MATH 380

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We have already seen models based on "proportionality"

e.g. change in population $\propto$ current population gives $P_{n+1} = k P_n$

→ Linear relation → exponential growth

e.g. $\Delta p_n \propto (K - P_n) P_n$ gives $P_{n+1} = P_n + k(K - P_n) P_n$

→ Nonlinear relation → logistic growth

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- $y \propto x \Leftrightarrow x \propto y$ $\quad (y = kx \Rightarrow x = \frac{1}{k} y$, and conversely$)$
- $y \propto x$ and $x \propto z \Rightarrow y \propto z$ $\quad (y = k_1 x,\ x = k_2 z \Rightarrow y = k_1 k_2 z)$
- $y \propto x^2 \Leftrightarrow x \propto y^{1/2}$ $\leftarrow$ Be careful! (not always invertible)
- $y \propto x^n$
- $y \propto f(x)$

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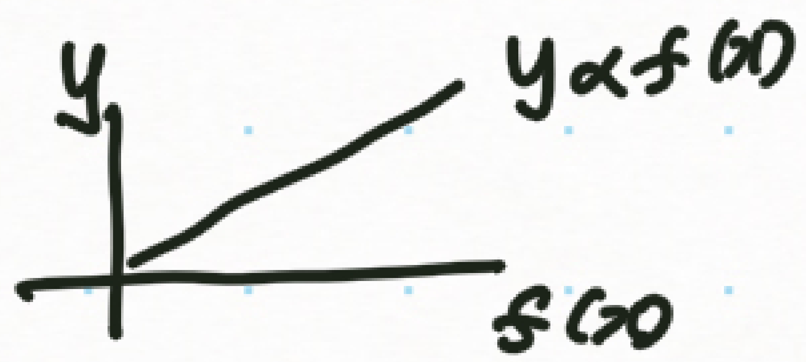
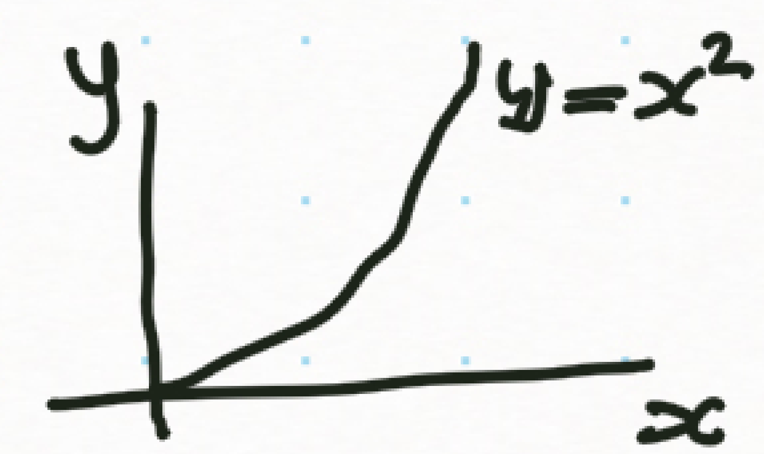
- $y \propto x \Leftrightarrow x \propto y$ $\quad$ ($y = kx \Rightarrow x = \frac{1}{k} y$, and conversely)
- $y \propto x$ and $x \propto z \Rightarrow y \propto z$ $\quad$ ($y = k_1 x$, $x = k_2 z \Rightarrow y = k_1 k_2 z$)
- $y \propto x^2 \Leftrightarrow x \propto y^{1/2}$ $\quad$ ($y = kx^2 \Rightarrow x = \sqrt{\frac{1}{k}}\, y^{1/2}$ if $k > 0$)
- $y \propto x^n$
- $y \propto f(x)$



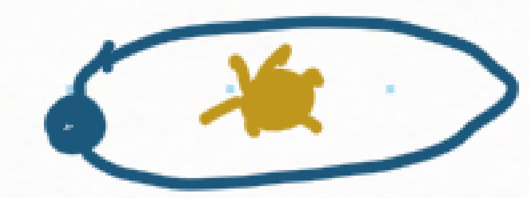


Many famous laws from Physics are "proportionalities"

- **Hooke's Law:** $F \propto S$, Force for restoring a spring stretched distance S
- **Newton's Law:** $F \propto a$, infact $F = ma$
- **Ohm's law:** $V = iR$
- **Boyle's law:** $V \propto \frac{1}{P}$, volume of gas is inversely proportional to pressure P at fixed temperature k.

  $V = k/P$
- **Einstein:** $E \propto m$, infact $E = c^2 M$
- **Kepler's Third law:** $T \propto R^{3/2}$, T period (days) & R mean distance to sun

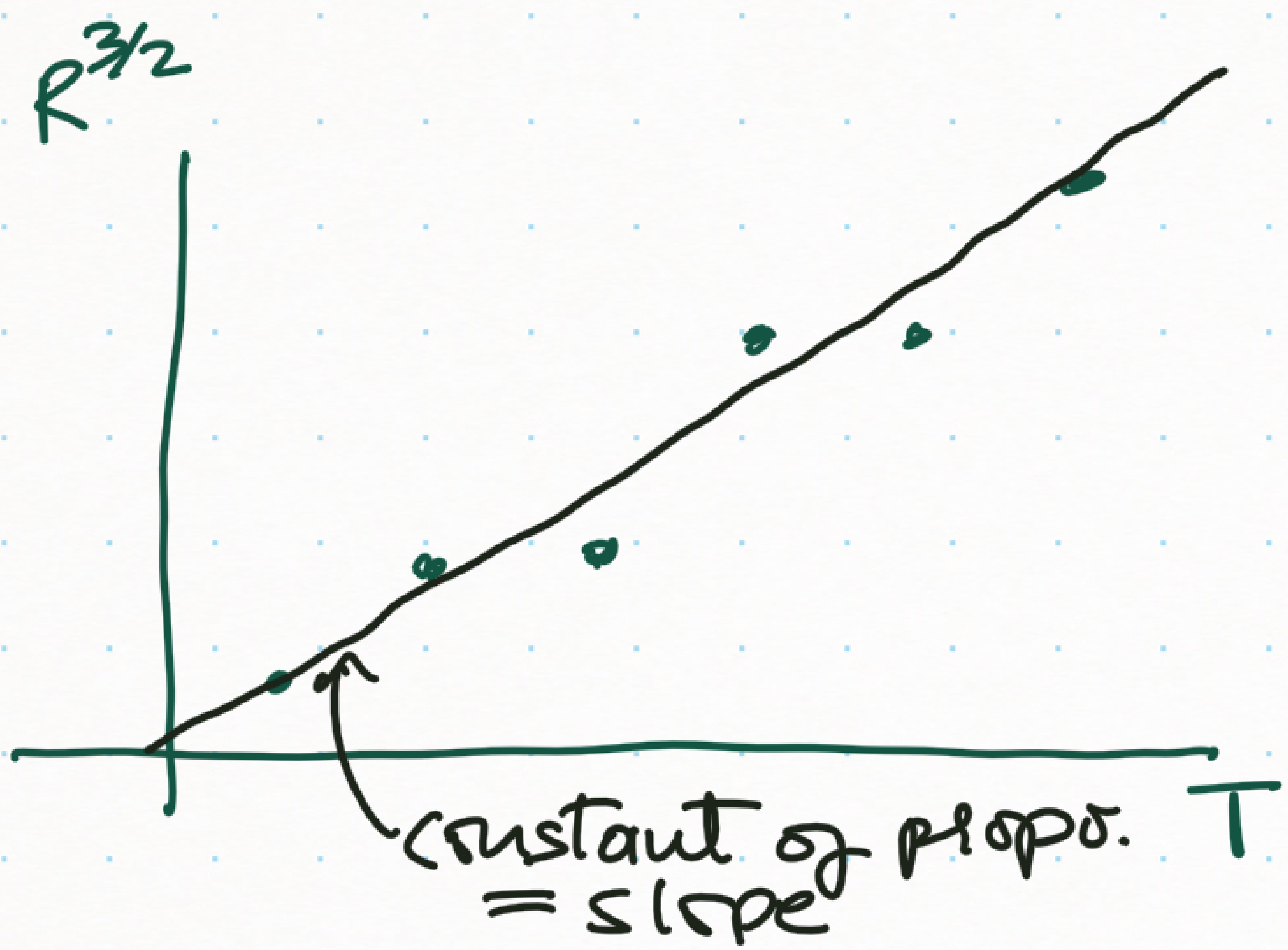
Kepler's Third Law of Planetary Motion



The square of orbital period of a planet is directly proportional to the cube of the semi-major axis of its orbit.

