MATH 380

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Vertex cover problem as an optimization problem

Let $G = (V(G), E(G))$ be the given graph.

Suppose $V(G) = \{v_1, v_2, \ldots, v_n\}$.

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Let $x_i = \begin{cases} 1 & \text{if } v_i \in S \\ 0 & \text{if } v_i \notin S \end{cases}$

Using the variables x_i, how will you model the requirement that each edge must be incident to at least one vertex in S?

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$$x_i + x_j \geq 1 \quad \text{for all } v_iv_j \in E(G)$$

Objective function?

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$$\min \sum_{i=1}^{n} x_i$$

[minimize total # vertices picked]

s.t.

$$x_i + x_j \geq 1 \quad \forall v_i v_j \in E(G)$$

[each edge is "covered" by at least one vertex]

$$x_i \in \{0, 1\} \quad \forall i = 1, \ldots, n$$

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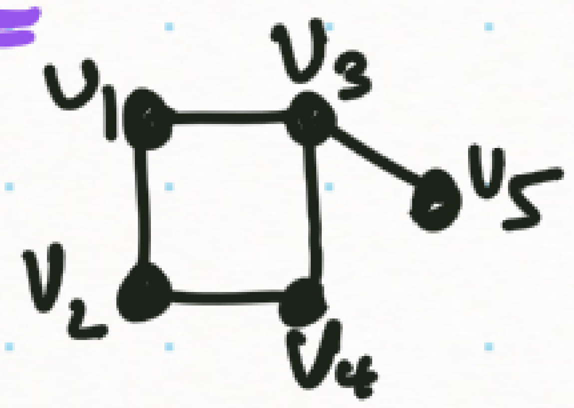
e.g.



$\min\ x_1+x_2+x_3+x_4+x_5$ s.t.

$x_1+x_2 \geq 1$	$x_2+x_4 \geq 1$	$x_3+x_5 \geq 1$
$x_1+x_3 \geq 1$	$x_3+x_4 \geq 1$	

$x_1, x_2, x_3, x_4, x_5 \in \{0,1\}$

Most Binary Optimization Problems are very hard to solve.
We use different ideas to get approximate / non optimal solutions

1. Heuristics based on greedy algorithms / local search.

2. Randomized Rounding schemes.

Consider $\min \vec{c}^T \vec{x}$ s.t. $A\vec{x} \le \vec{b}$, $x_1, x_2, \ldots, x_n \in \{0,1\}$ $\xrightarrow{\text{LP relaxation}}$ $\min \vec{c}^T \vec{y}$ s.t. $A\vec{y} \le \vec{b}$, $y_1, \ldots, y_n \in [0,1]$

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$\xleftarrow{\text{"round"}}$ each $y_i \in [0,1]$ to $x_i \in \{0,1\}$

e.g. $x_i = \begin{cases} 1 & \text{if } y_i \geq 0.5 \\ 0 & \text{if } y_i < 0.5 \end{cases}$

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Check whether these variables satisfy $A\vec{x} \le \vec{b}$

Issues
- may not be satisfied
- this solution might be far from optimal solution.

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Flip a coin with $\mathbb{P}[\text{heads}] = y_i$
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Compute the probability of "failure" & Expected value of objective function

Analyze this to guarantee a good / close to optimal solution with high probability

(Monte Carlo) Randomized Algorithm:

Assigning jobs to qualified applicants / processors / machines

Let $a_1, a_2, \ldots, a_n$ be jobs to be carried out.

Let $b_1, b_2, \ldots, b_m$ be a pool of available applicants / processors / machines...

Each b_j is capable of carrying out only certain jobs a_i.

We have to assign each job to one applicant qualified to do it.

How can we maximize the number of assigned jobs?

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Let $G=(V(G),E(G))$ be a graph that captures the given information as
$V(G) = V_a \cup V_b$ where $V_a = \{a_1, a_2, \dots, a_n\}$
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Edges between any pair a_i and b_j if applicant b_j is qualified for job a_i.

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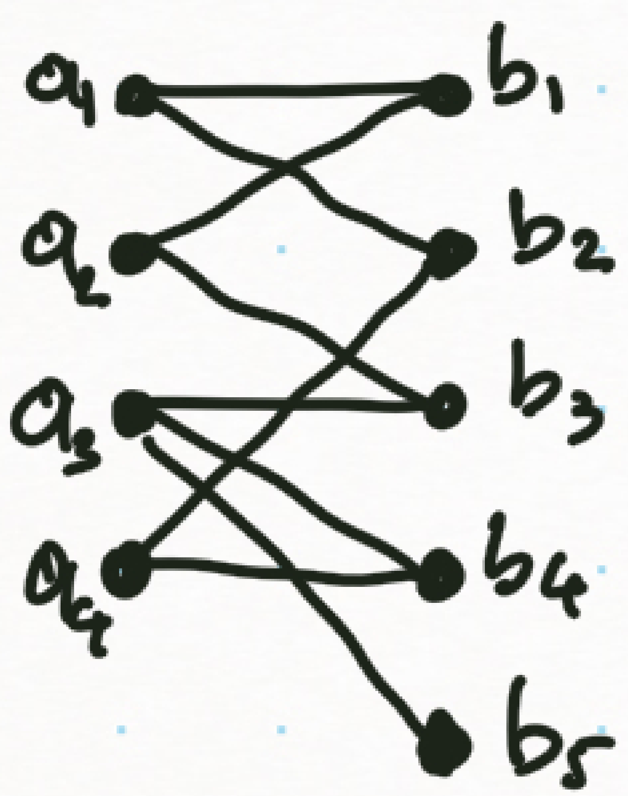

A graph whose vertices can be partitioned into two disjoint subsets such that all edges only go inbetween the two parts, is called a bipartite graph.

