2) = R(x_1, x_2) - C(x_1, x_2)$ over $x_1 \geq 0, x_2 \geq 0$

Sensitivity? e.g. "Price elasticity" of 40" TV

set $a = 0.01$

price of 40" TV $= 339 - ax_1 - 0.003x_2$

Costs, $C(x_1,x_2) = 195x_1 + 225x_2 + 400000$

Revenue, $R_a(x_1,x_2) = \underline{(339 - \cancel{0.01}^{\,a} x_1 - 0.003x_2)x_1} + (399 - 0.004x_1 - 0.01x_2)x_2$

Maximize $P_a(x_1,x_2) = R_a(x_1,x_2) - C(x_1,x_2)$ over $x_1 \geq 0, x_2 \geq 0$

Sensitivity? e.g. "Price elasticity" of 40" TV

set $a = 0.01$

price of 40" TV $= 339 - ax_1 - 0.003x_2$

Setting $\nabla P_a = \vec{0}$ gives $x_1 = \dfrac{1662000}{400000a - 49}$, $x_2 = 8700 - \dfrac{581700}{400000a - 49}$

At $a = 0.01$, Sensitivity, $S(x_1, a) = \left(\dfrac{dx_1}{da}\right)\left(\dfrac{a}{x_1}\right) = \cdots = \dfrac{-400}{351} \approx -1.1$

$S(x_2, a) = \left(\dfrac{dx_2}{da}\right)\left(\dfrac{a}{x_2}\right) = \cdots = \dfrac{9695}{36123} \approx 0.27$

e.g. 10% increase in price elasticity of 40" TVs means we should make ~11% fewer 40" TVs and 2.7% more 50" TVs.

Costs, $C(x_1, x_2) = 195x_1 + 225x_2 + 400000$

Revenue, $R_a(x_1, x_2) = \underline{(339 - \cancel{0.01}\,a\, x_1 - 0.003x_2)x_1} + (399 - 0.004x_1 - 0.01x_2)x_2$

Maximize $P_a(x_1, x_2) = R_a(x_1, x_2) - C(x_1, x_2)$ over $x_1 \geq 0, x_2 \geq 0$

Sensitivity of Profit to price elasticity of 40" TVs

$$\frac{dP_a}{da} = \frac{\partial P_a}{\partial x_1}\frac{\partial x_1}{\partial a} + \frac{\partial P_a}{\partial x_2}\frac{\partial x_2}{\partial a} + \frac{\partial P_a}{\partial a}$$

$$= 0 + 0 + -x_1^2 \quad = -x_1^2 \quad \text{at } a = 0.01, \text{ since } \nabla P_a = 0$$

So, sensitivity $S(P_a, a) = \left(\frac{dP_a}{da}\right)\left(\frac{a}{P_a}\right)$ at $a = 0.01$

$$\approx -(4735)^2 \frac{0.01}{553641} \approx -0.40$$

A 10% increase in price elasticity of 40" TVs will cause about 4% drop in profits.