$$T^2 \propto R^3$$

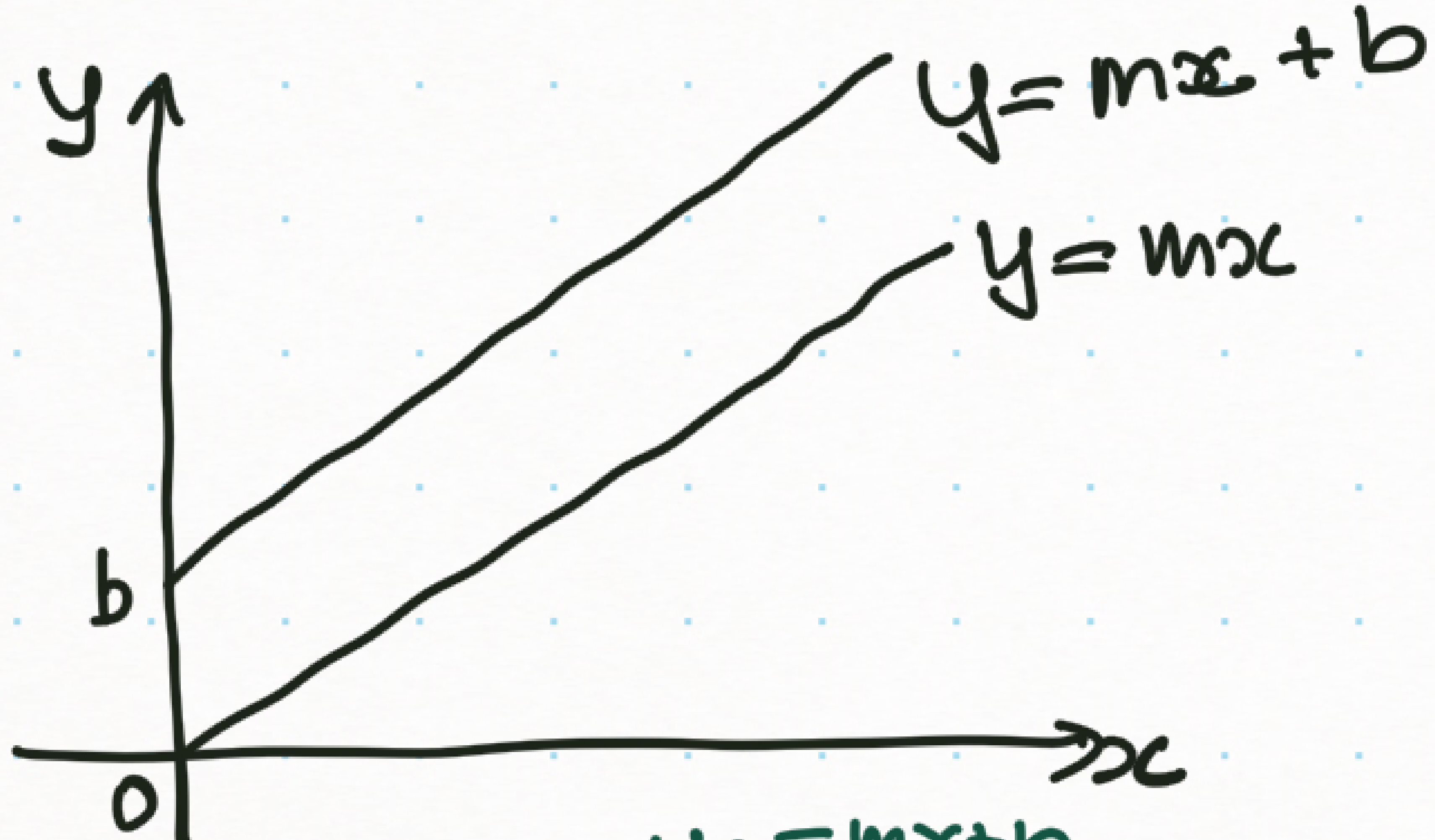
So, $$\frac{T_{Earth}^2}{R_{Earth}^3} = \frac{T_{Mars}^2}{R_{Mars}^3} = \frac{T_{Planet}^2}{R_{Planet}^3}$$



Planet	T (period)	R (mean dist.)
Mercury	88	36
Venus	224.7	67.25
Earth	365.3	93
Mars	687.0	141.75
Jupiter	4331.8	⋮
⋮	⋮	⋮

Comment Actual relation: $y = mx + b$

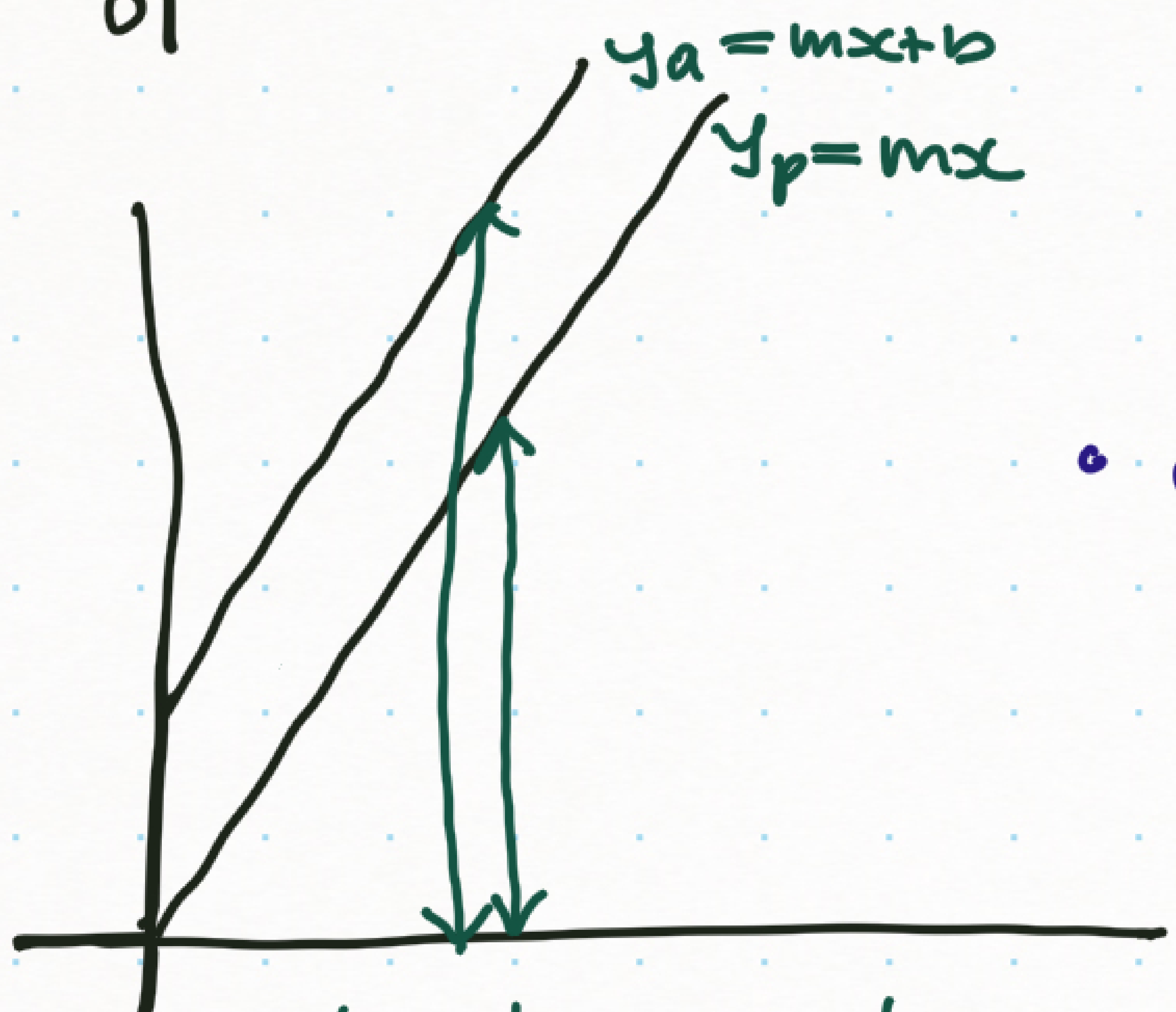
Simplified Proportionality model: $y = mx$



Lets give these different names for comparison:

$y_a = mx + b$

$y_p = mx$



Relative Error $= \dfrac{y_a - y_p}{y_a} = \dfrac{b}{y_a}$

- If slope m is relatively large (compared to 1) then $\dfrac{b}{y_a} \approx 0$, i.e., $\dfrac{y_a - y_p}{y_a} \approx 0$, i.e., $y_a - y_p \approx 0$

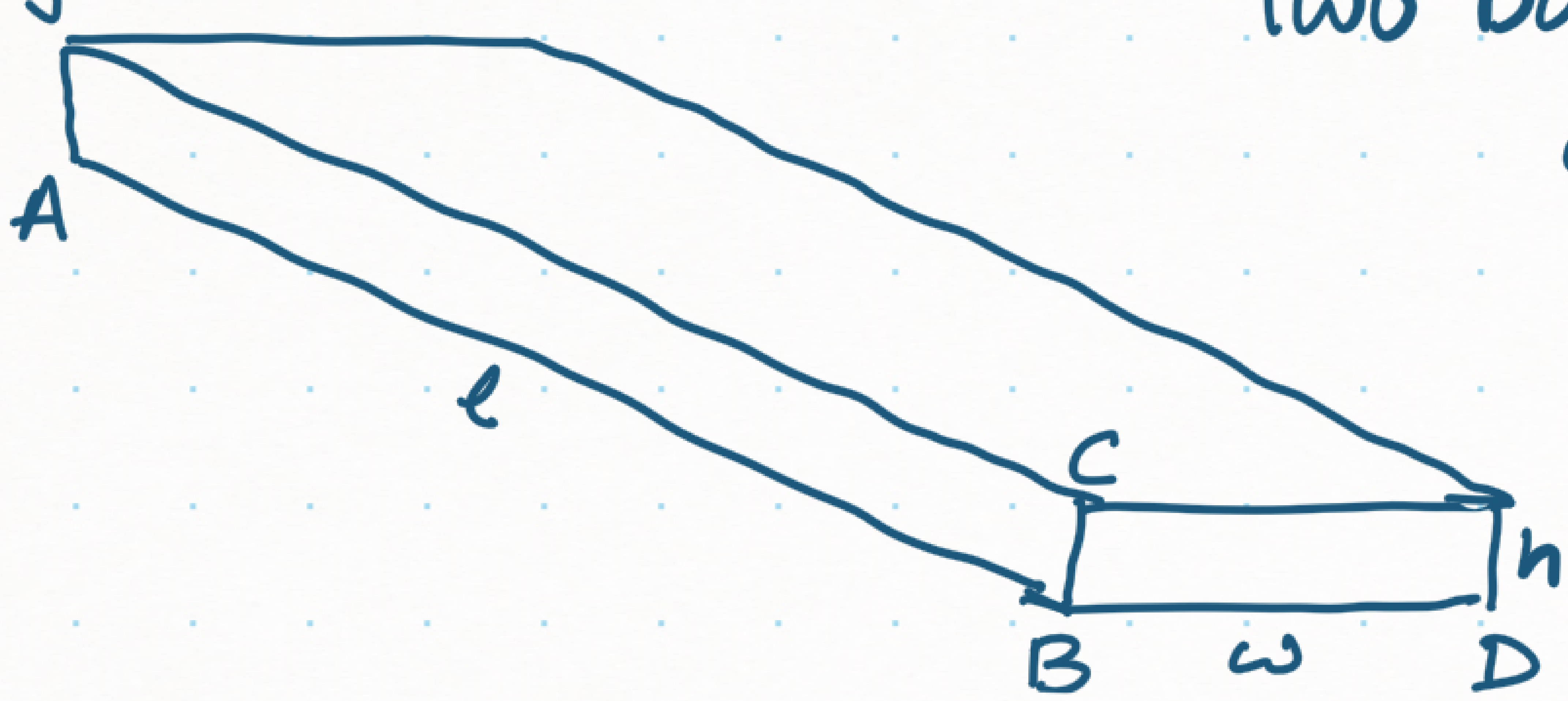
  i.e. $y_a \approx y_p$

with large slope, b (small) becomes irrelevant to y_a vs. y_p

Modeling using Geometric Similarity

Defn Two objects are geometrically similar if there is a one-to-one correspondence between points of the objects s.t. the ratio of distances is constant for every pair of corresponding points

e.g.