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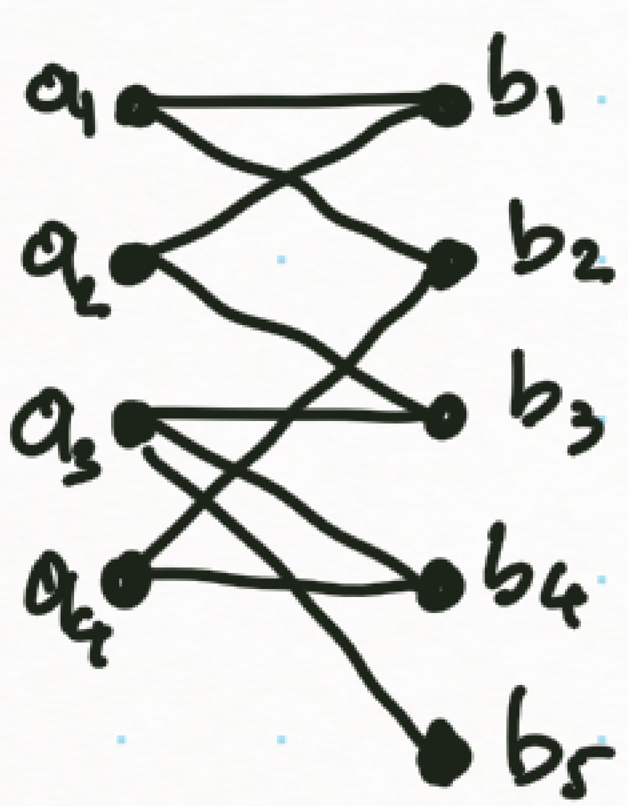

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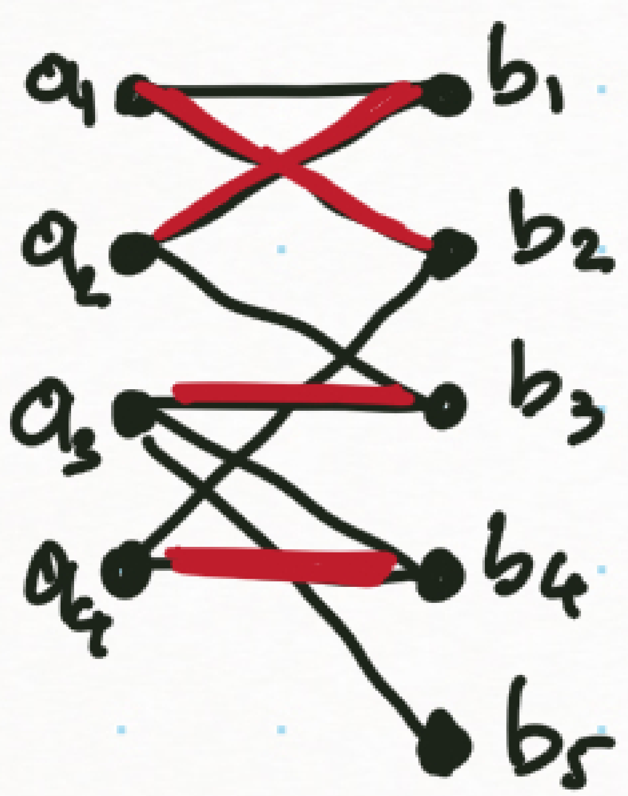


a_1 assigned to b_2

a_2 —— " —— b_1

a_3 —— " —— b_3

a_4 —— " —— b_4

A matching is subset of $E(G)$ such that all edges in it are vertex disjoint.

Maximum Matching: $\max_{M \text{ matching}} |M|$

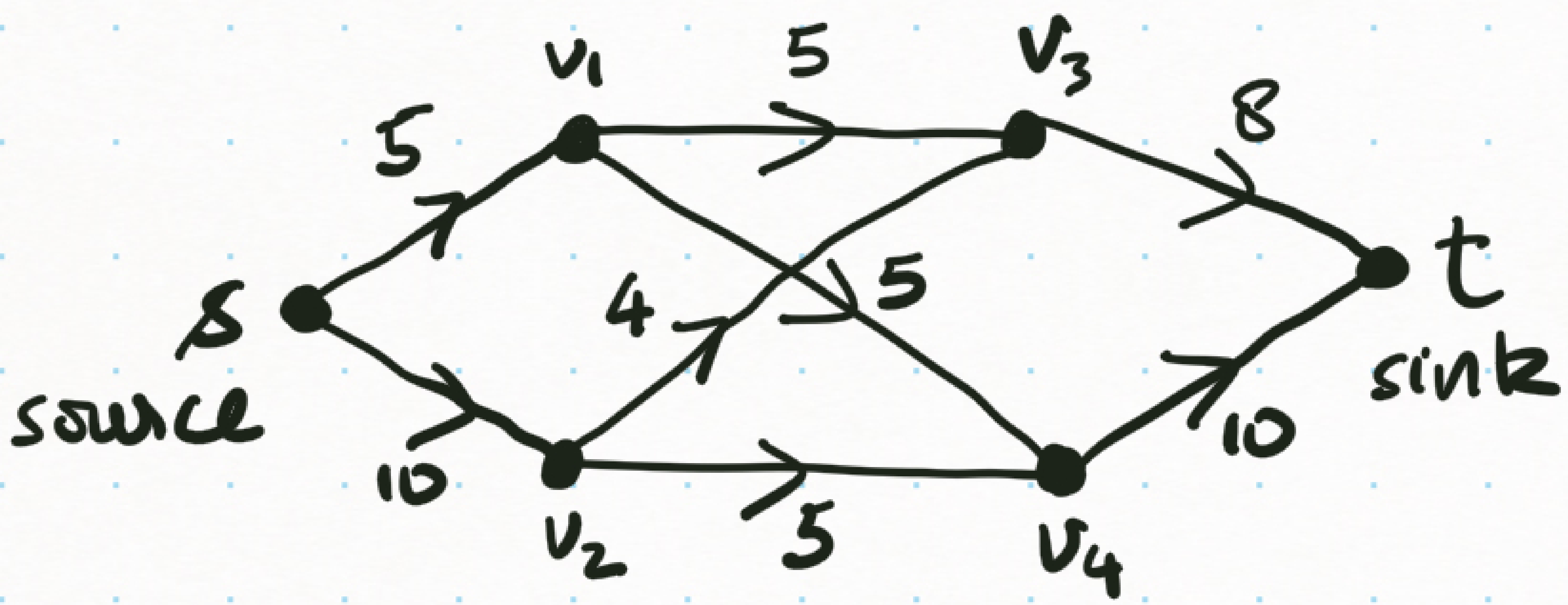
Maximum Matching problem in a bipartite graph can be modeled as a more general network optimization problem called the

