Welcome to

MATH 400 Real Analysis.

PART#1

We are concerned about

→ What is a real number?

→ How do we find the limit of a sequence of numbers?

→ Can we add infinitely many real numbers together and still get a finite number?

→ Can we rearrange the elements of an infinite sum, and still have the same sum?

→ What is a function?
What is does it mean for a function to be continuous? differentiable? integrable? bounded?

Limitations of Calculus

① Division by zero

Cancellation law $ac = bc \Rightarrow a = b$ does not work when $c = 0$

But often times such division by zero can be hidden.

In calculus, we have to be careful about defining functions and limits when a denominator could become zero (or "close" to zero).

② Divergent Series

Recall $S = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots\cdots$

then using the trick $2S = 2 + 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots\cdots$

$= 2 + S$

& hence $S = 2$

Apply the same trick to the series

$$S = 1 + 2 + 4 + 8 + 16 + \cdots\cdots$$

$\Rightarrow 2S = 2 + 4 + 8 + 16 + \cdots\cdots = S - 1$

$\Rightarrow S = -1$

Another example: $S = 1 - 1 + 1 - 1 + 1 - 1 + \cdots\cdots$

$S = (1-1) + (1-1) + \cdots\cdots = 0 + 0 + 0 + \cdots = 0$

$S = 1 - (1 - 1 + 1 - 1 + 1 - 1 + \cdots) = 1 - S \Rightarrow S = 1/2$ $\Leftrightarrow \frac{1}{2} = 0$??

③ Divergent sequences

Let r be a fixed real number.

Let $L = \lim_{n\to\infty} r^n$

Changing variables $n = m+1$,

$$L = \lim_{m+1\to\infty} r^{m+1} = \lim_{m+1\to\infty} r\, r^m = r \lim_{m+1\to\infty} r^m$$

$$= r \lim_{m\to\infty} r^m = rL$$

$\Rightarrow rL = L$

$\Rightarrow$ either $r=1$ or $L=0$

$\Rightarrow \lim_{n\to\infty} r^n = 0$ if $r \neq 1$??

What if $r = 20$?

(4) Limiting values of functions

Recall $\sin(y+\pi) = -\sin(y)$

So, $\lim\limits_{x\to\infty} \sin(x) = \lim\limits_{y+\pi\to\infty} \sin(y+\pi) = \lim\limits_{y\to\infty} (-\sin(y)) = -\lim\limits_{y\to\infty} \sin(y)$

$\Rightarrow \lim\limits_{x\to\infty} \sin(x) = -\lim\limits_{x\to\infty} \sin(x) \quad \Rightarrow \lim\limits_{x\to\infty} \sin(x) = 0$

Recall $\sin(z+\frac{\pi}{2}) = \cos(z)$

so, $\lim\limits_{x\to\infty} \sin(x) = \ ? \ ? \ ? \ ? = \lim\limits_{z\to\infty} \cos(z)$

$\Rightarrow \lim\limits_{x\to\infty} \cos(x) = 0$

$\therefore \lim\limits_{x\to\infty} (\sin^2(x) + \cos^2(x)) = 0^2 + 0^2 = 0$

BUT!!

$\sin^2 x + \cos^2 x = 1 \ \forall x \Rightarrow \lim\limits_{x\to\infty} \sin^2 x + \cos^2 x = 1$

$0 = 1$??

(5) Interchanging Sums

Take a matrix of numbers, e.g. $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ & compute the sums of all rows and all columns & then total those.

$$\begin{array}{cc} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} & \begin{array}{c} 6 \\ 15 \\ 24 \end{array} \\ \begin{array}{ccc} 12 & 15 & 18 \end{array} & 45 \end{array}$$

$$\sum_{j=1}^{n} \sum_{i=1}^{m} a_{ij} = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij}$$

But what about $\displaystyle\sum_{i=1}^{\infty} \sum_{i=1}^{\infty} a_{ij} = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij}$?

$$\begin{array}{cc} \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots \\ -1 & 1 & 0 & 0 & \cdots \\ 0 & -1 & 1 & 0 & \cdots \\ 0 & 0 & -1 & 1 & 0 \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix} & \begin{array}{c} 1 \\ 0 \\ 0 \\ 0 \\ \vdots \end{array} \\ \begin{array}{ccccc} 0 & 0 & 0 & 0 & 0 \end{array} & \end{array}$$

$1 = 0$? ?