Two boxes X and X' of length l & l', width w & w', & height h & h',

Modeling using Geometric Similarity

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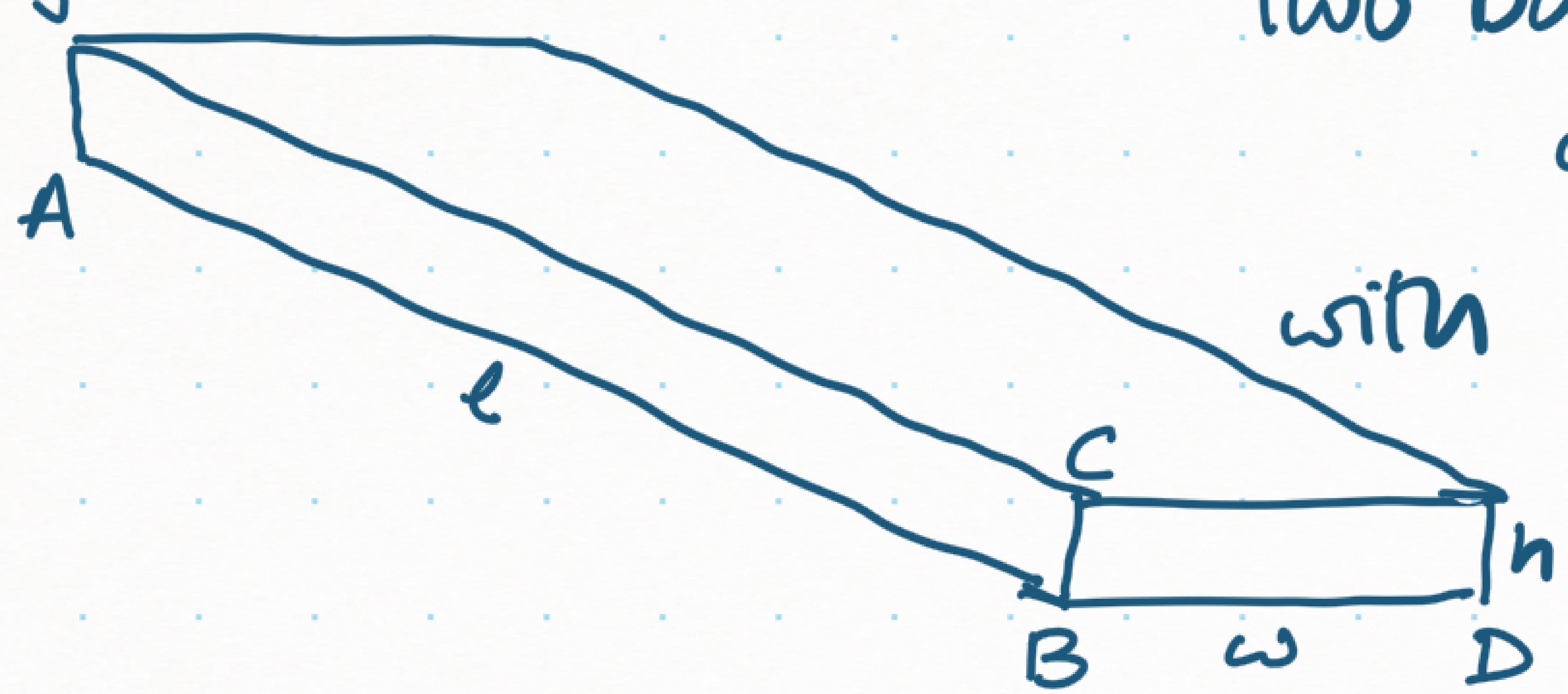

Two boxes X and X' of length l & l', width w & w', & height h & h',

with $\frac{l}{l'} = \frac{w}{w'} = \frac{h}{h'} = k$

where $k > 0$ is the "<u>scaling factor</u>".

Any geometric objects within these boxes will be geom. similar. including straight lines, triangles, etc.

For the two geom. similar boxes X and X',

Volume $(X) = V_X = lwh$, Volume $(X') = V_{X'} = l'w'h'$

So, $\dfrac{V_X}{V_{X'}} = \dfrac{lwh}{l'w'h'} = \left(\dfrac{l}{l'}\right)\left(\dfrac{w}{w'}\right)\left(\dfrac{h}{h'}\right) = k^3$, i.e., $V_X = k^3 V_{X'}$

i.e., $\boxed{V_X \propto V_{X'}}$

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Similarly for surface area,

$$\frac{S_X}{S_{X'}} = \frac{2(lw + wh + hl)}{2(l'w' + w'h' + h'l')} = \frac{(kl')(kw') + (kw')(kh') + (kh')(kl')}{l'w' + w'h' + h'l'} = k^2$$

i.e., $S_X = k^2 S_{X'}$, i.e., $\boxed{S_X \propto S_{X'}}$

& so on.

$\boxed{\text{Let's pick a "characteristic dimension" } \ell \text{ that dominates / determines the overall shape of } X}$

In our example, ℓ could be one of length, height, width depending on the overall shape & assumptions about X.

We can express the ratios in terms of ℓ & ℓ'.

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$$\frac{S_x}{S_{x'}} = k^2 = \frac{\ell^2}{\ell'^2} \quad , \text{ ie., } \frac{S_x}{\ell^2} = \frac{S_{x'}}{\ell'^2} = \text{constant}$$

ie., $S_x = (\text{constant})\,\ell^2$, i.e., $\boxed{S_x \propto \ell^2}$

Similarly, $S_{x'} \propto \ell'^2$

Let's pick a "characteristic dimension" ℓ that dominates / determines the overall shape of X

In our example, ℓ could be one of length, height, width depending on the overall shape & assumptions about X.

We can express the ratios in terms of ℓ & ℓ'.

$$\frac{V_X}{V_{X'}} = k^3 = \frac{\ell^3}{\ell'^3} \quad , \text{ ie., } \quad \frac{V_X}{\ell^3} = \frac{V_{X'}}{\ell'^3} \ (\leftarrow \text{constant})$$

$$\text{ie., } \boxed{V_X \propto \ell^3} \quad \& \quad V_{X'} \propto \ell^3$$

When we assume objects under study are geometrically similar, we get Volume $\propto \ell^3$ & Surf. Area $\propto \ell^2$ for a characteristic dim. ℓ.

example: Raindrop from a motionless Cloud

Want to understand Terminal velocity of a raindrop from a motionless cloud.

F_d ↑ Air resistance

↓ Gravity F_g

By Newton's second law of motion

Net Force $F = ma$

i.e., $F_g - F_d = ma$, m = mass, a = acceleration

At terminal velocity (max velocity), $a = ?$

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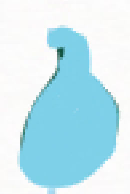

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i.e., $F_g - F_d = ma$, $\quad a = \text{acceleration}$

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Ques What is the relationship between terminal velocity and mass of the raindrop?

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Assumptions ① $F_d \propto S v^2$, S = surface area, v = velocity

② $F_g \propto w$, weight

③ Mass $m \propto w$, weight

④ All raindrops are geometrically similar.

Since $f_g \propto w$ and $m \propto w$, we have $\boxed{f_g \propto m}$ — ①

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i.e. diameter

Let l be a characteristic dimension for the raindrops, then

$S \propto l^2$ and Volume $V \propto l^3$, ie, $S \propto V^{2/3}$

Can we relate volume (V) to mass (m)?

Since $f_g \propto \omega$ and $m \propto \omega$, we have $\boxed{f_g \propto m}$ — ①

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Assuming constant water density, $V \propto m$ ($\because m = \rho V$)

$\therefore S \propto m^{2/3}$

So, $f_d \propto S v^2 \Rightarrow \boxed{f_d \propto m^{2/3} v^2}$ — ②

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By ① & ②, $f_g = k_1 m$ and $f_d = k_2 m^{2/3} v^2$ for constants k_1, k_2

At Terminal velocity, $f_g = f_d$,

Since $f_g \propto \omega$ and $m \propto \omega$, we have $\boxed{f_g \propto m}$ — ①

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By ① & ②, $f_g = k_1 m$ and $f_d = k_2 m^{2/3} v^2$ for constants k_1, k_2

At Terminal velocity, $f_g = f_d$, i.e., $k_1 m = k_2 m^{2/3} v^2$

i.e. $m^{1/3} = \frac{k_2}{k_1} v^2$, ie, $m^{1/3} \propto v^2$, i.e., $\boxed{v \propto m^{1/6}}$

Catch & Release Fishing Tournament

Fishers release fish soon after catching them.
Fishing award based on total weight of fish caught.
Weighing a live fish is not easy.

Problem Predict weight of a fish in terms of some easily measurable dimension.

Assumptions

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Problem Predict weight of a fish in terms of some easily measurable dimension.

