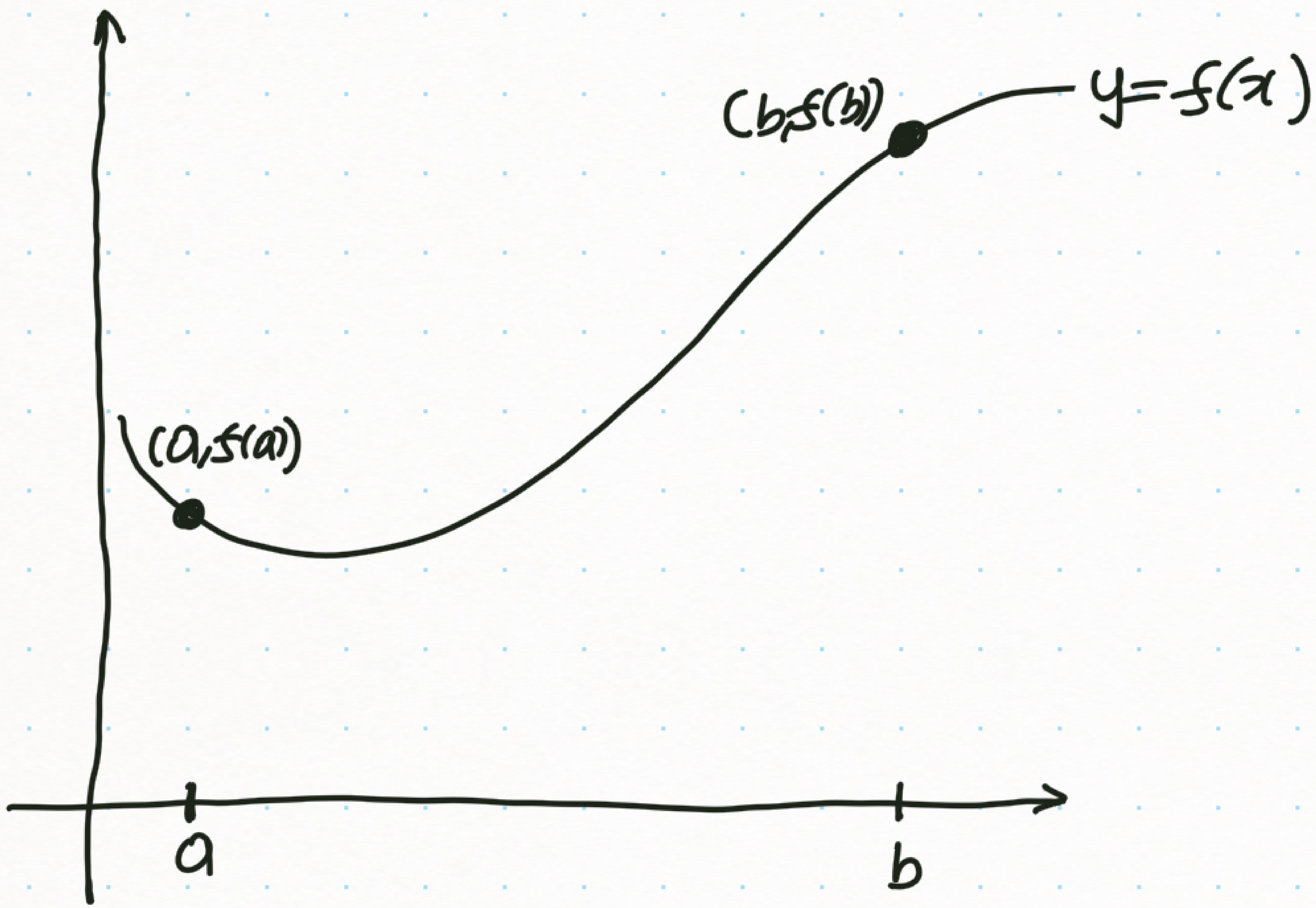
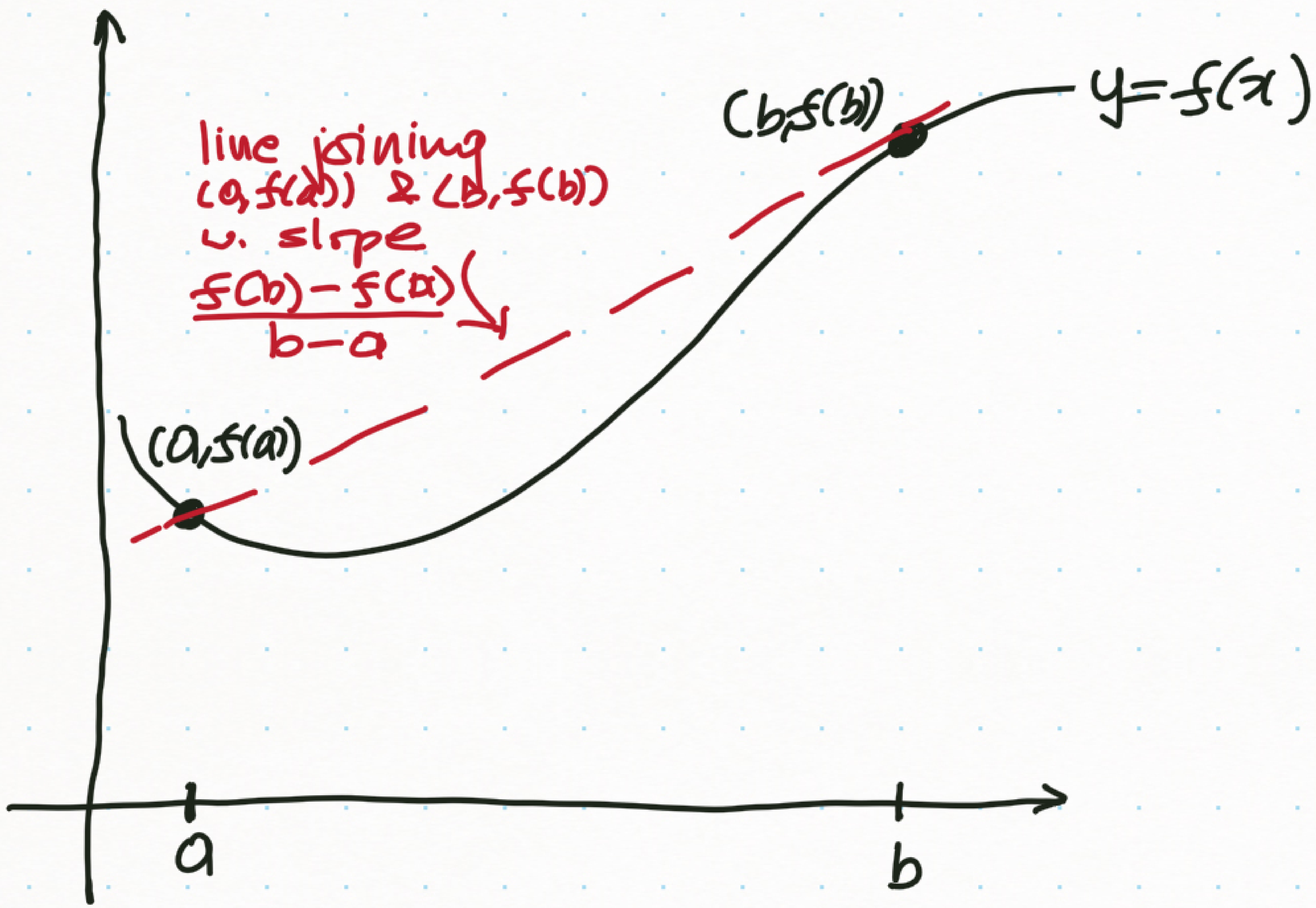
Math 400