⑥ Interchanging integrals

Recall computing volume under a surface $z = f(x,y)$

By slicing parallel to x-axis for each fixed value of y, we compute the area $\int f(x,y)\,dx$ & then integrate that area in y to get: Volume $= \int\int f(x,y)\,dx\,dy$

By slicing parallel to y-axis: Volume $= \int\int f(x,y)\,dy\,dx$

Can we always interchange integrals?

$$\int_0^1 (e^{-xy} - xy e^{-xy})\,dy = y e^{-xy}\Big|_{y=0}^{y=1} = e^{-x}$$

gives us

and $$\int_0^\infty (e^{-xy} - xy e^{-xy})\,dx = x e^{-xy}\Big|_{x=0}^{x=\infty} = 0$$

① $$\int_0^\infty \int_0^1 (e^{-xy} - xy e^{-xy})\,dy\,dx = \int_0^\infty e^{-x}\,dx = -e^{-x}\Big|_0^\infty = 1$$

② $$\int_0^1 \int_0^\infty (e^{-xy} - xy e^{-xy})\,dx\,dy = \int_0^1 0\,dx = 0$$

$1 = 0$??

(7) Interchanging Limits

$$\lim_{x\to 0}\lim_{y\to 0}\frac{x^2}{x^2+y^2} = \lim_{y\to 0}\lim_{x\to 0}\frac{x^2}{x^2+y^2}$$

Note $\rightarrow$ $\lim_{y\to 0}\frac{x^2}{x^2+y^2} = \frac{x^2}{x^2+0^2} = 1 \Rightarrow \lim_{x\to 0}\lim_{y\to 0}\frac{x^2}{x^2+y^2} = 1$

$\lim_{x\to 0}\frac{x^2}{x^2+y^2} = \frac{0^2}{0^2+y^2} = 0 \Rightarrow \lim_{y\to 0}\lim_{x\to 0}\frac{x^2}{x^2+y^2} = 0$

$0 = 1$??

(8) Interchanging limits and integrals

For any $y \in \mathbb{R}$, $\displaystyle\int_{-\infty}^{\infty} \frac{1}{1+(x-y)^2}\,dx = \arctan(x-y)\Big|_{x=-\infty}^{\infty} = \frac{\pi}{2} - \left(-\frac{\pi}{2}\right)$

$= \pi$

Taking limits as $y \to \infty$, we get

$$\int_{-\infty}^{\infty} \lim_{y\to\infty} \frac{1}{1+(x-y)^2}\,dx = \lim_{y\to\infty} \int_{-\infty}^{\infty} \frac{1}{1+(x-y)^2}\,dx = \pi$$

But for every x, $\displaystyle\lim_{y\to\infty} \frac{1}{1+(x-y)^2} = 0$

$$\Rightarrow \int_{-\infty}^{\infty} \lim_{y\to\infty} \frac{1}{1+(x-y)^2}\,dx = \int_{-\infty}^{\infty} 0\,dx = 0$$

$\pi = 0$??

9 L'Hopital's Rule

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)} \quad \text{if } f(x) \,\&\, g(x) \to 0 \text{ as } x \to x_0$$

e.g. $$\lim_{x \to 0} \frac{x^2 \sin(x^{-4})}{x} = \lim_{x \to 0} x \sin(x^{-4})$$