Assumptions ① Single species of fish (often Bass)

② All geometrically similar

③ Uniform (avg.) density

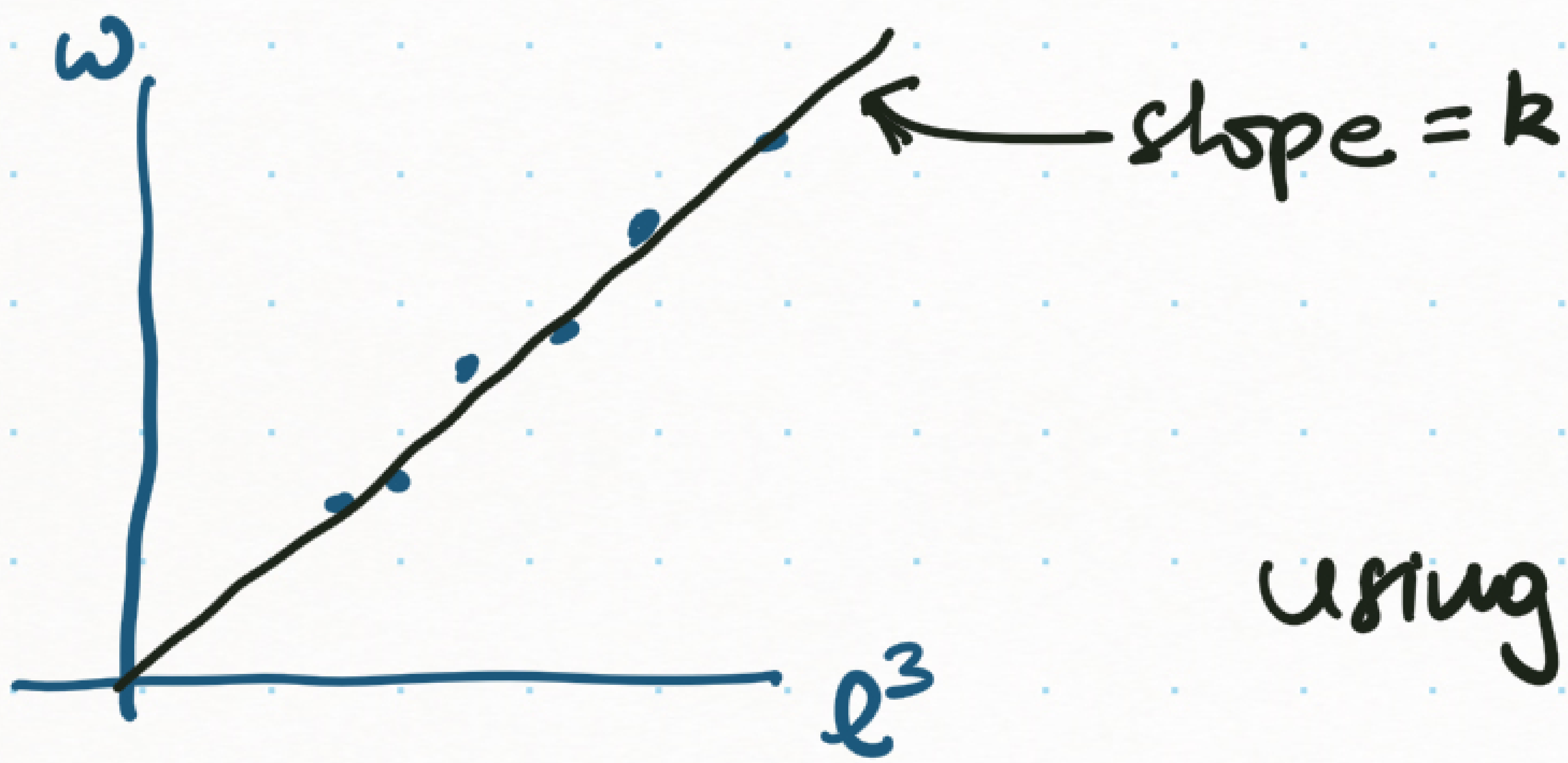
④ There is a characteristic dimension, say length l.

By Geom. similarity, volume $V \propto l^3$

By assumption of constant density, Weight, $w \propto V$

(since $w = \rho V$)

$\therefore$ $w \propto l^3$ i.e., $w = k l^3$



using actual data to estimate $k \approx 0.00853$

Finally, check the error = |actual wt. − predicted wt.|

using the given data & the model $w = (0.00853) l^3$

Potential problems?

Model $w = (0.00853)\, l^3$, where the l = length is characteristic dimension

treats ~~s~~kinny and fat fish alike if their length is same.

What "characteristic dimension" would capture this difference?

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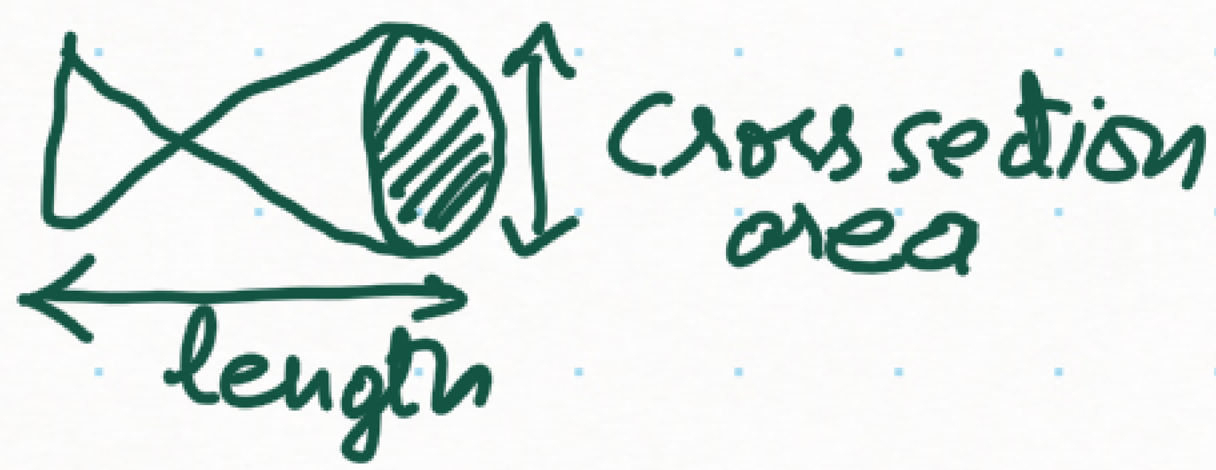
Volume $\approx$ 

$\overline{\text{length}} \times \overline{\text{Cross section area}}$

length: measure the length, ℓ

Cross section area: how to estimate?

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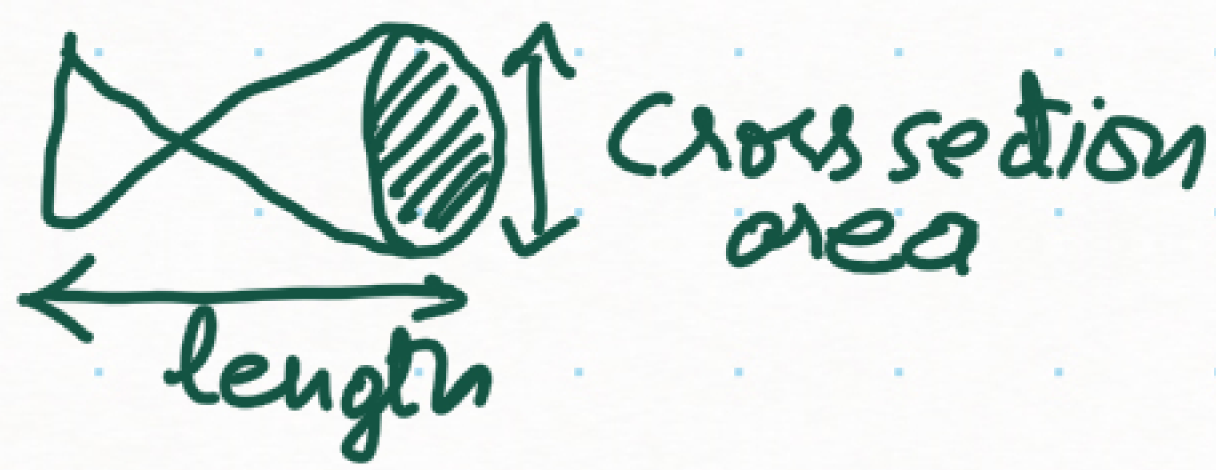

length $\times$ Cross section area

how to estimate?

measure the length, ℓ

measure the girth, g "circumference at widest"

Then, Volume, $V \propto \ell g^2$

why? "girth $\propto$ circumference $\propto$ radius"

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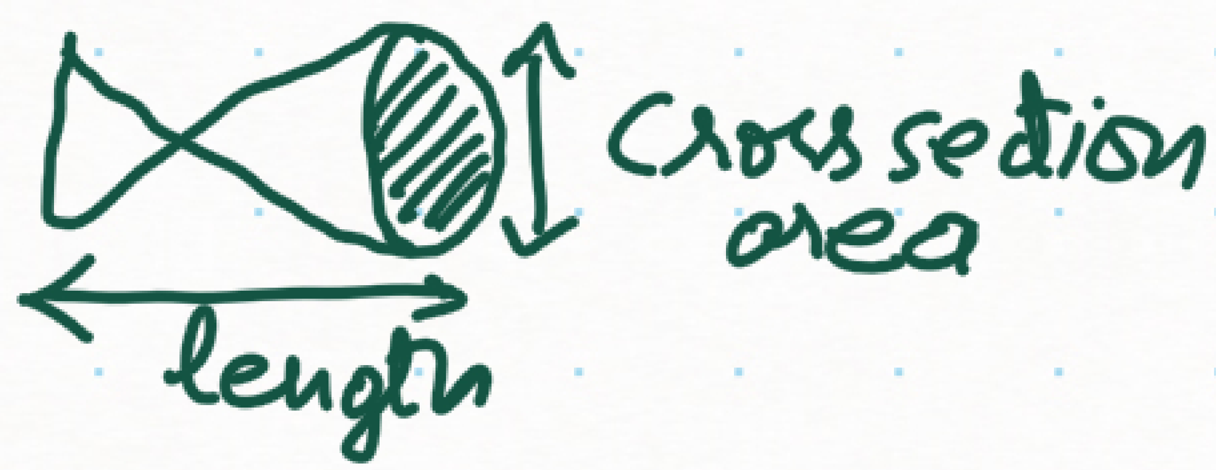

$\overline{\text{length}} \times \overline{\text{Cross section area}}$

measure the length, l

how to estimate?

measure the girth, g "circumference at widest"

Then, Volume, $V \propto l g^2$

why? "girth $\propto$ circumference $\propto$ radius"

As before, assuming constant density, $w \propto V$

$\therefore\ w = k\, l g^2$ & estimate k as usual using available data, ...