Real Analysis

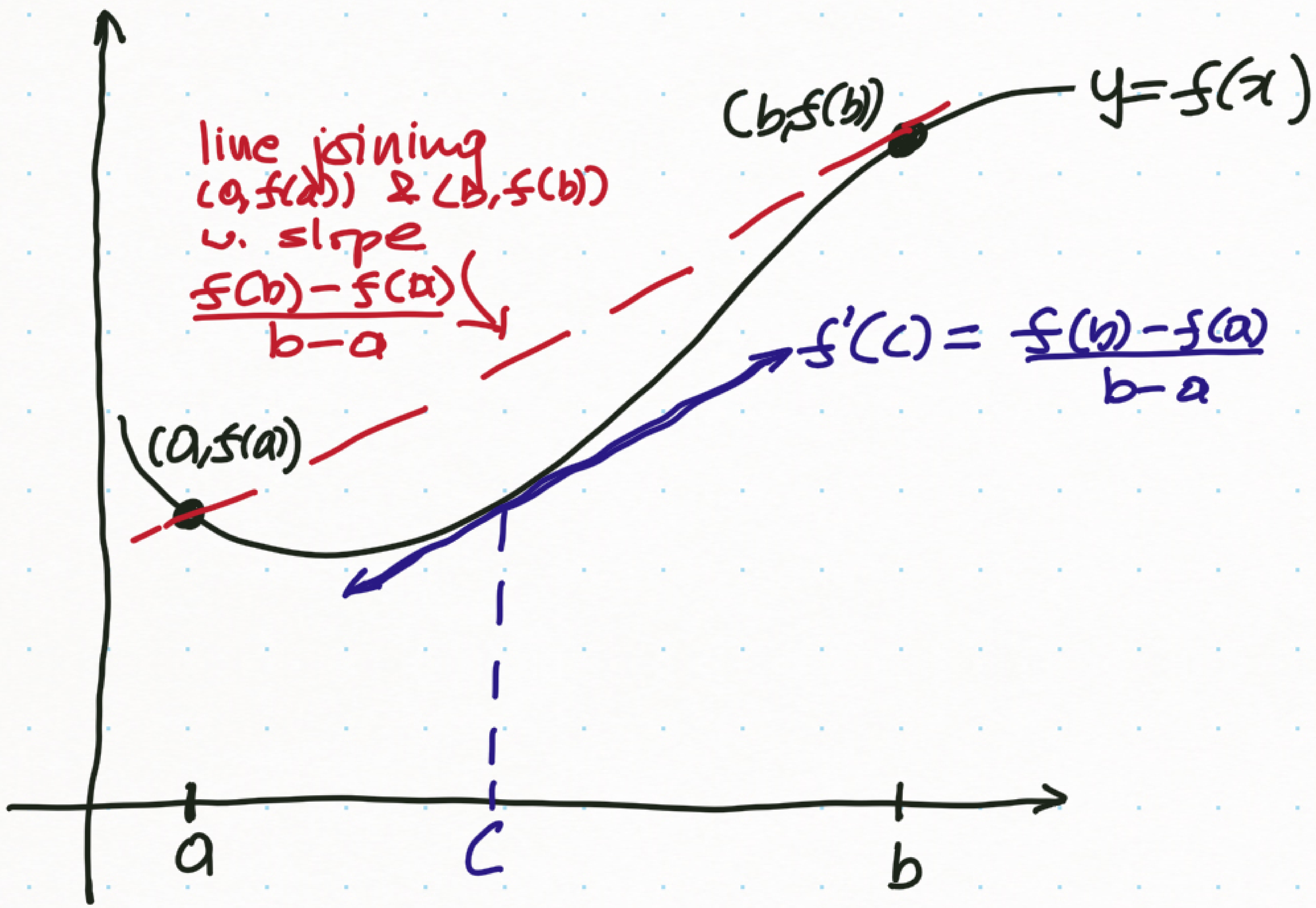
Part #29



(b,f(b))
y=f(x)
(a,f(a))
a
b



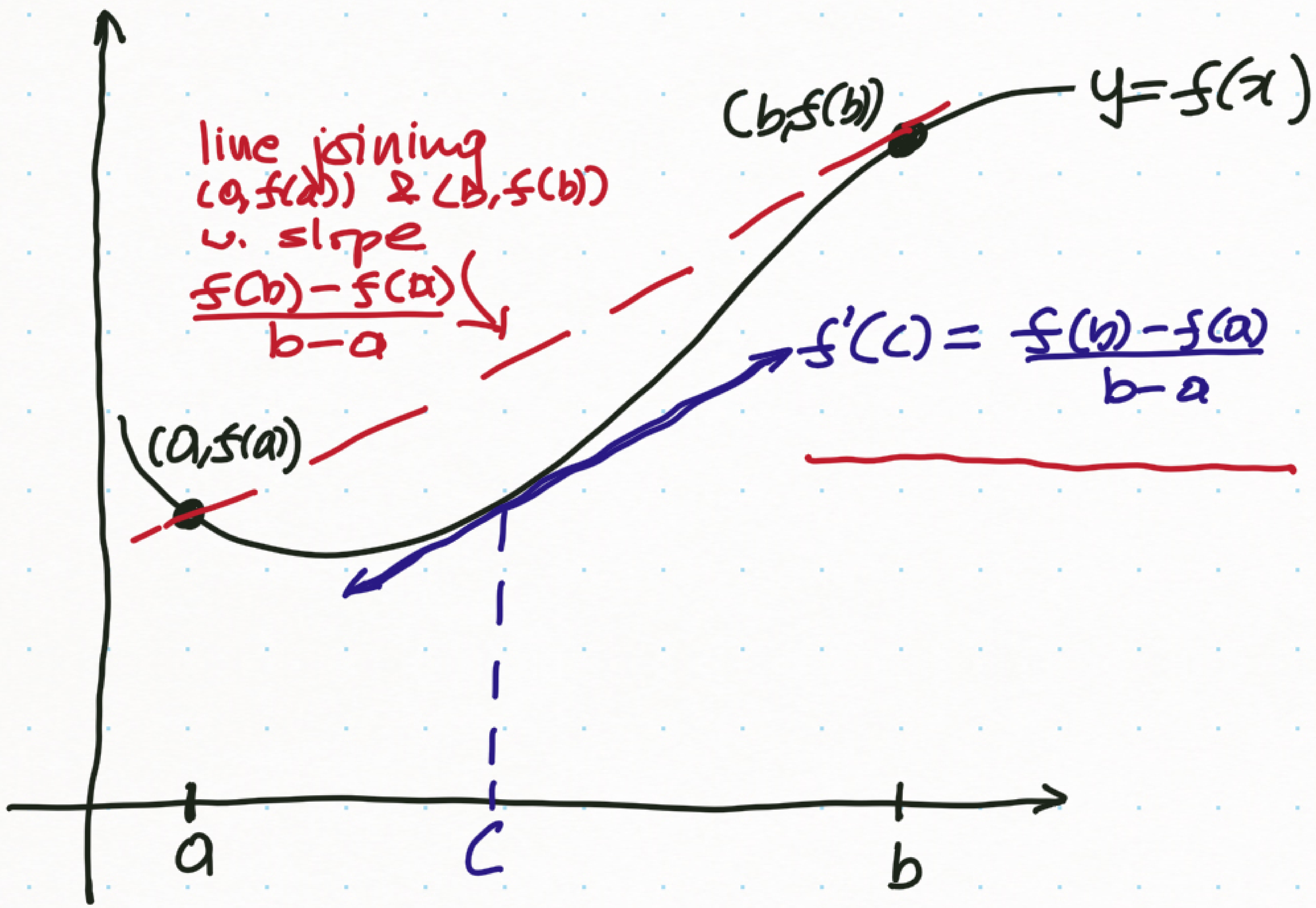
line joining
(a, f(a)) & (b, f(b))
w. slope
$\frac{f(b)-f(a)}{b-a}$
(b, f(b))
y = f(x)
(a, f(a))
a
b



Mean Value Theorem

$\exists\, c \in (a,b)$ s.t. $f'(c) = \dfrac{f(b)-f(a)}{b-a}$

A sort of IVT for derivatives



We have already done the hard work.

f on $[a,b]$ achieves it max & min by EVT

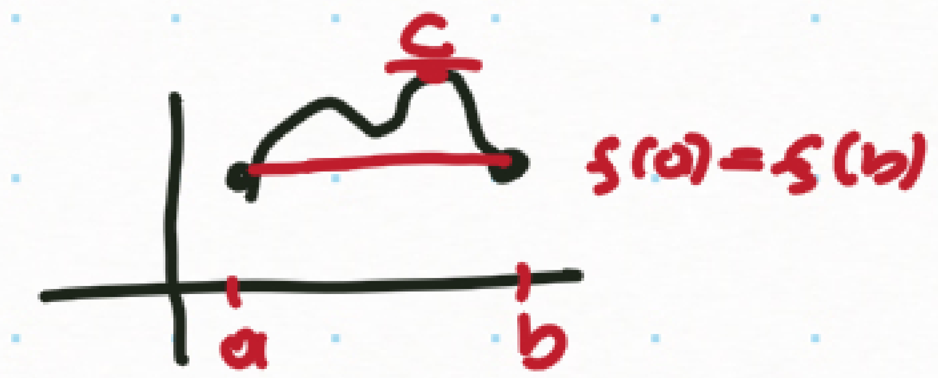
combine with Interior Extremum Thm. that $f'(c)=0$ when c is max/min

Mean Value Theorem

$\exists\, c \in (a,b)$ s.t. $f'(c)=\frac{f(b)-f(a)}{b-a}$

Rolle's Theorem

Let $f:[a,b]\to\mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b).
If $f(a)=f(b)$ then $\exists\, c\in(a,b)$ s.t. $f'(c)=0$



Note $f'(c)=\dfrac{f(b)-f(a)}{b-a}=0$

Proof f is cont. on a compact set, so f attains its max & min

If max & min occur on a or b then f is a constant function and $f'(c)=0\ \forall c\in(a,b)$

If max or min occur on $c\in(a,b)$ then by IFT $f'(c)=0$.