$$= 0$$

$$= \lim_{x \to 0} \frac{2x \sin(x^{-4}) - 4x^{-3} \cos(x^{-4})}{1}$$

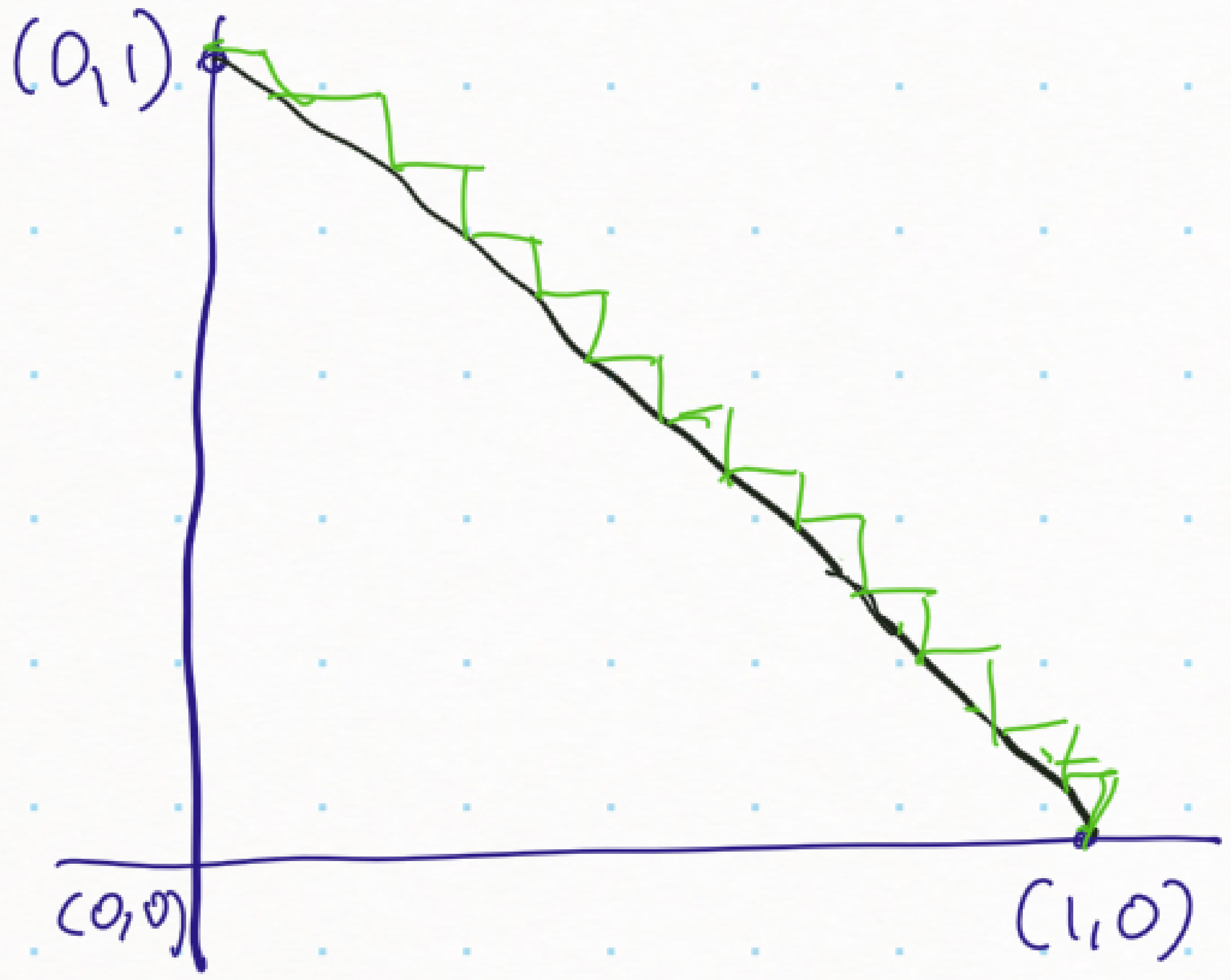
$$= \underbrace{\lim_{x \to 0} 2x \sin(x^{-4}) - \lim_{x \to 0} \frac{4 \cos(x^{-4})}{x^3}}_{\text{is divergent}}$$

$\downarrow$ 0 by squeeze test (try it!)

$\downarrow$ divergant

is divergent

(10) Limits and Lengths



What is the length of the hypotenuse?

Pick a large N and approximate the hypotenuse by a "staircase" of N horizontal edges & N vertical edges of the same length $\left(= \frac{1}{N}\right)$

$\therefore$ Total length of the staircase $= 2\left(\frac{1}{N}\right)(N)$

As $N \to \infty$, "staircase" becomes closer & closer to the hypotenuse, so length of the hypotenuse

$$= \lim_{N \to \infty} \left(2 \tfrac{1}{N} N\right) = \lim_{N \to \infty} 2 = 2$$

By Pythagoras, $\sqrt{2}$.

$2 = \sqrt{2}$??

MATH 400 Real Analysis

PART #2

"Natural numbers are the work of God. All the rest is the work of mankind"

— Kronecker (1823-1891)

$\mathbb{N} = \{1, 2, 3, \ldots\ldots\}$ sometimes w. 0

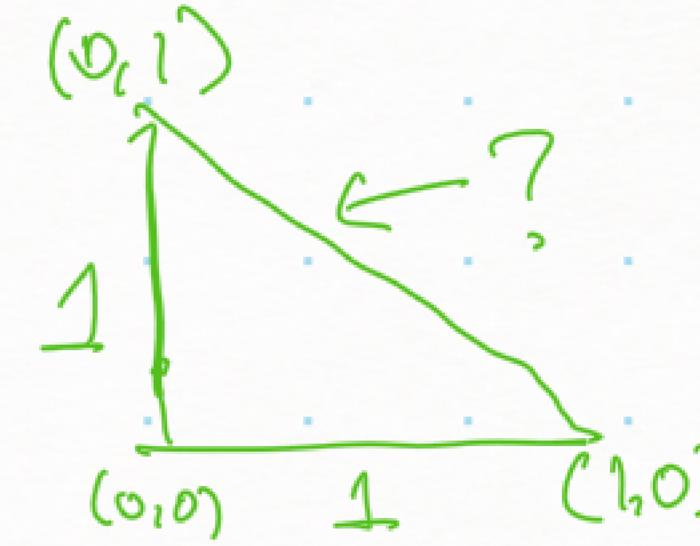
$\mathbb{Z}$ as solutions to $x + k = 0$, $k \in \mathbb{N}$