Have you been doing the 'Reading HW'?
Particularly, Section 1.4 has many interesting models.

→ How rental cars move between and within two locations.

$$O_{n+1} = 0.6\,O_n + 0.3\,T_n \quad \text{and} \quad T_{n+1} = 0.4\,O_n + 0.7\,T_n .$$

→ Ship survival in a naval battle.

$$B_{n+1} = B_n - 0.1F_n \quad \text{and} \quad F_{n+1} = F_n - 0.1B_n$$

→ Passenger transfer between airlines at an airport.

→ Spread of disease (Epidemic) models.

Recovered / Removed → $R(n+1) = R(n) + 0.6\,I(n)$

Infected → $I(n+1) = I(n) - 0.6\,I(n) + 0.0014\,I(n)S(n)$

Susceptible → $S(n+1) = S(n) - 0.0014\,S(n)I(n)$

$I(0), S(0), R(0)$ given.

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$\underset{?}{I(0)}$, $\underset{?}{S(0)}$, $\underset{?}{R(0)}$ given.

Strategy for Astronaut Docking Procedure

- Astronauts have to train in a manual docking procedure: bring a spacecraft to rest relative to another craft.
- Hand controls for variable acceleration and decceleration, and a device that measures the rate of closing between two crafts.
- Proposed strategy:

Strategy for Astronaut Docking Procedure

- Astronauts have to train in a manual docking procedure: bring a spacecraft to rest relative to another craft.
- Hand controls for variable acceleration and decceleration, and a device that measures the rate of closing between two crafts.
- Proposed strategy:
  - Look at closing velocity. If zero then done, else
  - Move the acceleration control opposite to closing velocity (i.e., positive closing velocity means decellerate, ...) and in proportion to magnitude of the closing velocity (i.e., brake twice as hard if we are closing twice as fast, ...)
  - Pause, and repeat.

Is this a reasonable strategy? Under what circumstances?

Problem Find the relation between velocity & time (under this strategy) and determine if/when velocity goes to zero.

Variables & Assumptions

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Variables & Assumptions

t_n = time of n^{th} velocity observation (s)

v_n = velocity at time t_n (m/s)

c_n = time to make n^{th} control adjustment (s)

a_n = acceleration after n^{th} adjustment (m/s²)

w_n = wait time before repeating & $(n+1)^{th}$ observation (s)

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- $t_{n+1} = t_n + c_n + w_n$ (in words?)
- $v_{n+1} = v_n + a_{n-1}c_n + a_n w_n$ (in words?)
- Spacecraft follows standard control law, $a_n \propto -v_n$, i.e., $a_n = -k v_n$
- With training, we ensure $c_n = c$ & $w_n = w$ constants.

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Combining: $t_{n+1} - t_n = c + w$, fixed.

$v_{n+1} - v_n = (-k v_{n-1})c + (-k v_n)w$.

Model

$v_{n+1} - v_n = (-k v_{n-1})c + (-k v_n)\omega$

can be expressed as

(in our eqn. v_{n+1} depends on both v_n & v_{n-1})

Define $x_1(n) = v_n$ and $x_2(n) = v_{n-1}$

Then

$$x_1(n+1) - x_1(n) = -k\omega\, x_1(n) - kc\, x_2(n)$$

$$x_2(n+1) - x_2(n) = x_1(n) - x_2(n)$$

we want to relate $x_1(n)$ to $x_2(n)$

Model $v_{n+1} - v_n = (-k v_{n-1})c + (-k v_n)\omega$

can be expressed as

Define $x_1(n) = v_n$ and $x_2(n) = v_{n-1}$ (in our eqn. v_{n+1} depends on both v_n & v_{n-1})

Then

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if $c \neq \omega$

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can be expressed as

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Solving gives $\boxed{v_n = v_0(1-k\omega)^n}$ If $\underline{0 < k\omega < 2}$ then $v_n \to 0$.

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Control strategy will work if c is small; ω is a bit larger & $\approx 1/k$.

What does it mean to find a model that best fits the data?
And how do we find it?

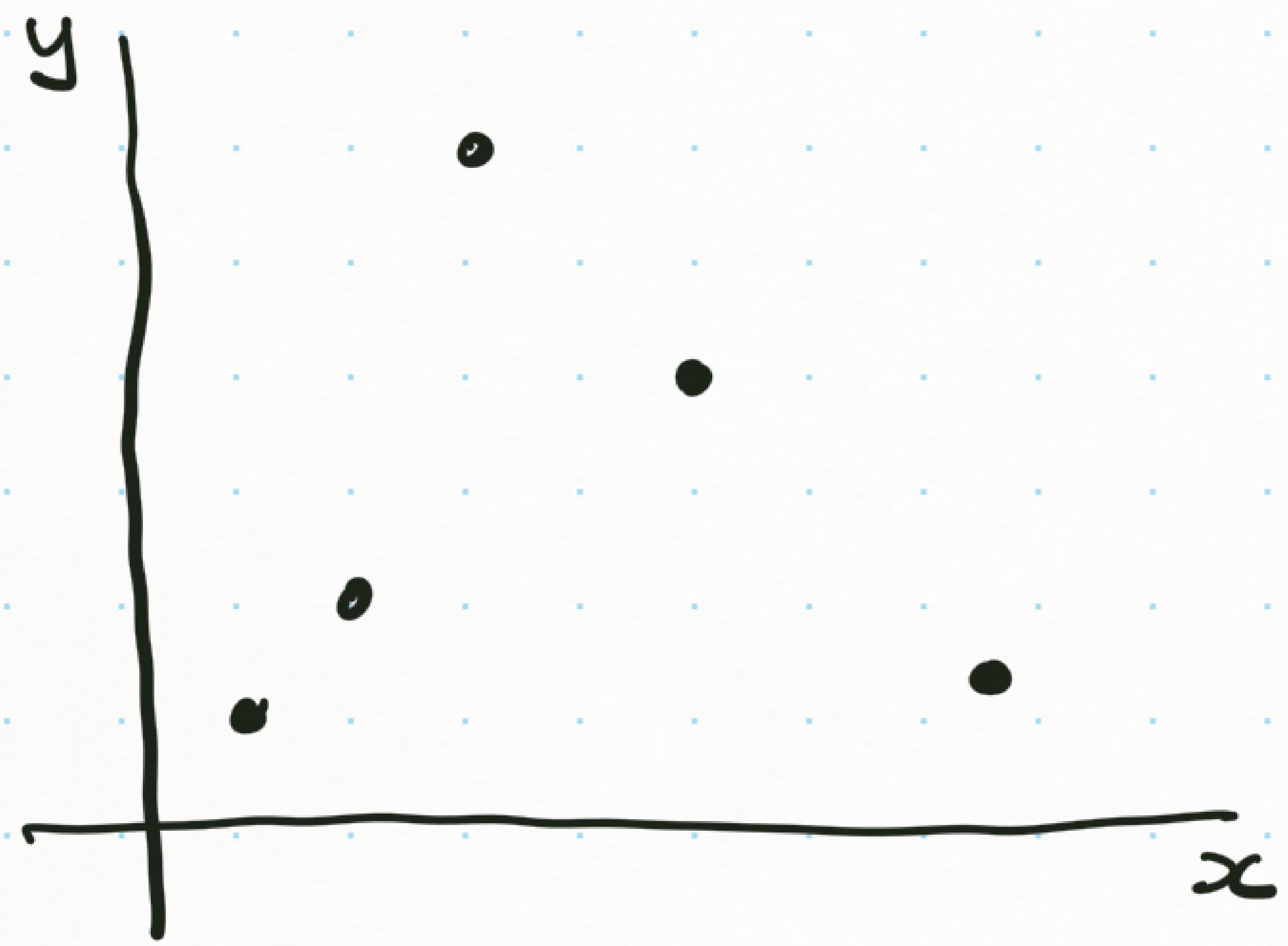
Given some observations (data) of a phenomenon under study, we want to:

1. Find one or more models that describe that phenomenon

2. Find the best "parameters"/constants for the models based on given data / information.

3. Make predictions based on these "fitted" models & test how well they do compared to actual observations.
   Is one model "better" than the other?
   Is it a reasonable model?

We can create a purely empirical model based purely on the given data.