∎

Mean Value Theorem

If $f:[a,b]\to\mathbb{R}$ is continuous on $[a,b]$ and differentiable on (a,b), then $\exists\, c\in(a,b)$ s.t.

$$f'(c)=\frac{f(b)-f(a)}{b-c}$$

Proof (Idea: Reduce to Rolle's Thm)

The equation of the line through $(a,f(a))$ and $(b,f(b))$ is

$$y=\left(\frac{f(b)-f(a)}{b-a}\right)(x-a)+f(a)$$

Consider the difference between this line and $y=f(x)$:

$$d(x)=f(x)-\left[\left(\frac{f(b)-f(a)}{b-a}\right)(x-a)+f(a)\right]$$

d is continuous on $[a,b]$ & differentiable on (a,b) [By Algebra of cont. & diff. functions]

Mean Value Theorem If $f:[a,b]\to\mathbb{R}$ is continuous on $[a,b]$ and differentiable on (a,b), then $\exists\, c\in(a,b)$ s.t.

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d is continuous on $[a,b]$ & differentiable on (a,b) [By Algebra of cont. & diff. function]

and $d(a)=0=d(b)$

By Rolle's Thm applied to d, $\exists\, c\in(a,b)$ s.t. $d'(c)=0$.

$d'(c)=f'(c)-\dfrac{f(b)-f(a)}{b-a}$, i.e., $\exists\, c$ s.t. $f'(c)=\dfrac{f(b)-f(a)}{b-a}$

Cor If $g: I \to \mathbb{R}$ is differentiable on an interval I and $g'(x) = 0 \ \forall x \in I$, then $g(x) = k$ for some constant k

Proof Let $x_1, x_2 \in I$ with $x_1 < x_2$. We want to show $g(x_1) = g(x_2)$.

By MVT applied to g on $[x_1, x_2]$:

$$\exists\, c \in (x_1, x_2) \subseteq I \text{ s.t. } g'(c) = \frac{g(x_2) - g(x_1)}{x_2 - x_1}$$

Since $g'(c) = 0$, we get $g(x_1) = g(x_2)$.

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Since $g'(c) = 0$, we get $g(x_1) = g(x_2)$.

Cor If f and g are differentiable functions on an interval I and satisfy $f'(x) = g'(x) \ \forall x \in I$, then $f(x) = g(x) + k$ for some constant k.

Proof Try it! Can you reduce it to Cor above?

Recall, f increasing means $\underline{f(x_1) \leq f(x_2)}$ for any $\underline{x_1 < x_2}$.

__Cor__ Let $f: I \to \mathbb{R}$ be differentiable on the interval I.

(i) f is increasing $\iff f'(x) \geq 0 \ \forall x \in I$

(ii) f is decreasing $\iff f'(x) \leq 0 \ \forall x \in I$

__Proof__ Assume f is increasing.

This means $f(x) - f(c)$ and $x - c$ for any $x, c \in I$ are either both nonnegative (≥ 0), or both nonpositive (≤ 0),

$\therefore$ for any $x \neq c$, $\boxed{\dfrac{f(x) - f(c)}{x - c}} \geq 0 \quad \forall x, c \in I$

Hence $f'(c) = \lim\limits_{x \to c} \dfrac{f(x) - f(c)}{x - c} \geq 0$ as needed.

Recall, f increasing means $f(x_1) \leq f(x_2)$ for any $x_1 < x_2$.

Cor Let $f: I \to \mathbb{R}$ be differentiable on the interval I.

(i) f is increasing $\iff$ $f'(x) \geq 0 \ \forall x \in I$

(ii) f is decreasing $\iff$ $f'(x) \leq 0 \ \forall x \in I$

Proof Assume $f'(x) \geq 0 \ \forall x \in I$

For any $x_1, x_2 \in I$ with $x_1 < x_2$,

by MVT $\exists\, c \in (x_1, x_2) \subseteq I$ s.t. $f'(c) = \dfrac{f(x_2) - f(x_1)}{x_2 - x_1}$

That is, $f(x_2) - f(x_1) = f'(c)\,(x_2 - x_1)$

≥ 0 since $x_2 \geq x_1$

and $f'(c) \geq 0$

ie, $f(x_2) \geq f(x_1)$.

Generalized Mean Value Theorem

If f and g are continuous on $[a,b]$ and differen. on (a,b) then $\exists\, c \in (a,b)$ s.t. $[f(b)-f(a)]g'(c) = [g(b)-g(a)]f'(c)$

If g' is never zero on (a,b) then we can say

$$\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$$

Proof Apply MVT to $h(x) = [f(b)-f(a)]g(x) - [g(b)-g(a)]f(x)$

L'Hospital's Rules for evaluating $\lim_{x\to c} \frac{f(x)}{g(x)}$

$$= \frac{\lim f(x)}{\lim g(x)}$$ (assuming both limits exist)

Theorem [L'Hospital for 0/0 form]

Let I be an open interval containing pt. a.
Suppose f and g are differentiable on I, except possibly a.