$\mathbb{Q}$ as solutions to $k_1 x + k_2 = 0$, $k_1, k_2 \in \mathbb{Z}$

$\mathbb{R}$ as ??

$\mathbb{C} = \{r_1 + i r_2 \mid r_1, r_2 \in \mathbb{R}\}$
arose out of attempts to define solutions to equations such as $x^2 + 1 = 0$, etc.
[But there is more than that in $\mathbb{C}$]

Is $\mathbb{R}$ the same as $\mathbb{Q}$?



Thm There is no solution in $\mathbb{Q}$ for the equation $x^2 = 2$

There is no rational number whose square is 2.

Proof For contradiction, assume $\exists\, p, q \in \mathbb{Z}$ s.t. $\left(\frac{p}{q}\right)^2 = 2$

we may assume that p & q have no common factor & write the fraction in lowest terms.

$p^2 = 2q^2 \Rightarrow p^2$ is even $\Rightarrow p$ is even

$p = 2k+1$

$p^2 = (2k+1)^2$

$= 4k^2 + 1 + 4k$

$= 2(2k^2+2k) + 1$

$\Rightarrow 2 \mid p$

$\Rightarrow p = 2r$

$\Rightarrow (2r)^2 = 2q^2$

$\Rightarrow 4r^2 = 2q^2$

$\Rightarrow q^2 = 2r^2 \Rightarrow q^2$ is even $\Rightarrow q$ is even

$\Rightarrow 2 \mid q$

$\rightarrow$ p & q have a common factor

$\times$

$\mathbb{N}$ allows addition but no subtraction

$\mathbb{Z}$ allows addition & subtraction and multiplication but no division

$\mathbb{Q}$ allows addition & subt. and mult. & division.

? closed under the op.
additive id. & additive inverses
multiplicative id. & mult. inverses
commutative $xy = yx$ $\quad x+y = y+x$
associative $(xy)z = x(yz)$
distributive $a(b+c) = ab+ac$

Field

$\mathbb{Q}$ also has a natural order on it : $r, s \in \mathbb{Q} \Rightarrow$ exactly one of following is true

$r < s$,
$r = s$
$r > s$.

Plus $r < s$, $s < t \Rightarrow r < t$
Transitive property.

$\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ all have a natural order

$\mathbb{N}$, $\mathbb{Z}$ are not fields but $\mathbb{Q}$ is a field

$\mathbb{N}$, $\mathbb{Z}$ have intervals of gaps, but $\mathbb{Q}$ is dense

Given any two $r < s \in \mathbb{Q}$, there is a $t \in \mathbb{Q}$ s.t. $r < t < s$

$$t = \frac{r+s}{2}$$

So what does $\mathbb{Q}$ lack?

It has lots of "holes" like $\sqrt{2}$, $\sqrt{3}$, ...

Although note that we can approximate these irrational numbers quite well using rationals

e.g. $(1.41)^2 = 1.9881$

$(1.414)^2 = 1.999396$ closer & closer to "$\sqrt{2}$"

. . . . .

MATH 400

Real Analysis

PART #3

Review notation & Terminology from Section 1.2

→ **Induction** example 1.2.7

→ **Sets** examples 1.2.1 & 1.2.2

$A \cup B$, $A \cap B$, A^c

$A \subseteq B$

$\bigcup_{i=1}^{\infty} A_i$ $\bigcap_{i=1}^{\infty} A_i$

De Morgan's Laws

① $(A \cap B)^c = A^c \cup B^c$

② $(A \cup B)^c = A^c \cap B^c$

Functions

(Dirichlet 1830s)

Domain   coDomain

$$f : A \longrightarrow B$$

$$\underset{\in A}{x} \longmapsto \underset{\in B}{f(x)}$$

$\text{Range}(f) = \{ y \in B \mid y = f(x) \text{ for some } x \in A \}$

Beyond the notion of "f is a formula/expression"

example (Dirichlet 1829)

$$g(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

$g : \mathbb{R} \longrightarrow \mathbb{R}$

$\text{Range}(g) = \{0, 1\}$

A special function

Absolute value function

$|\cdot| : \mathbb{R} \longrightarrow \mathbb{R}^+ \cup \{0\}$ } non-negative reals

$\mathbb{R}^+$: positive reals $= \{x \in \mathbb{R} \mid x > 0\}$

$|\cdot|$: distance of x from 0

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

e.g.