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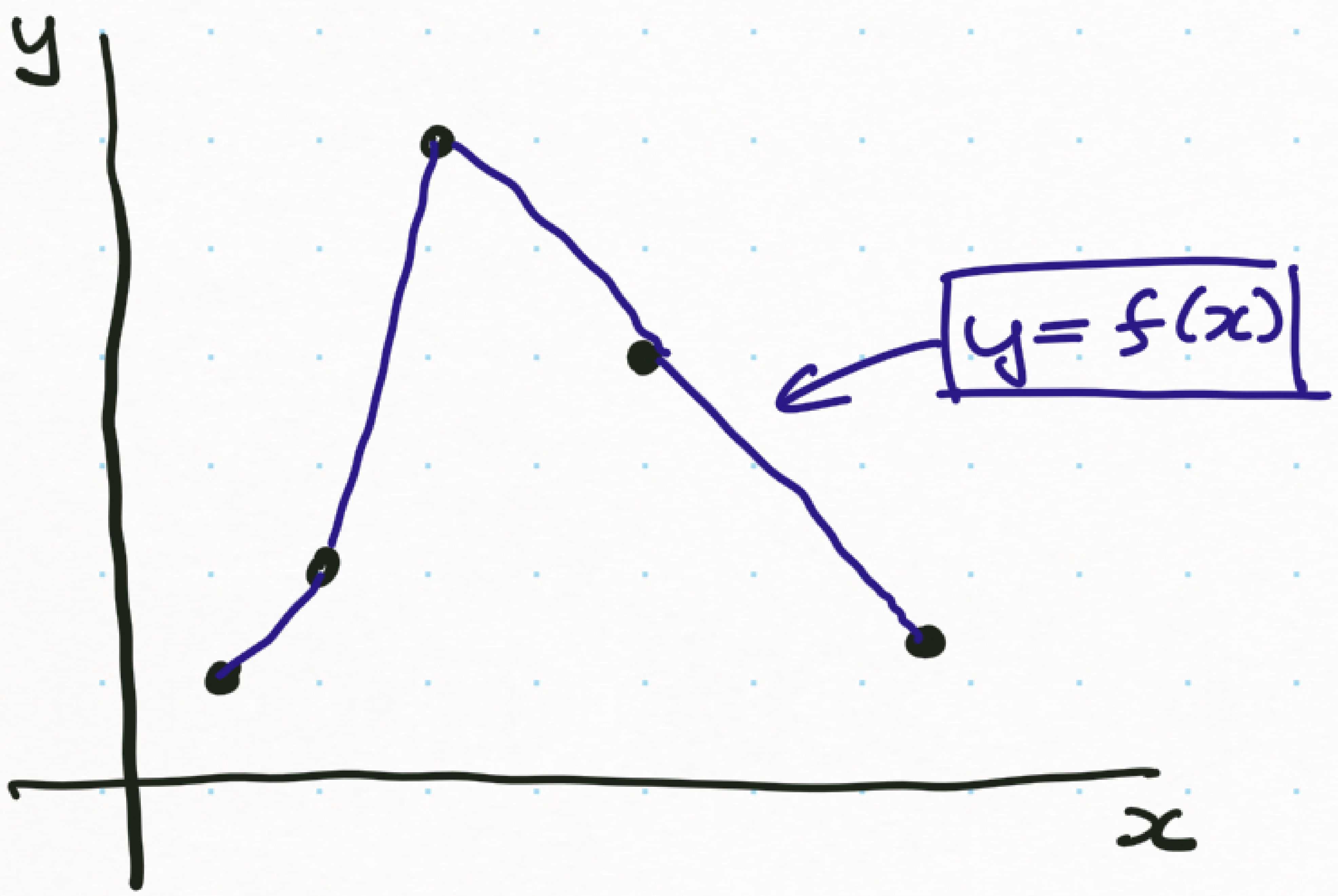

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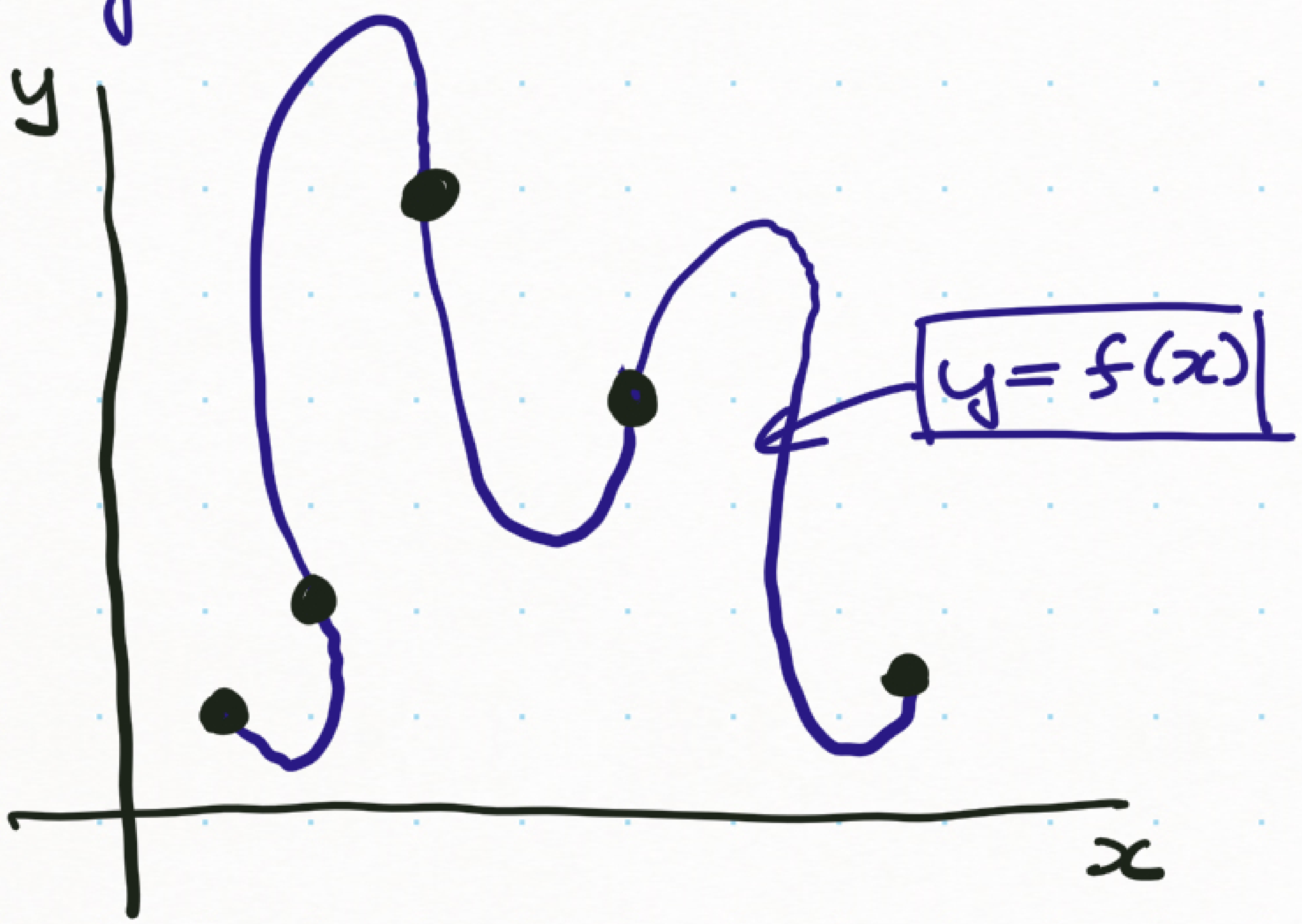

A model like this is rarely "explanatory" or generalizable, it only tries to be "predictive" but that's also tricky due to "overfitting".

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Sources of error in the Modeling process

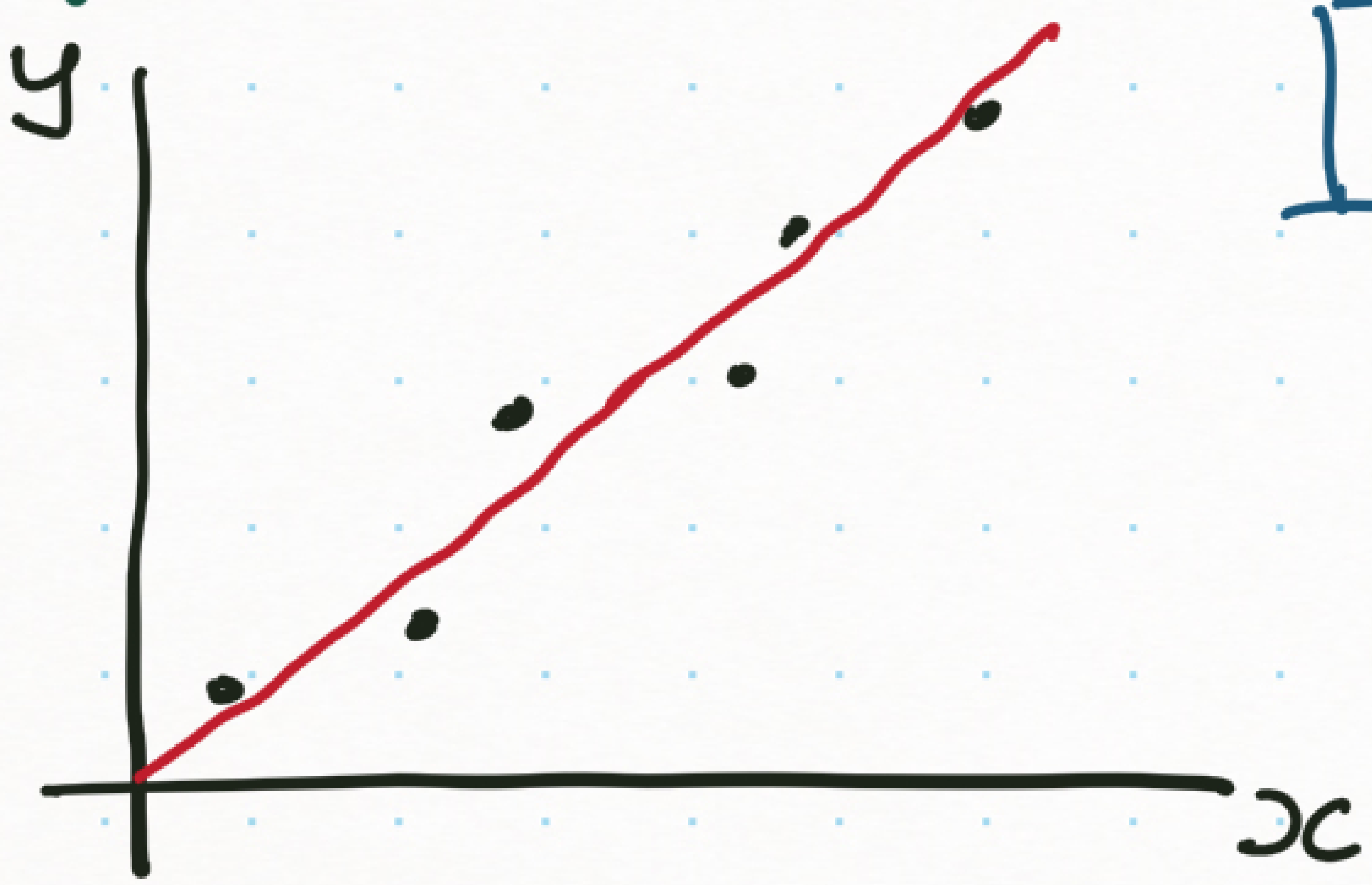
I. Qualitative

1. Oversimplification / Incorrect assumptions.
2. Missing variables / parameters.

II. Quantitative

1. Truncation errors.
   e.g. use $\sin x \approx x - x^3/3!$
2. Round-off errors.
   e.g. $0.333333 - 1/3 \neq 0$
3. Measurement errors.
   e.g. human / instrument errors in making observations.