If $f(a) = g(a) = 0$, and $g'(x) \neq 0 \ \forall x \neq a$,

then $\lim_{x\to a} \frac{f'(x)}{g'(x)} = L \implies \lim_{x\to a} \frac{f(x)}{g(x)} = L$

Proof [HW?]

- Write the definition of $\lim_{x\to a} \frac{f(x)}{g(x)} = L$ to find a δ
- Apply GMVT to f and g in $[a, x]$ (& $[x, a]$)
- Verify the definition of $\lim_{x\to a} \frac{f'(x)}{g'(x)} = L$

Definition $\left[\lim_{x\to c} f(x)=\infty\right]$

Given $f: I \to \mathbb{R}$ and a limit point c of I,

we say $\lim_{x\to c} f(x) = \infty$ if $\forall M>0$ $\exists \delta>0$ s.t.

$$0<|x-c|<\delta \Rightarrow f(x)\geq M.$$

Similarly, define $\lim f(x) = -\infty$

Definition $\left[\lim_{x\to c} f(x)=\infty\right]$

Given $f: I \to \mathbb{R}$ and a limit point c of I, we say $\lim_{x\to c} f(x) = \infty$ if $\forall M>0\ \exists \delta>0$ s.t.

$$0<|x-c|<\delta \Rightarrow f(x) \geq M.$$

Theorem [L'Hospital for ∞/∞ form]

Suppose f and g are differentiable on (a,b) and $g'(x) \neq 0\ \forall x \in (a,b)$.

If $\lim_{x\to a} g(x) = \infty$ (or $-\infty$), then

$$\lim_{x\to a} \frac{f'(x)}{g'(x)} = L \Rightarrow \lim_{x\to a} \frac{f(x)}{g(x)} = L$$

Note this is the only req.

MATH 400

Real Analysis

Part # 30

We want understand convergence of

— sequence of functions

— series of functions

and build towards justifying Taylor / Mclaurin series expansions of functions.

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

Let $f_n : A \to \mathbb{R}$ be a function for each $n \in \mathbb{N}$

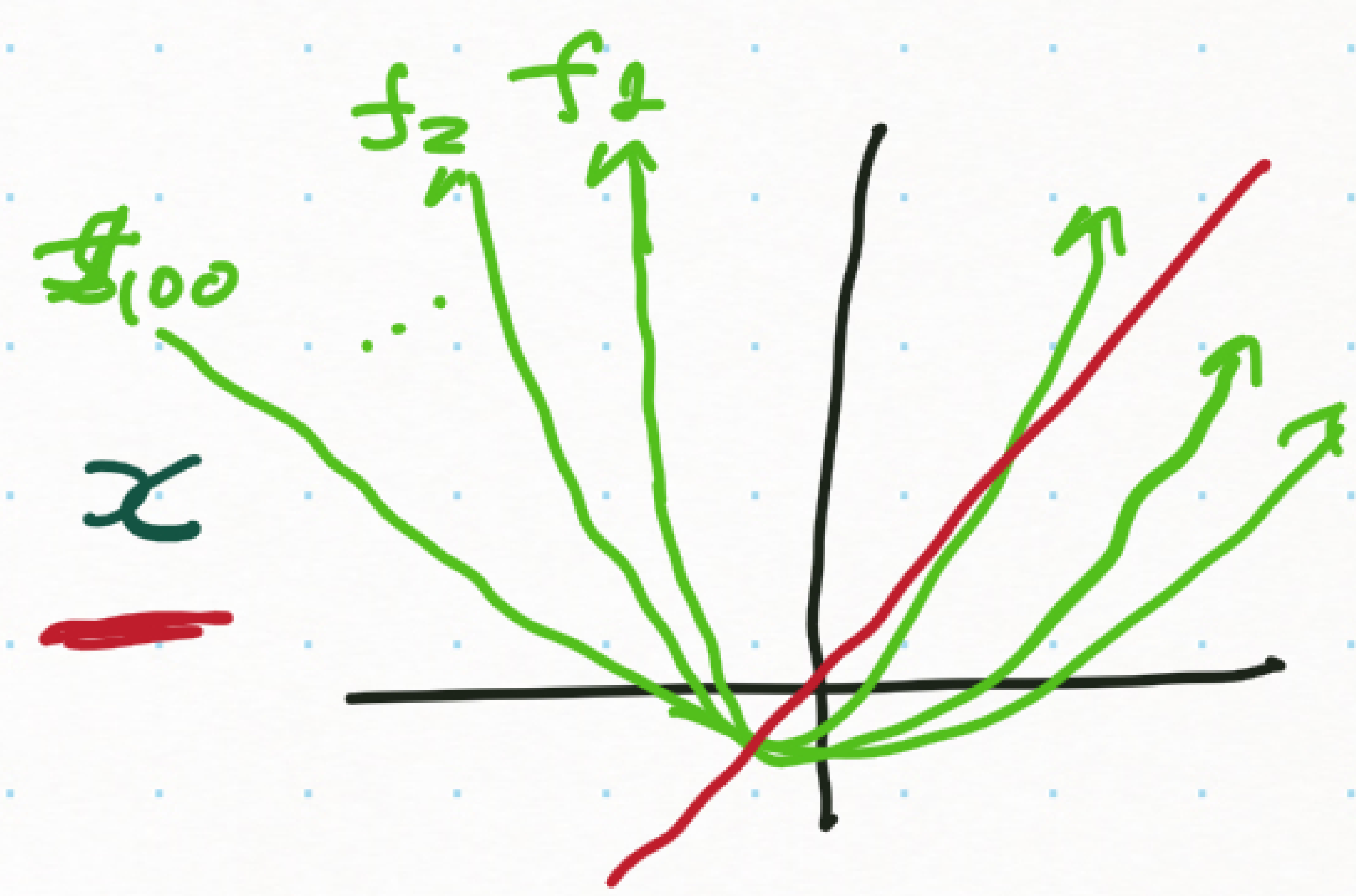
<u>Defn</u> The sequence (f_n) of functions <u>converges pointwise</u> on A to a function f if for all $x \in A$, the sequence of numbers $f_n(x)$ converges to $f(x)$ as $n \to \infty$.