Properties

① $|ab| = |a||b|$

② Triangle Inequality $|a+b| \leq |a| + |b|$

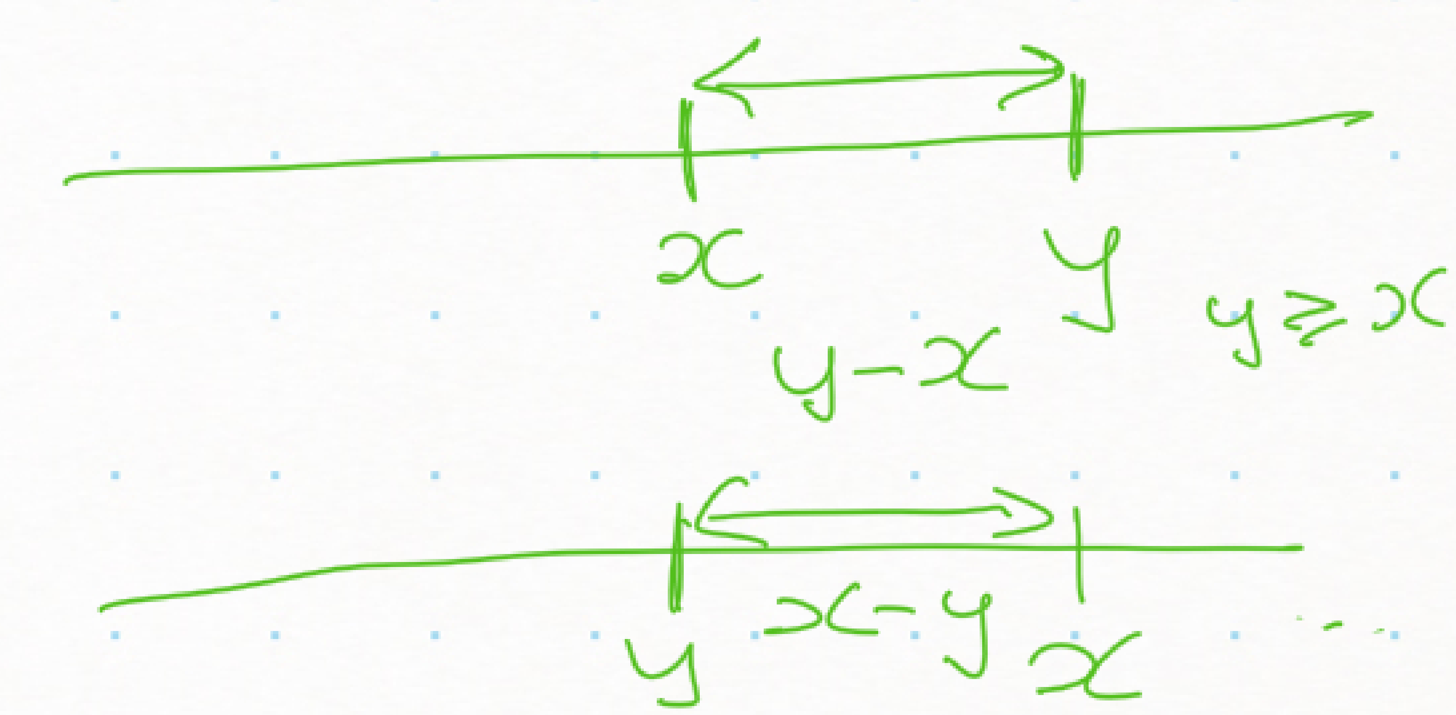
Corollary Let $a, b, c \in \mathbb{R}$

$|a-b| \leq |a-c| + |c-b|$

Pf: $|a-b| = |(a-c) + (c-b)|$
$\leq |a-c| + |c-b|$

distance between x & y

$\boxed{d(x,y) = |x-y|}$ ☆



An example of a "Real Analysis-type" proof

Thm Let $a, b \in \mathbb{R}$.
$a = b$ if and only if $|a-b| < \epsilon \ \forall \epsilon > 0$

Two real numbers a & b are equal iff for every real number $\epsilon > 0$ we have $|a-b| < \epsilon$

$1 = 0.999\ldots$
$\epsilon = 0.00001$
$|1 - 0.9999\ldots|$
$= 0.0000000\ldots$
< 0.0001

Proof $\boxed{\Rightarrow}$ Assume $a = b$
$|a-b| = 0 < \epsilon$ for every $\epsilon > 0$.