Maximum Flow Problem

Given a network $G = (N, A)$ (nodes: N, arcs: A)

two special nodes $s \equiv$ source & $t =$ sink

capacity on each arc, $u_{ij} \geq 0$ for each arc $(v_i, v_j) \in A$



source $\equiv$ factory
sink $\equiv$ retail store
network $\equiv$ transportation netwk
capacity $\equiv$ how much can be transported using that link
$v_1, v_2, v_3, v_4 \equiv$ warehouses/ transhipment nodes.

Aim maximize the amount of flow from s to t while respecting the capacities

Flow $\equiv$ quantity of goods transported across each arc.

Max Flow Problem

Decision variables $\quad x_{ij}$ = amount of flow from v_i to v_j for each arc (v_i, v_j)

$$\max \sum_{i \in N^+(s)} x_{si}$$

such that

$$\sum_{v_k \in N^-(v_i)} x_{ki} - \sum_{v_j \in N^+(v_i)} x_{ij} = 0 \quad \forall v_i \in N$$

$$x_{ij} \leq u_{ij} \quad \forall (v_i, v_j) \in A$$

$$x_{ij} \geq 0 \quad \forall (v_i, v_j) \in A$$

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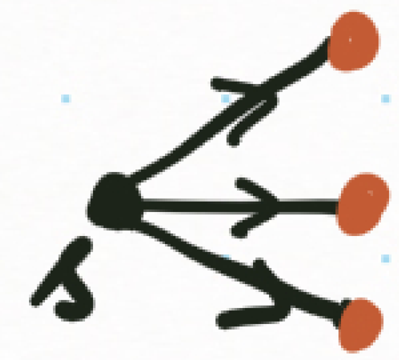

[total flow out of node s]

$N^+(s)$
out-neighborhood of s

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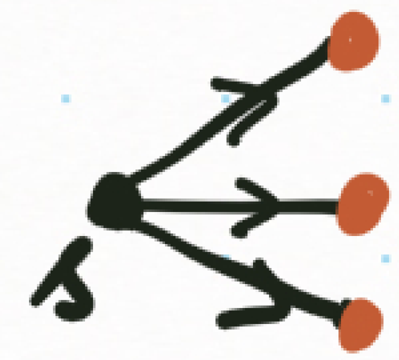

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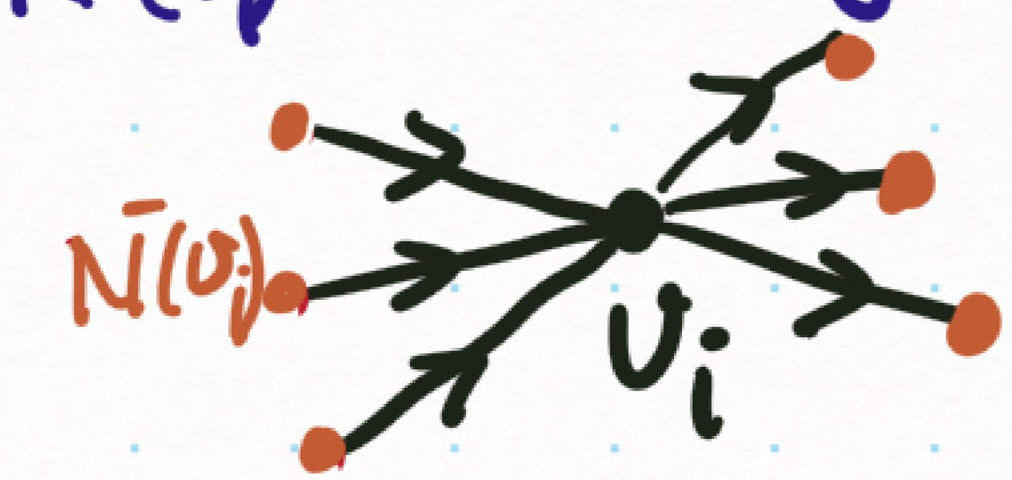
[In-flow = Out-flow at each node v_i]



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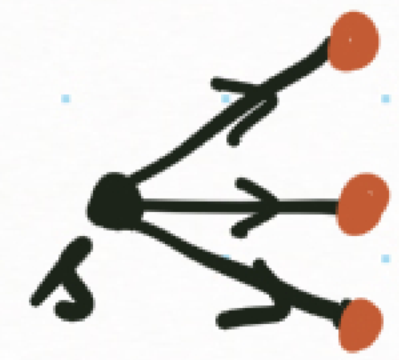