we write $\lim_{n\to\infty} f_n(x) = f(x)$ or $f_n \to f$

<u>Examples</u>

① $f_n(x) = (x^2+nx)/n$ on $\mathbb{R}$

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{x^2+nx}{n} = \lim_{n\to\infty}\left(\frac{x^2}{n}+x\right) = \underline{x}$$



$\therefore f_n \to f$ pointwise where $f(x) = x$

If f_n is continuous for each n, Then is f (where $f_n \to f$) also continuous?

Let $f_n \to f$ where each f_n is continuous.

To show: f is continuous we have to show $\underline{|f(x)-f(c)|} < \epsilon$ when $|x-c|<\delta$

$$|f(x)-f(c)| = |f(x)-f_k(x)+f_k(x)-f_k(c)+f_k(c)-f(c)|$$

$$\leq \underbrace{|f_k(x)-f(x)|}_{<\epsilon/3} + \underbrace{|f_k(x)-f_k(c)|}_{<\epsilon/3} + \underbrace{|f_k(c)-f(c)|}_{<\epsilon/3}$$

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since $f_k \to f$ for $k \geq k_0$

If f_n is continuous for each n, Then is f (where $f_n \to f$) also continuous?

Let $f_n \to f$ where each f_n is continuous.

To show: f is continuous we have to show $|f(x) - f(c)| < \epsilon$ when $|x-c| < \delta$

$$|f(x) - f(c)| = |f(x) - f_k(x) + f_k(x) - f_k(c) + f_k(c) - f(c)|$$

$$\leq \underbrace{|f_k(x) - f(x)|}_{< \epsilon/3} + \underbrace{|f_k(x) - f_k(c)|}_{< \epsilon/3 \text{ since } f_k \text{ is continuous}} + \underbrace{|f_k(c) - f(c)|}_{< \epsilon/3 \text{ since } f_k \to f}$$

?? $\leftarrow$ for $k \geq k_0$ $\leftarrow$ for $k \geq k_0$

same k_0 may not work for every such x

If f_n is continuous for each n, Then is f (where $f_n \to f$) also continuous?

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$$|f(x)-f(c)| = |f(x)-f_k(x)+f_k(x)-f_k(c)+f_k(c)-f(c)|$$

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?? $\leftarrow$ for $k \geq k_0$ $\leftarrow$ for $k \geq k_0$

same k_0 may not work

We need $k = \sup\{k_x, k_0 : x \in (c-\delta, c+\delta)\}$ But!

Examples

(2) Let $g_n(x) = x^n$ on $[0,1]$.

We know $x^n \to 0$ if $x \in [0,1)$ & $x^n \to 1$ if $x = 1$

$\therefore g_n \to g$ pointwise where $g(x) = \begin{cases} 0 & \text{if } 0 \le x < 1 \\ 1 & \text{if } x = 1 \end{cases}$

Each g_n is continuous but g is not.

Examples

② Let $g_n(x) = x^n$ on $[0,1]$.

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Each g_n is continuous but g is not.

③ Let $h_n(x) = x^{1 + \frac{1}{2n-1}}$ on $[-1,1]$

For each fixed $x \in [-1,1]$

$$\lim_{n\to\infty} h_n(x) = x \overbrace{\lim_{n\to\infty} x^{\frac{1}{2n-1}}}^{= \begin{cases} -1 & \text{if } x<0 \\ +1 & \text{if } x>0 \end{cases}} = |x| = h(x)$$

Each h_n is differentiable but h is not.

ϵ-N definition of pointwise convergence of f_n

$f_n \to f$ means For x, $\forall \epsilon > 0$ $\exists N$ (possibly dependent on x) s.t.

$|f_n(x) - f(x)| < \epsilon \quad \forall n \geq N.$

we want N to work for all x simultaneously.

Definition $f_n : A \to \mathbb{R}$ be a sequence of functions (f_n) converges uniformly on A to f defined on A if $\forall \epsilon > 0$ $\exists N$ s.t. $|f_n(x) - f(x)| < \epsilon$ for all $n \geq N$ and $x \in A$

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Definition $f_n : A \to \mathbb{R}$ be a sequence of functions

(f_n) converges uniformly on A to f defined on A

if $\forall \epsilon > 0$ $\exists N$ s.t. $|f_n(x) - f(x)| < \epsilon$ for all $n \geq N$ and $x \in A$

Examples ① $g_n(x) = \dfrac{1}{n(1+x^2)}$ on $\mathbb{R}$.

For any fixed $x \in \mathbb{R}$, $g_n(x) \to 0$ as $n \to \infty$, so $g(x) = 0$ is the pointwise limit.