$\boxed{\Leftarrow}$ $|a-b| < \epsilon \ \forall \epsilon > 0 \Rightarrow a = b$

Contrapositive $a \neq b \Rightarrow \exists \epsilon_0 > 0$ s.t. $\boxed{|a-b| \geq \epsilon_0}$

Let $\epsilon_0 = |a-b| > 0$ (which is positive since $a \neq b$)

$\Rightarrow |a-b| \geq \epsilon_0$ ✓

scratch
define $\epsilon_0 > 0$ using a, b
& compare it to $|a-b|$

$|a-b| \neq 0$
$|a-b| > 0$

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Real Analysis

PART #4

Recall, we think of $\mathbb{R}$ as an extension of $\mathbb{Q}$ in which there are no "holes" (like $\sqrt{2}$); every length along the number line corresponds to a real number.

<u>More formally</u>

$\mathbb{R}$ is a complete ordered field which contains $\mathbb{Q}$

Field addition, multiplication, inverses under both, 0 & 1, commutative, associative, distributive.

Ordered For $r, s, t \in \mathbb{R}$

$\rightarrow r < s$ or $r = s$ or $r > s$

$\rightarrow r \leq s$ & $s \leq r \Rightarrow r = s$

$\rightarrow r \leq s$ & $s \leq t \Rightarrow r \leq t$

$\rightarrow r \leq s \Rightarrow r + t \leq s + t$

$\rightarrow r \geq 0$ & $s \geq 0 \Rightarrow rs \geq 0$

There is a notion of positive numbers

Complete Satisfies "Axiom" of completeness

Every non empty set of real numbers that is bounded above has a least upper bound.

upper bound

Defn $A \subseteq \mathbb{R}$ is <u>bounded above</u> if $\exists\, b \in \mathbb{R}$ s.t. $a \le b \ \forall a \in A$

$A \subseteq \mathbb{R}$ is <u>bounded below</u> if $\exists\, l \in \mathbb{R}$ s.t. $l \le a \ \forall a \in A$

For all

There exists   lower bound

Defn $s \in \mathbb{R}$ is the <u>least upper bound</u> (also called <u>supremum</u>) for a set $A \subseteq \mathbb{R}$ if

(i) s is an upper bound for A

(ii) If b is any upper bound for A then $s \le b$

We write <u>$s = \sup A$</u>

Similarly, greatest lower bound infimum. $t = \inf A$

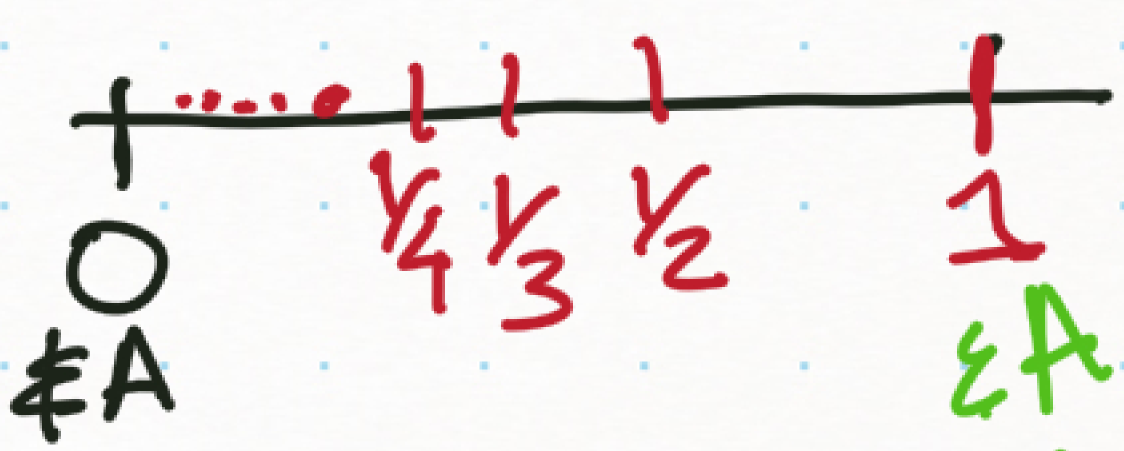
Observation A set can only have one supremum

If $s_1 = \sup A$ & $s_2 = \sup A$ then $s_1 \le s_2$ (by prop (ii) of s_1, prop (i) of s_2)

$s_2 \le s_1$ (— " — of s_2, of s_1)

so, $s_1 = s_2$

Example $A = \{\frac{1}{n} ; n \in \mathbb{N}\} = \{1, \frac{1}{2}, \frac{1}{3}, \dots\}$



$\sup A = 1$

(i) $\frac{1}{n} \le 1 \quad \forall n \in \mathbb{N}$

(ii) consider $b < 1$, then b is not an upper bound because $\underset{\in A}{1} > b$.