To fit our model to given data, we have so far been doing it "visually" / "ad hoc" manner.



$\boxed{y \propto x \text{, ie, } y = kx}$

we estimate k as the slope of a line visually close to the data points

or we estimate k by taking the average of

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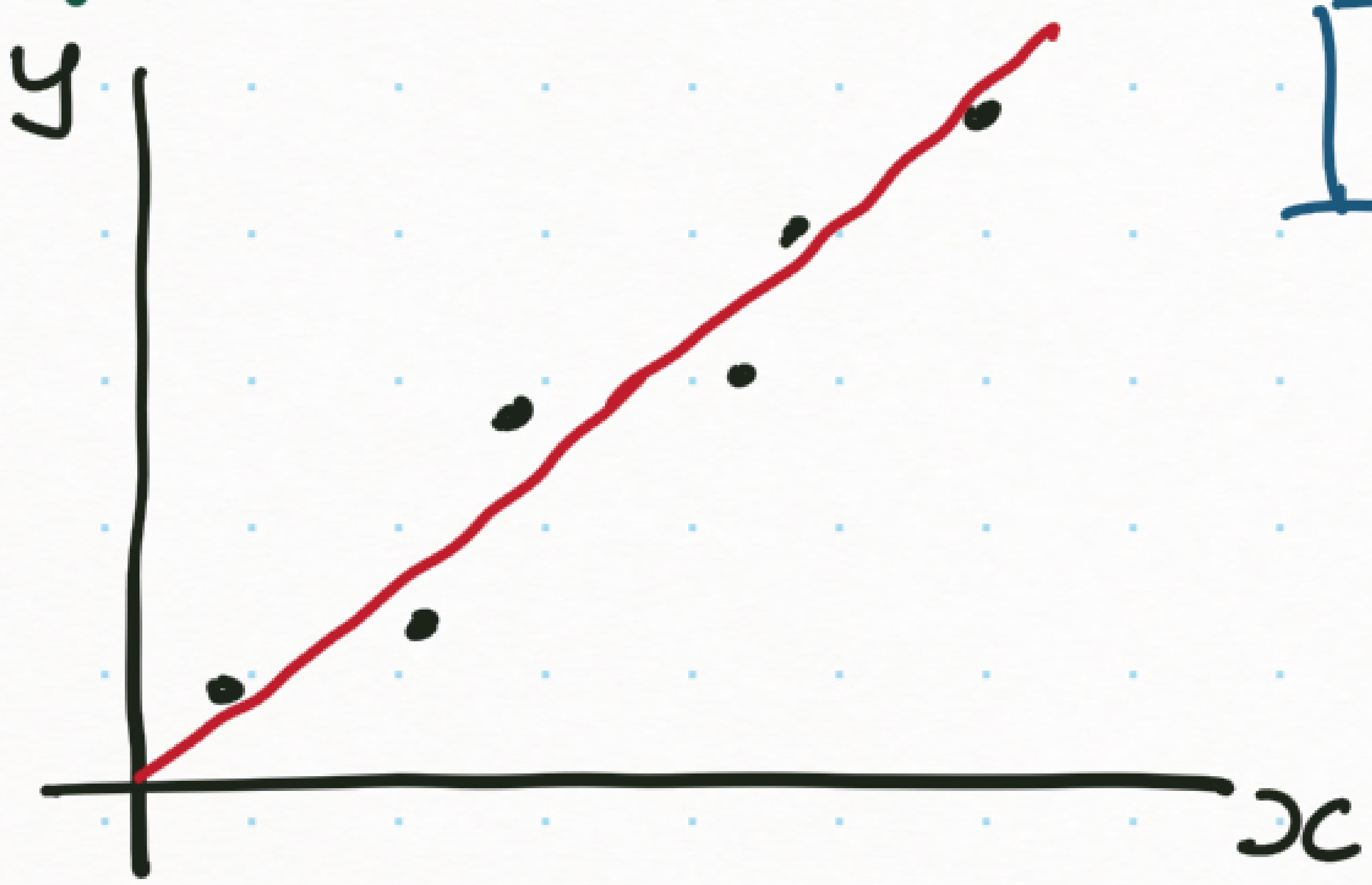

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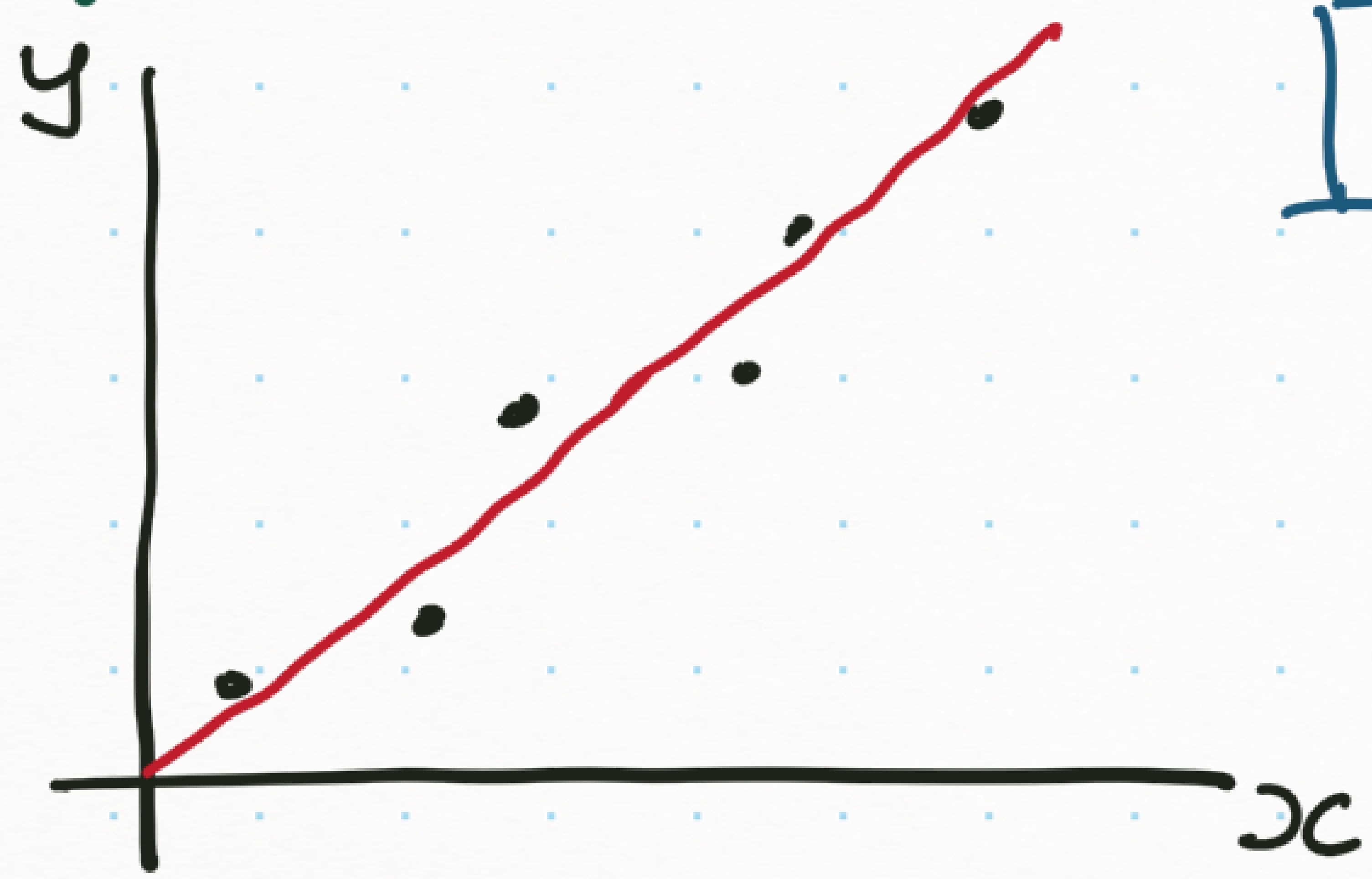

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e.g.

x	1	2	3	4
y	8.1	23	61	162

exponential growth $\Rightarrow$ <u>Model?</u>

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How to find it?