Uniform? $|g_n(x) - g(x)| = \left|\dfrac{1}{n(1+x^2)} - 0\right| = \left|\dfrac{1}{n(1+x^2)}\right| \leq \dfrac{1}{n}$ $\forall x \in \mathbb{R}$

Given $\epsilon > 0$, $\exists$ $N > 1/\epsilon$ s.t. $|g_n(x) - g(x)| < \epsilon$ for $n \geq N$

(2) $f_n(x) = (x^2+nx)/n \longrightarrow f(x) = x$ pointwise

On $\mathbb{R}$, this convergence is not uniform

$|f_n(x) - f(x)| = \left|\frac{x^2+nx}{n} - x\right| = \frac{x^2}{n} < \epsilon$ means $N > \frac{x^2}{\epsilon}$

On $[-b, b]$, this convergence is uniform

$|f_n(x) - f(x)| = \frac{x^2}{n} \leq \frac{b^2}{n} < \epsilon$ means $N > \frac{b^2}{\epsilon}$

Theorem [Cauchy Criterion for Uniform Convergence]

(f_n) seq. defined on $A \subseteq \mathbb{R}$ converges uniformly on A

iff $\underbrace{\forall \epsilon > 0 \ \exists N \text{ s.t. } |f_n(x) - f_m(x)| < \epsilon \ \forall m, n > N \text{ and } x \in A}_{(f_n) \text{ is Cauchy}}$

Proof

Using Cauchy criterion for convergence of sequence (of numbers)

Continuous Limit Theorem

Let (f_n) converge uniformly on A to f.
If each f_n is continuous at $c \in A$ then f is continuous at c.

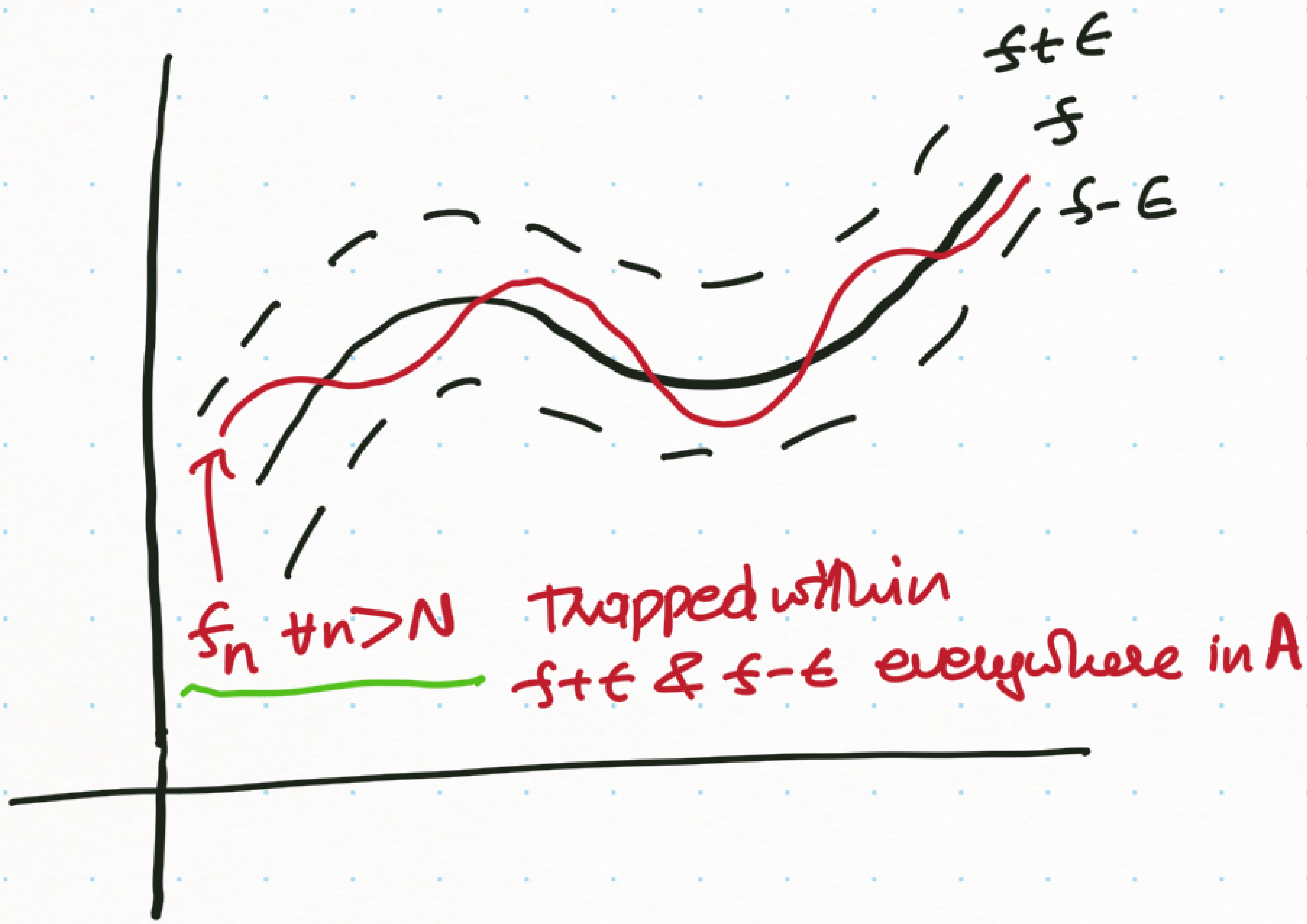
Proof Fix $c \in A$ & let $\epsilon > 0$.