$\therefore$ 1 least upper bd.

$\inf A = \ ?$

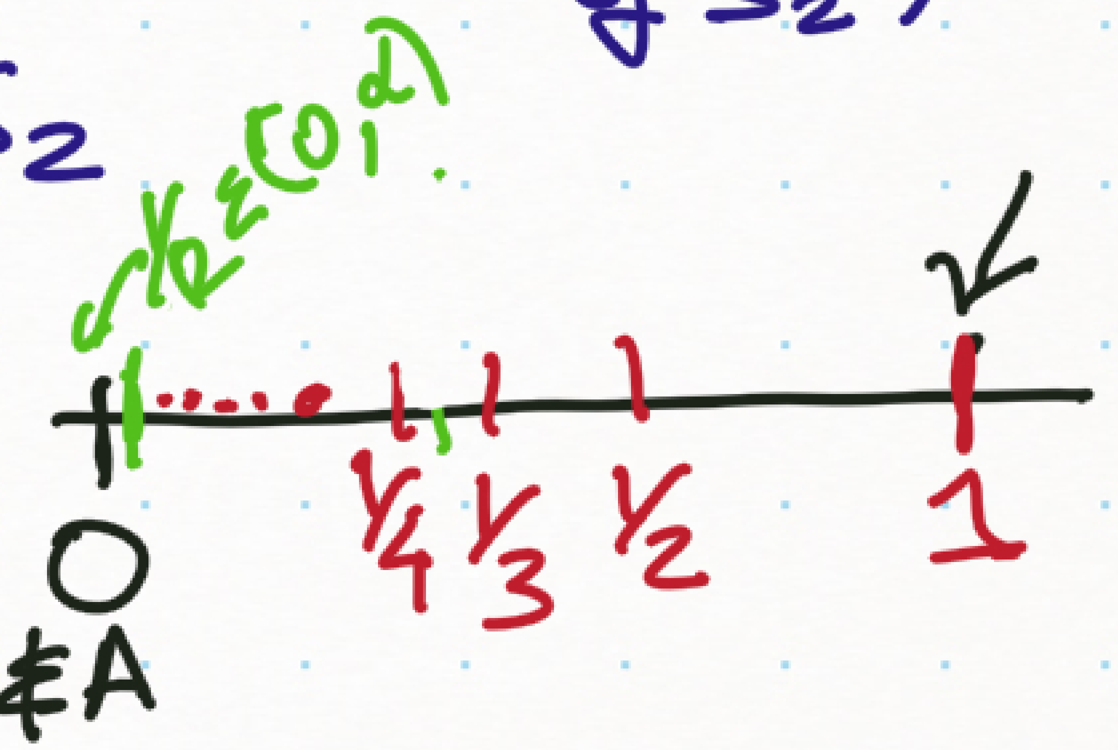
Observation A set can only have one supremum

If $s_1 = \sup A$ & $s_2 = \sup A$ then $s_1 \le s_2$ (by prop (ii) of s_1)

$s_2 \le s_1$ (—— " —— of s_2)

so, $s_1 = s_2$

Example $A = \left\{\frac{1}{n} ; n \in \mathbb{N}\right\} = \left\{1, \frac{1}{2}, \frac{1}{3}, \dots\right\}$



$\sup A = 1$

$\inf A = 0$ (i) $0 \le a$ $\left(= 1/n\right)$ for every $a \in A$ (lower bound)

(ii) It is not possible for $\alpha > 0$ to be a lower bd. of A.

α is not a lower bd of A

What about $\frac{1}{\alpha}$?