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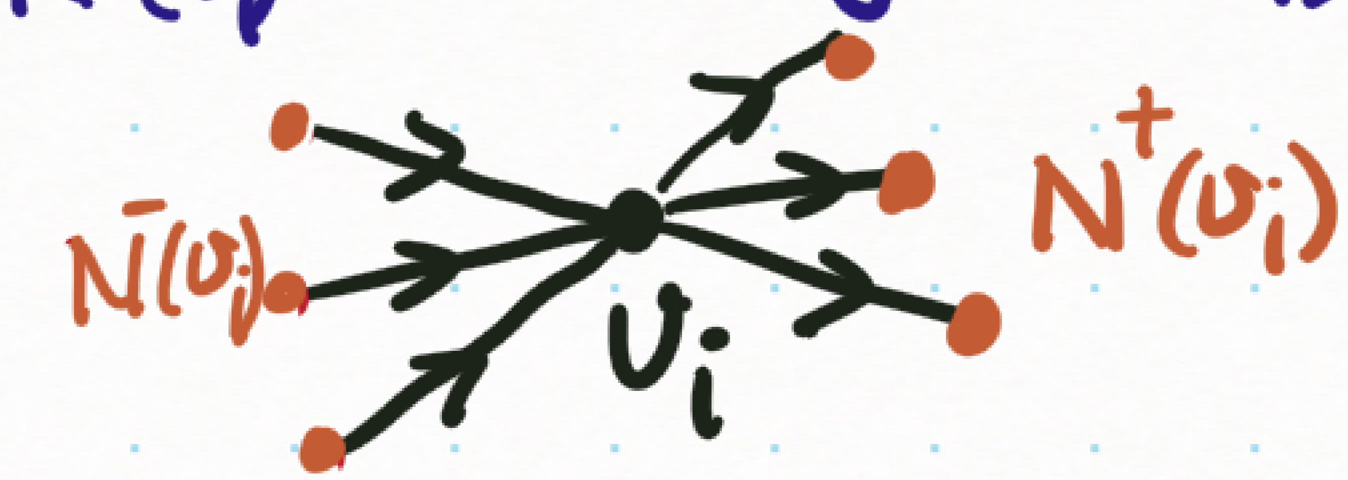


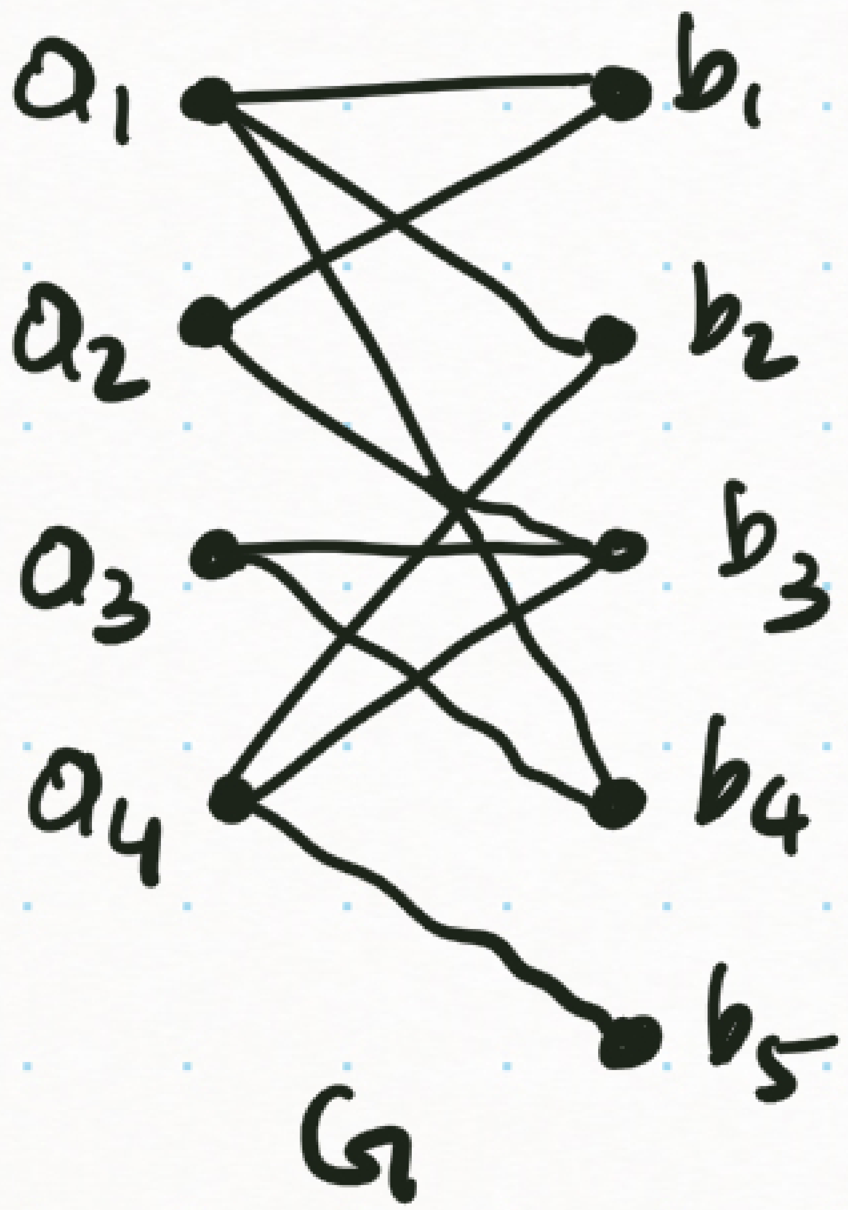
$$x_{ij} \leq u_{ij} \quad \forall (v_i, v_j) \in A$$

[capacity on each arc $v_i v_j$]