By Unif. Convg. Choose N s.T. $|f_k(x) - f(x)| < \epsilon/3 \quad \forall k \geq N$ & $x \in A$

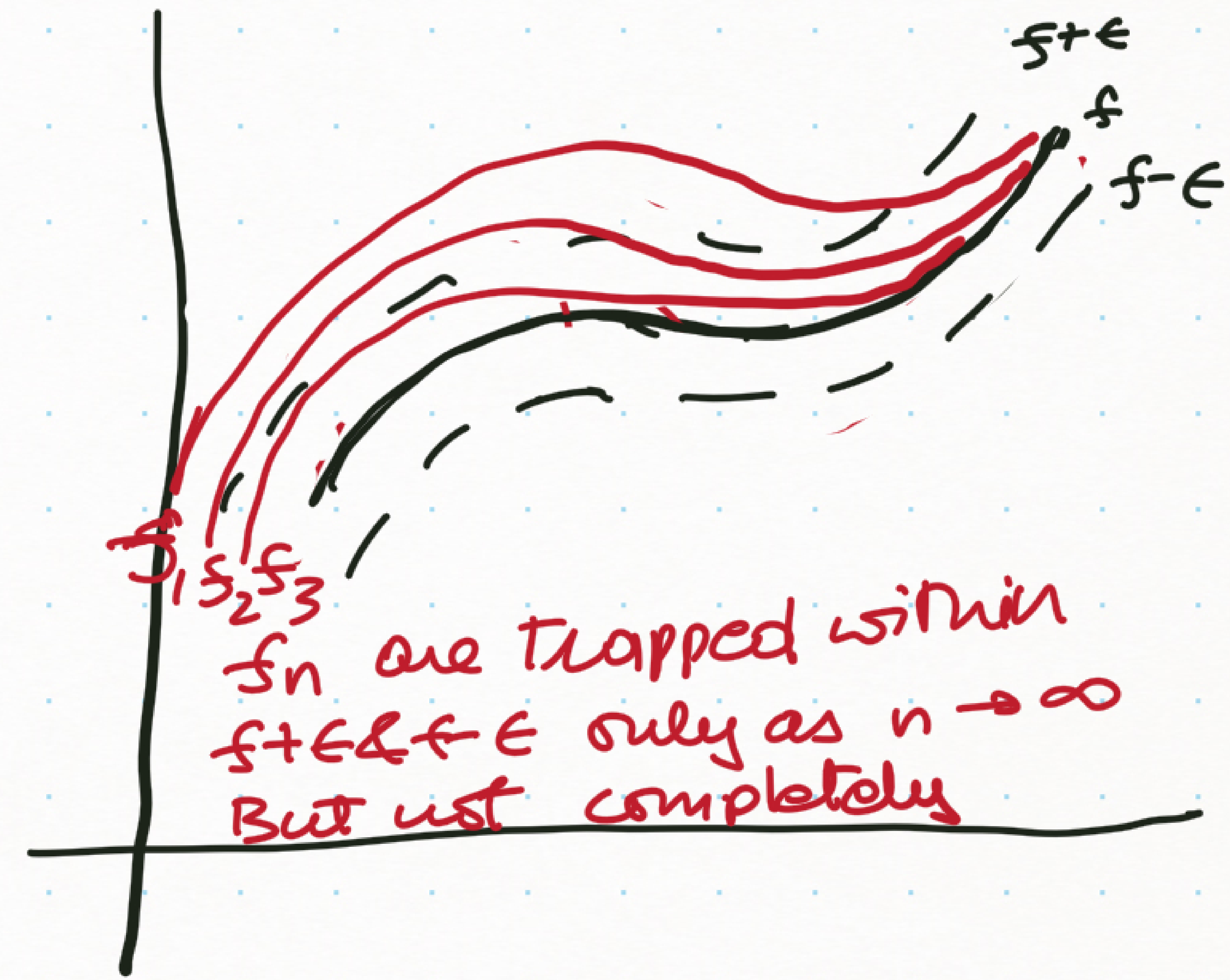
so $|f_N(x) - f(x)| < \epsilon/3 \quad \forall x \in A$. $(k = N)$

By f_N continuous) $\exists \delta > 0$ s.T. $|f_N(x) - f_N(c)| < \epsilon/3$ for $|x-c| < \delta$

$$\therefore |f(x) - f(c)| = |f(x) - f_N(x) + f_N(x) - f_N(c) + f_N(c) - f(c)|$$
$$\leq |f(x) - f_N(x)| + |f_N(x) - f_N(c)| + |f_N(c) - f(c)|$$
$$< \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon$$



Uniform Convergence



Pointwise convergence

We already saw pointwise convergence does not preserve differentiability.

Does uniform convergence preserve differentiability?

<u>Example</u> $f_n : [-1, 1] \to \mathbb{R}$ with $f_n(x) = x^{1 + 1/2n-1}$

we saw $(f_n) \longrightarrow f(x) = |x|$ pointwise

Also check (!) (f_n) uniformly converges to f

Each f_n is differentiable on $[-1, 1]$

but f is not differentiable at 0.

We already saw pt.wise/Unif. convergence does not preserve differentiability.

Does uniform convergence "preserve derivatives"?

<u>Example</u> let $g_n : [-2,2] \to \mathbb{R}$ as $g_n(x) = \dfrac{x}{1+nx^2}$

(g_n) converges to $g(x) = 0$ both pointwise and uniformly.

However, note $g_n'(0) = 1 \;\; \forall n$ $\left(g_n'(x) = \dfrac{1-nx^2}{(1+kx^2)^2}, \text{ by Q. Rule} \right)$

But $g'(0) = 0$

So the derivatives may not match.

Differentiability Limit Theorem

Let $f_n \to f$ pointwise on $[a,b]$, and assume f_n' exists for all n.

If (f_n') converges uniformly on $[a,b]$ to g

then f is differentiable and $f' = g$.

we need uniform convergence of (f_n') to ensure that limit of (f_n) preserves differentiability and the derivatives match.

Differentiability Limit Theorem (stronger)

Let ~~$f_n \to f$ pointwise on $[a,b]$~~, and assume f_n' exists for all n.

If (f_n') converges uniformly on $[a,b]$ to g

then f is differentiable and $f' = g$.

This can be replaced by a weaker requirement:

$\exists\, x_0 \in [a,b]$ s.t. $\underbrace{f_n(x_0)}_{\text{sequence of numbers}}$ is convergent

This gives us: $f_n \to f$ uniform convergence. & $f' = g$