$\exists\, k \in \mathbb{N}$ s.t. $k > 1/\alpha \Rightarrow \frac{1}{k} < \alpha$, $\frac{1}{k} \in A$

so

Defn $a_0 \in \mathbb{R}$ is a maximum of $A \subseteq \mathbb{R}$ if $a_0 \in A$ and $a_0 \geq a \ \forall a \in A$

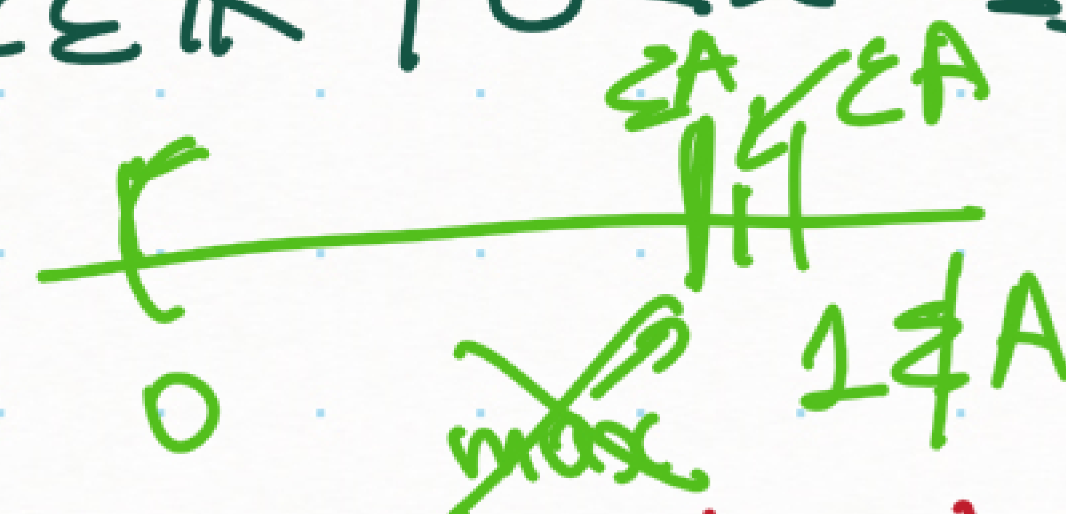
$a_1 \in \mathbb{R}$ is a minimum of $A \subseteq \mathbb{R}$ if $a_1 \in A$ and $a_1 \leq a \ \forall a \in A$

e.g. $[0,1] = \{x \in \mathbb{R} \mid 0 \leq x \leq 1\}$ max $= 1 =$ sup

min $= 0 =$ inf

$(0,1) = \{x \in \mathbb{R} \mid 0 < x < 1\}$ max does not exist sup $= 1$

min does not exist inf $= 0$



Although not every nonempty bounded ~~set~~ in $\mathbb{R}$ contains max, Axiom of completeness tells us that it will always have a sup.

But what about $\mathbb{Q}$?

$\sqrt{2} = 1.4142.....$

<u>example in $\mathbb{Q}$</u> $S = \{ q \in \mathbb{Q} : q^2 < 2 \}$

Is S bounded above? Yes $10, 5, 2, 3/2 \ldots$

Does S have a supremum in $\mathbb{Q}$? No. Why?

Does S have a supremum in $\mathbb{R}$? Yes $\sqrt{2}$

<u>Thm</u> There exists an unique complete ordered field.

We call this field $\mathbb{R}$.

Try to prove: Let $A \subseteq \mathbb{R}$ be nonempty & bounded above.

(Example 1.3.7)

Let $s = \sup A$.

Define $c + A = \{c + a \mid a \in A\}$

Prove that $\sup(c+A) = c + s$.

upper bound of $c+A$? ✓

least upper bound of $c+A$? ✓

Alternate way of thinking about "sup A is the least upper bound"

Lemma Let $s \in \mathbb{R}$ be an upper bound for $A \subseteq \mathbb{R}$.

Then, $s = \sup A \iff$ For each $\epsilon > 0$, $\exists\, a \in A$ s.t. $s - \epsilon < a$

any number smaller than s is not an upper bd. of A

$\boxed{\Rightarrow}$ $s = \sup A$

$s - \epsilon < s$

$\Rightarrow$ $s - \epsilon$ is not an upper bound of A

since s is the least u.b. of A (prop. (ii) of $s = \sup A$)

$\Rightarrow \exists\, a \in A$ s.t. $\underbrace{s - \epsilon}_{\text{not an u.b.}} < a$.

Alternate way of thinking about "sup A is the least upper bound"

Lemma Let $s \in \mathbb{R}$ be an upper bound for $A \subseteq \mathbb{R}$.
Then, $s = \sup A \iff$ For each $\epsilon > 0$, $\exists a \in A$ s.t. $s - \epsilon < a$ (*)

$\boxed{\Leftarrow}$ verify property (i) of defn. of $s = \sup A$

If b is any number less than s

then claim b is not an upper bound
$\iff \exists a \in A$ s.t. $b < a$
$\Leftarrow$ Take $\epsilon = s - b > 0$
apply (*), $\exists a \in A$ s.t. $s - (s-b) < a$
$\iff b < a$.
b is not an u.b.