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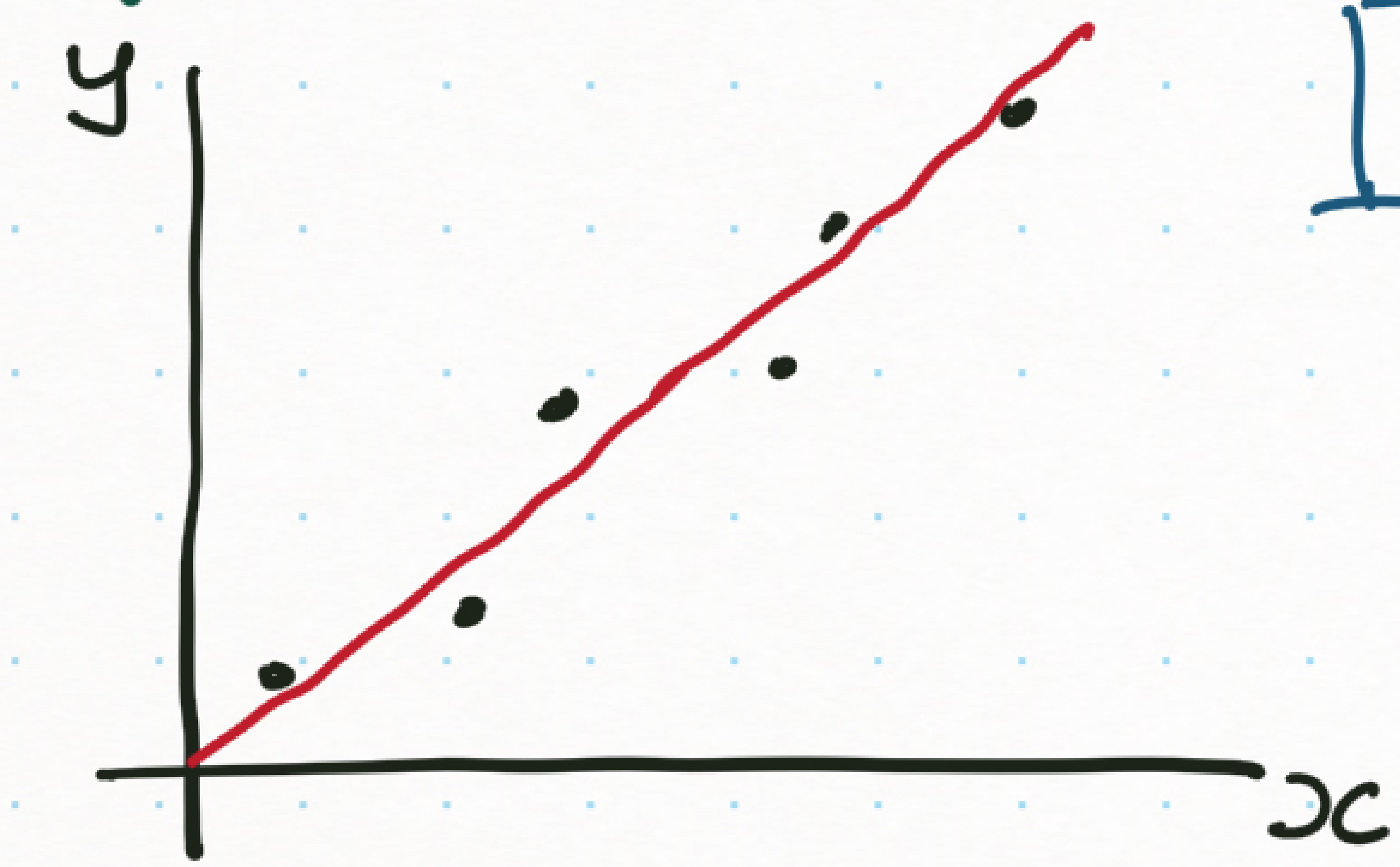

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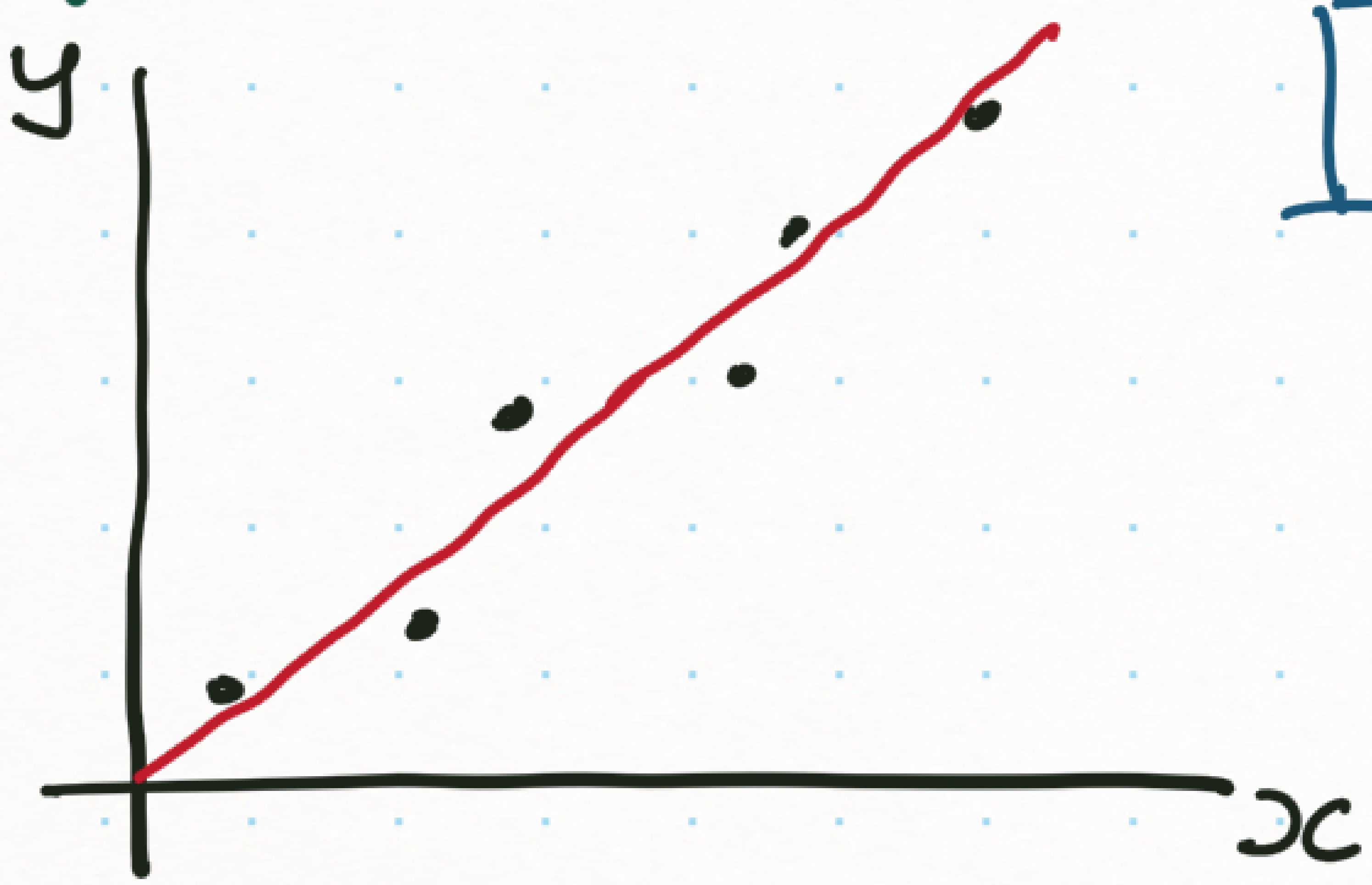

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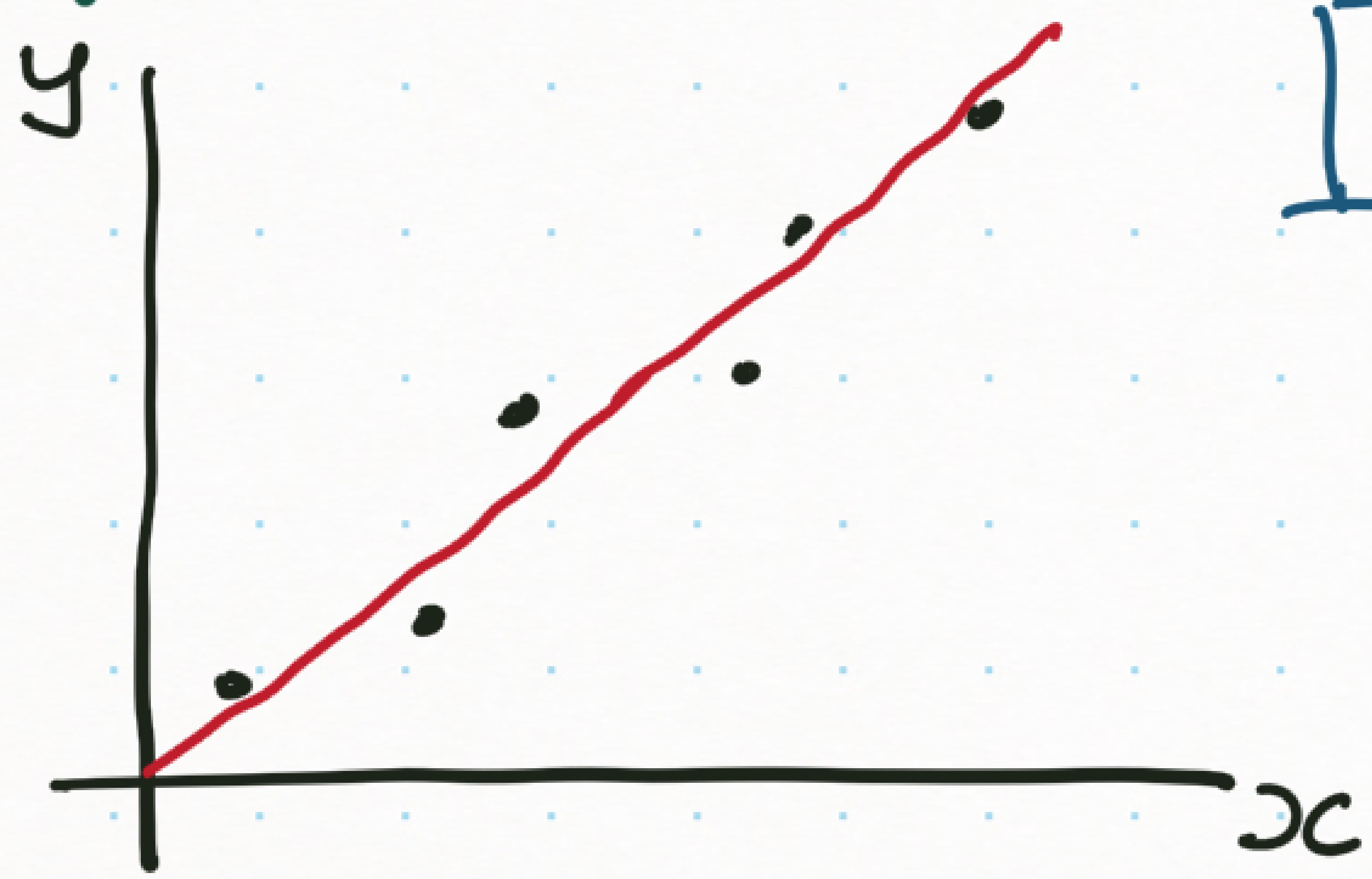

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