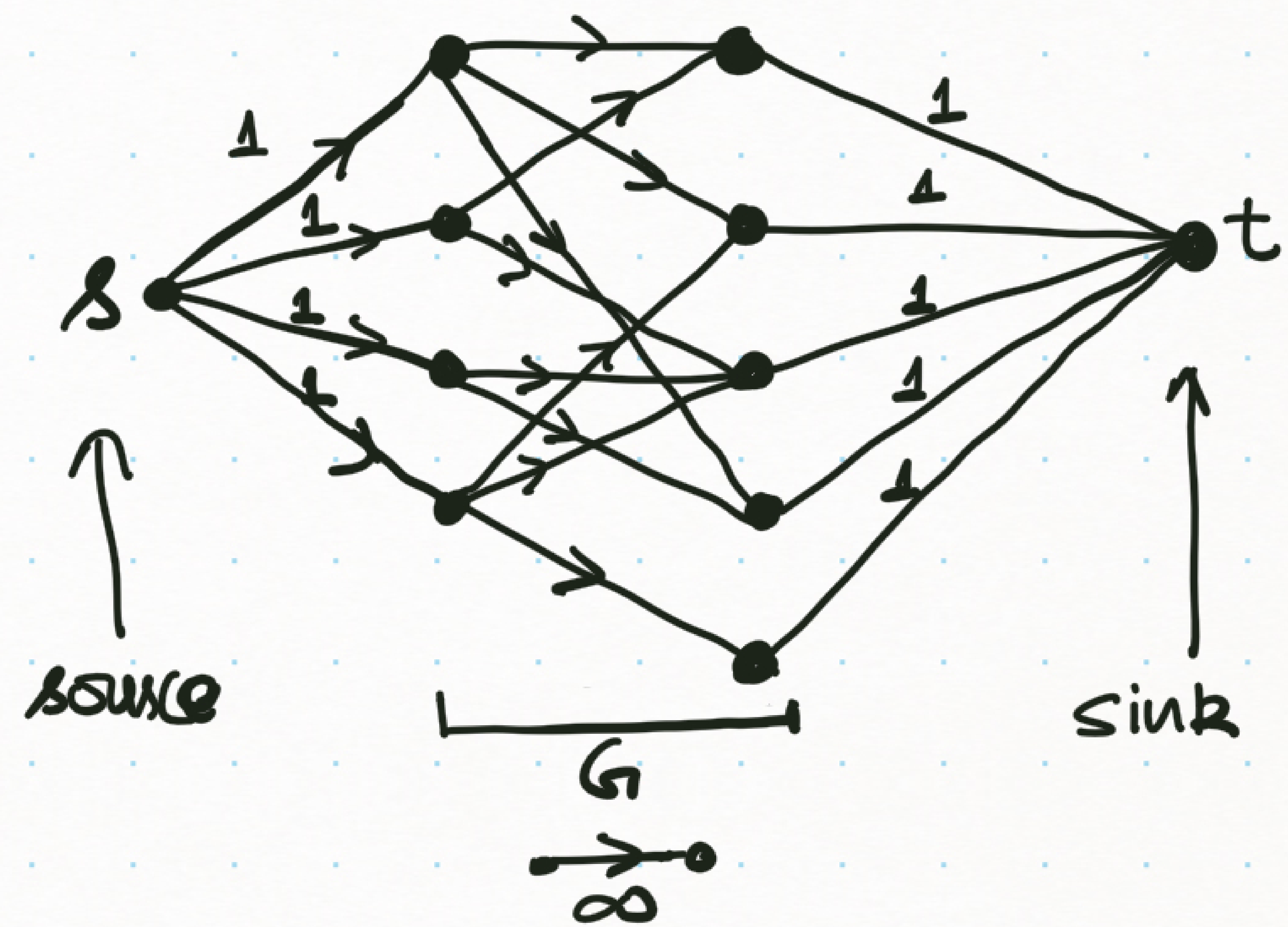
$$x_{ij} \geq 0 \quad \forall (v_i, v_j) \in A$$

[non-negativity]

Finding max matching on



is the same as finding the max flow on



Modeling Logistics problem on a transportation network

We have a transportation network connecting factories, warehouses and retail stores. Each factory can supply certain amount of goods and Each store has a demand for a certain amount of goods. How can we satisfy demand while minimizing transportation costs?

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Given a network $G=(N,A)$

for each arc (v_i, v_j), we are given capacity bounds $l_{ij} \le x_{ij} \le u_{ij}$

and there is cost per unit amount of transportation on that arc c_{ij}

for each node v_i, There is supply/demand $b(i)$ associated with it [$b(i) > 0$ means supply from factory

$b(i) < 0$ means demand from store

$b(i) = 0$ means transhipment point]

Decision variables x_{ij} = flow over the arc (v_i, v_j) for all arcs $(v_i, v_j) \in A$. $\quad v_i \longrightarrow v_j$

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Transportation Problem

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[capacity bound on flow over each arc]

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[Outflow − Inflow = "supply" at node v_i]

Transportation Problem

<u>Read</u> The Seven Bridges of Königsberg,
Euler's problem, and
how to solve this topological problem
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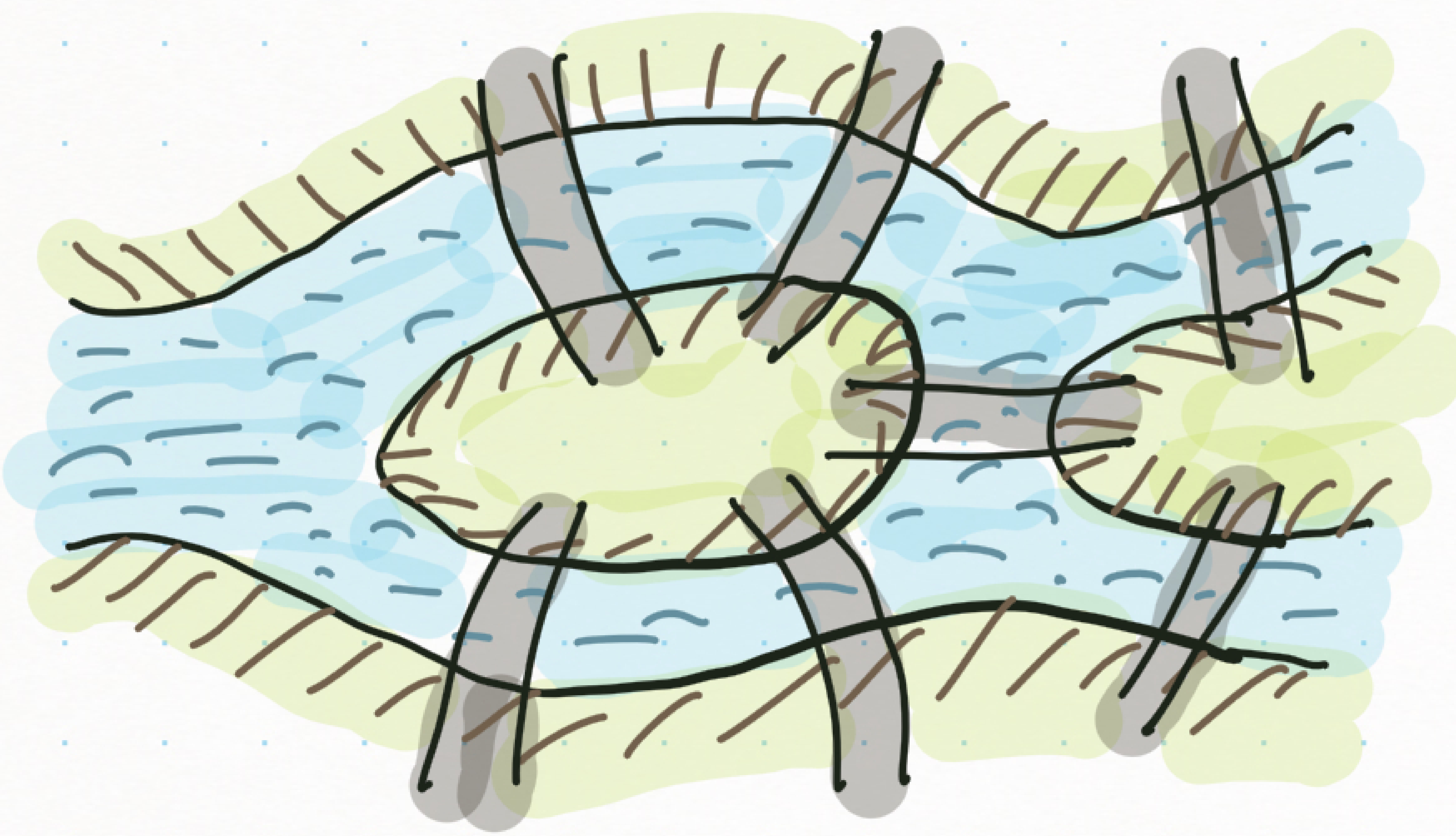

Start & end a walk at the same spot and cross each of the 7 bridges exactly once during the walk. ← Can this be done?

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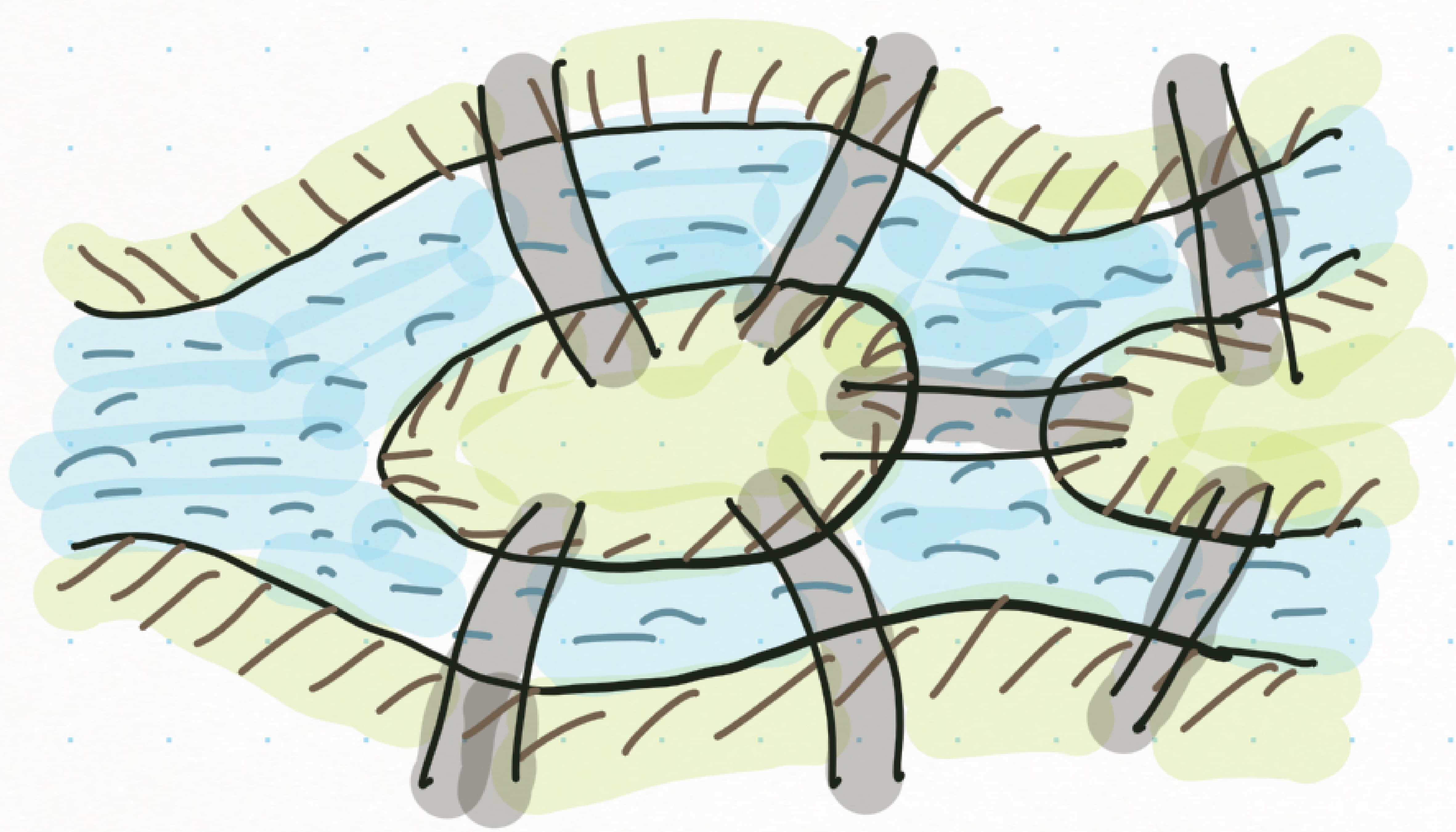
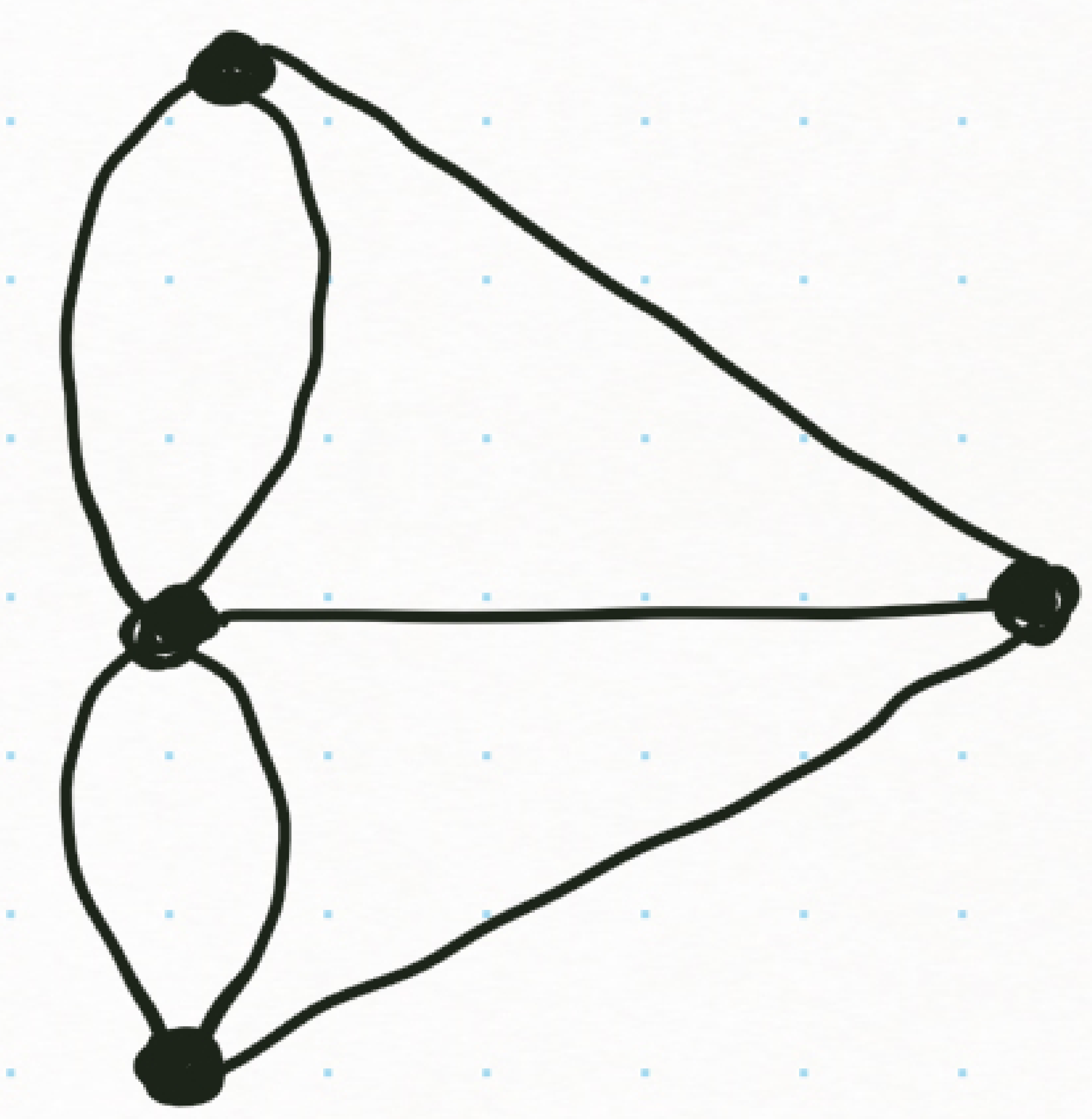
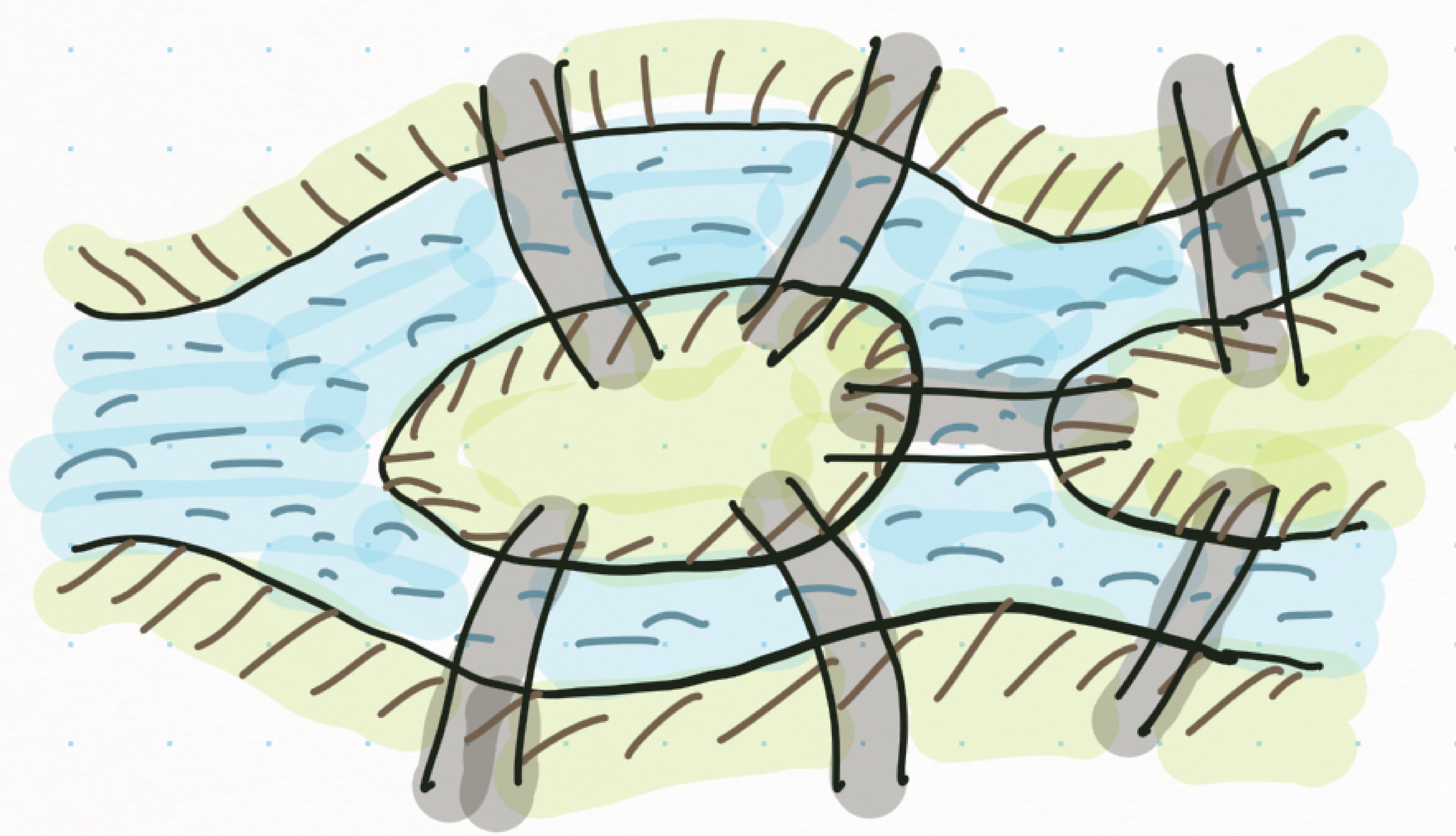
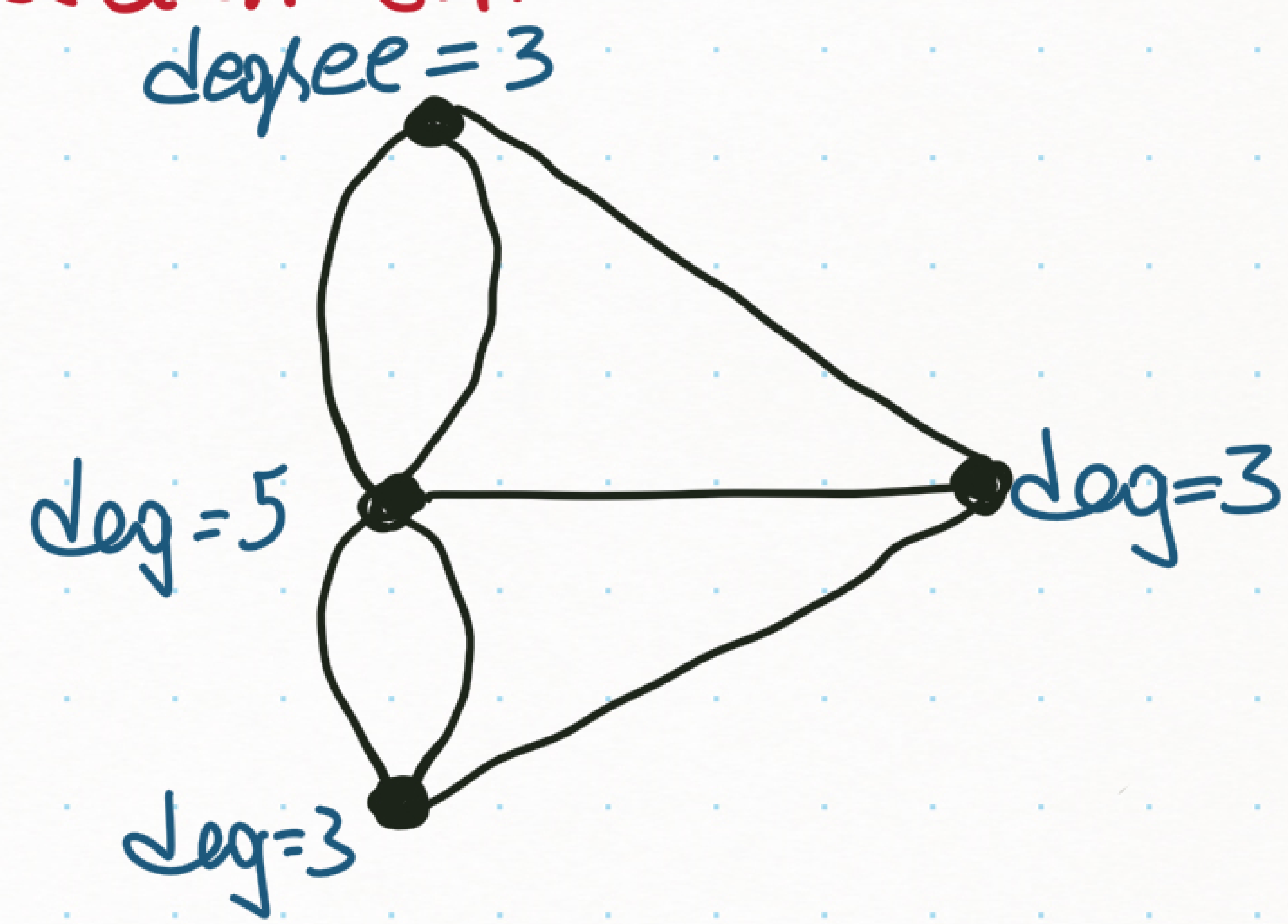




Start & end a walk at the same spot and cross each of the 7 bridges exactly once during the walk. ← Can this be done?

Read The Seven Bridges of Königsberg, Euler's problem, and how to solve this topological problem using graph theory in section 8.1.





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